SELF EXCITATION OF WAVES WITH DIFFERENT POLARIZATIONS IN NONLINEAR MEDIA

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Equations are derived for envelope waves with different circular polarizations in an isotropic nonlinear medium. The stability and self-focusing of such waves and the steady-state envelope waves are investigated. The results are applied to the problem of nonlinear electromagnetic waves in an isotropic plasma.

1. INTRODUCTION

OWING to the tensor character of certain important nonlinearity mechanisms (e.g., the high-frequency Kerr effect[1]), it is necessary to take into account the polarization of the electromagnetic waves when their self-action is considered. In this paper we consider the general problem of self-action of waves with different polarizations in an isotropic and nongyrotropic medium. In Sec. 2 we derive a system of nonstationary parabolic equations for the envelopes of the waves with two circular polarizations. With the aid of these equations we analyze in Secs. 3 and 4 the instability and the self-focusing of waves with different polarizations, and also the "envelope waves" that are established.

We shall show that under rather general assumptions the total "number of quanta" with each of the circular polarizations is a conserved quantity. We conclude therefore that a wave with specified circular polarization can break up into waves with linear polarization. To the contrary, a wave with linear polarization can under definite conditions be unstable against decay into waves with opposite circular polarizations. As to the steady-state envelope waves, it turns out that, besides previously known periodic and solitary waves[2], there can exist many more complicated types of steady-state waves, which can be treated as "nondissipative envelope shock waves."

In Sec. 5 we consider the instabilities of electromagnetic waves with circular and linear polarizations in an isotropic plasma.

2. FUNDAMENTAL EQUATIONS

We consider an isotropic non-gyrotropic medium having cubic nonlinearity. The equations of the electromagnetic field in such a medium are of the form

$$\left\{k^2\delta_{\alpha\beta}-k_\alpha k_\beta-\frac{\omega^2}{c^2}\varepsilon^{(0)}_{\alpha\beta}(\omega)\right\}E_\beta(\mathbf{k}\omega)$$
$$=\frac{\omega^2}{c^2}\int\varepsilon^{(1)}_{\alpha\beta\gamma\delta}(\mathbf{k}\omega,\mathbf{k}_1\omega_1,\mathbf{k}_2\omega_2,\mathbf{k}_3\omega_3)E_\beta(\mathbf{k}_1\omega_1)$$
$$\times E_\gamma(\mathbf{k}_2\omega_2)E_\delta(\mathbf{k}_3\omega_3)\,\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}\delta_{\omega-\omega_1-\omega_2-\omega_3}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,d\omega_1\,d\omega_2\,d\omega_3. \quad (1)$$

Here

$$\varepsilon^{(0)}_{\alpha\beta}(\omega)=\varepsilon^{tr}(\omega)\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)+\varepsilon^{l}(\omega)\frac{k_\alpha k_\beta}{k^2}$$

and $\epsilon^{(1)}_{\alpha\beta\gamma\delta}(\mathbf{k}\omega, \mathbf{k}_1\omega_1, \mathbf{k}_2\omega_2, \mathbf{k}_3\omega_3)$ are respectively the linear and nonlinear parts of the dielectric tensor. The tensor $\epsilon^{(1)}_{\alpha\beta\gamma\delta}$ is symmetrical in the last three indices.

We shall henceforth assume that the frequencies and the wave vectors of all the waves are close to one another. We break up the electromagnetic field into positive-frequency and negative-frequency parts:

$$\mathbf{E}_{k\omega}=\mathbf{A}_{\mathbf{k}\omega}+\mathbf{A}^*_{-\mathbf{k},-\omega}.$$

For the new variables we have

$$\left\{k^2\delta_{\alpha\beta}-k_\alpha k_\beta-\frac{\omega^2}{c^2}\varepsilon^{(0)}_{\alpha\beta}\right\}A_\beta=\frac{\omega^2}{c^2}\int\eta_{\alpha\beta\gamma\delta}(\mathbf{k}\omega,\mathbf{k}_1\omega_1,\mathbf{k}_2\omega_2,\mathbf{k}_3\omega_3)$$
$$\times A_\beta^*(\mathbf{k}_1\omega_1)A_\gamma(\mathbf{k}_2\omega_2)A_\delta(\mathbf{k}_3\omega_3)\,\delta_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}$$
$$\times\delta_{\omega+\omega_1-\omega_2-\omega_3}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,d\omega_1\,d\omega_2\,d\omega_3. \quad (2)$$

Here

$$\eta_{\alpha\beta\gamma\delta}(\mathbf{k},\omega,\mathbf{k}_1,\omega_1,\mathbf{k}_2,\omega_2,\mathbf{k}_3,\omega_3)=3\varepsilon^{(1)}_{\alpha\beta\gamma\delta}(\mathbf{k},\omega,-\mathbf{k}_1,-\omega_1,\mathbf{k}_2,\omega_2,\mathbf{k}_3,\omega_3).$$

In a transparent medium the tensor $\eta_{\alpha\beta\gamma\delta}$ has the following symmetry properties:

$$\eta_{\alpha\beta\gamma\delta}(\mathbf{k},\omega,\mathbf{k}_1,\omega_1,\mathbf{k}_2,\omega_2,\mathbf{k}_3,\omega_3)=\eta_{\beta\alpha\gamma\delta}(\mathbf{k}_1,\omega_1,\mathbf{k},\omega,\mathbf{k}_2,\omega_2,\mathbf{k}_3,\omega_3)$$
$$=\eta_{\alpha\beta\delta\gamma}(\mathbf{k},\omega,\mathbf{k}_1,\omega_1,\mathbf{k}_3,\omega_3,\mathbf{k}_2,\omega_2)=\eta_{\gamma\delta\alpha\beta}(\mathbf{k}_2,\omega_2,\mathbf{k}_3,\omega_3,\mathbf{k},\omega,\mathbf{k}_1,\omega_1). \quad (3)$$

We assume that there are no longitudinal oscillations in the medium, i.e., the equation $\epsilon^l = 0$ has no solutions. In this case the longitudinal component of the electric field is the result of only a small nonlinear term and can be neglected.

We introduce a unit polarization vector $\mathbf{n}(\mathbf{k})$ such that $(\mathbf{k}\cdot\mathbf{n}(\mathbf{k})) = 0$ and $|\mathbf{n}(\mathbf{k})|^2 = 1$, and we construct the vector*

$$\mathbf{S}(\mathbf{k})=\frac{1}{\sqrt{2}}\left[\mathbf{n}(\mathbf{k})+\frac{i[\mathbf{k},\mathbf{n}(\mathbf{k})]}{|\mathbf{k}|}\right],$$

having the following properties,

$$\mathbf{S}(-\mathbf{k})=\mathbf{S}^*(\mathbf{k}),$$
$$(\mathbf{S}(\mathbf{k}),\mathbf{S}(\mathbf{k}))=(\mathbf{S}^*(\mathbf{k}),\mathbf{S}^*(\mathbf{k}))=0,$$
$$(\mathbf{S}(\mathbf{k}),\mathbf{S}^*(\mathbf{k}))=1,$$
$$\left[\frac{\mathbf{k}}{|\mathbf{k}|},\mathbf{S}(\mathbf{k})\right]=i\mathbf{S}(\mathbf{k}),\quad\left[\frac{\mathbf{k}}{|\mathbf{k}|},\mathbf{S}^*(\mathbf{k})\right]=-i\mathbf{S}^*(\mathbf{k}).$$

The vector $\mathbf{S}(\mathbf{k})$ defined in this manner is the eigenvector of the Fresnel tensor

$$\Pi_{\alpha\beta}=k^2\delta_{\alpha\beta}-k_\alpha k_\beta-\frac{\omega^2}{c^2}\varepsilon^{(0)}_{\alpha\beta}.$$

We represent the Fourier component $\mathbf{A}_{\mathbf{k}\omega}$ in the form

*$[\mathbf{k},\mathbf{n}(\mathbf{k})]\equiv\mathbf{k}\times\mathbf{n}(\mathbf{k})$.

$$A_{k\omega} = S(k)\, b_{k\omega}^{+} + S^{*}(k)\, b_{k\omega}^{-}.$$

Then

$$\hat{\Pi} A_{k\omega} = S(k)\left(k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}\right) b_{k\omega}^{+} + S^{*}(k)\left(k^2 - \frac{\omega^2}{c^2}\varepsilon^{tr}\right) b_{k\omega}^{-}.$$

With such a definition, $b_{k\omega}^{+}$ and $b_{k\omega}^{-}$ are the complex amplitudes of the waves having different circular polarizations. In the case of sufficiently weak nonlinearity, we can assume that

$$\hat{\Pi} A_{k\omega} \approx -(\omega - \omega_k) S(k)\left(\frac{\omega^2}{c^2}\varepsilon^{tr}\right)'_{\omega_k} b_{k\omega}^{+} - (\omega - \omega_k) S^{*}(k)\left(\frac{\omega^2}{c^2}\varepsilon^{tr}\right)'_{\omega_k} b_{k\omega}^{-}, \quad (4)$$

where ω_k are the dispersion laws of the waves.

We introduce the mean wave number k_0 ($|k_i - k_0| \ll k_0$). It is then possible to put approximately $S(k_i) \approx S(k_0)$ and $\eta_{\alpha\beta\gamma\delta}(k, k_1, k_2, k_3) \approx \eta_{\alpha\beta\gamma\delta}(k_0, k_0, k_0, k_0) = \tilde{\eta}_{\alpha\beta\gamma\delta}$. Under these assumptions, in the general case of an isotropic and non-gyrotropic medium the nonlinear polarization tensor, is given by[3]:

$$\tilde{\eta}_{\alpha\beta\gamma\delta} = a\delta_{\alpha\beta}\delta_{\gamma\delta} + {}^1\!/_4 b(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}). \quad (5)$$

Multiplying (2) by $S(k)$ and $S^{*}(k)$ and taking (4) and (5) into account, we obtain

$$(\omega - \omega_k) b_{k\omega}^{+} + \int [Q b_{k_1\omega_1}^{+*} b_{k_2\omega_2}^{+} + P b_{k_1\omega_1}^{-*} b_{k_2\omega_2}^{-}]\, b_{k_3\omega_3}^{+} \times \delta_{k+k_1-k_2-k_3}\delta_{\omega+\omega_1-\omega_2-\omega_3}\, dk_1\, dk_2\, dk_3\, d\omega_1\, d\omega_2\, d\omega_3 = 0,$$

$$(\omega - \omega_k) b_{k\omega}^{-} + \int [Q b_{k_1\omega_1}^{-*} b_{k_2\omega_2}^{-} + P b_{k_1\omega_1}^{+*} b_{k_2\omega_2}^{+}]\, b_{k_3\omega_3}^{-} \times \delta_{k+k_1-k_2-k_3}\delta_{\omega+\omega_1-\omega_2-\omega_3}\, dk_1\, dk_2\, dk_3\, d\omega_1\, d\omega_2\, d\omega_3 = 0, \quad (6)$$

where

$$Q = \frac{1}{2} b \frac{\omega^2}{[\omega^2 \varepsilon^{tr}]'}, \quad P = \left(2a + \frac{1}{2} b\right) \frac{\omega^2}{[\omega^2 \varepsilon^{tr}]'} \quad (7)$$

In the system of equations (6), we expand ω_k in powers of $\delta_k = k - k_0$ up to second order, and take the inverse Fourier transforms with respect to the time and the coordinates. We introduce further the envelopes $a^{\pm}$ instead of the quantities $b^{\pm}$:

$$a^{\pm}(r, t) = e^{i\omega(k_0)t - ik_0 r} b^{\pm}(r, t).$$

The equations for the envelopes are

$$i\left(\frac{\partial a^+}{\partial t} + v\frac{\partial a^+}{\partial x}\right) + \hat{\mathscr{L}} a^+ = [Q|a^+|^2 + P|a^-|^2]a^+,$$

$$i\left(\frac{\partial a^-}{\partial t} + v\frac{\partial a^-}{\partial x}\right) + \hat{\mathscr{L}} a^- = [Q|a^-|^2 + P|a^+|^2]a^-. \quad (8)$$

Here

$$\hat{\mathscr{L}} = \frac{\omega_k''}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2k}\frac{\partial\omega}{\partial k}\Delta_\perp, \quad v = \frac{\partial\omega}{\partial k}.$$

The system (8) has the solutions

$$a^+ = A_1 e^{-i\Omega t + i\varkappa_1 x}, \quad a^- = A_2 e^{-i\Omega t + i\varkappa_2 x}, \quad (9)$$

where

$$\Omega = Q|A_1|^2 + P|A_2|^2 + \frac{\omega''}{2}\varkappa_1^2 + v\varkappa_1 = Q|A_2|^2 + P|A_1|^2 + \frac{\omega''}{2}\varkappa_2^2 + v\varkappa_2.$$

The solution (9) is a wave with elliptic polarization. As can be readily seen, when $Q \neq P$ we have $\kappa_1 \neq \kappa_2$, so that the plane of polarization of the elliptically-polarized wave rotates. This effect (in the case of a Kerr nonlinearity mechanism) was pointed out by Zel'dovich and Raĭzer[3].

When $a^+ = a^- = a/\sqrt{2}$, we have a linearly polarized wave satisfying the equation

$$i\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) a + \hat{\mathscr{L}} a = \frac{1}{2}(Q + P)\,|a|^2 a. \quad (10)$$

This equation was used earlier by one of the authors in[4].

We note that the quantities

$$G^+ = \int a^{+*} a^+ dr, \quad G^- = \int a^{-*} a^- dr \quad (11)$$

are integrals of motion of equations (8). This result can be interpreted as follows. The classical electromagnetic field described by us is the limiting state of a quantized field. The quantities a^+ and a^- are the classical analogs of the operators of annihilation of the quanta of the electromagnetic field, having specified circular polarizations, while Eqs. (8) are the analog of the Heisenberg equations of motion for these operators. The cubic nonlinearity in these equations corresponds to allowance for the processes of scattering of quanta by one another. In each scattering act, the total angular momentum should be conserved; this angular momentum consists of the spin (polarization) and orbital (relative motion) momenta. However, in our case of an almost monochromatic wave, the average orbital momentum is quite small (of the order of $\Delta k/k$) compared with the spin. Thus, the principal role is played by scattering processes that conserve the quantum polarization. In such a case, the total number of quanta of each polarization is conserved, corresponding to conservation of the quantities in (11). To take into account the nonlinear terms describing the scattering with change of polarization, in which the quantities in (11) are not conserved, it is necessary to expand the tensor $\eta_{\alpha\beta\gamma\delta}$ and the vectors $S(k_i)$ in powers of Δk_i.

The simplest nonlinearity mechanism is the scalar nonlinearity $D_{nl} = \epsilon_2 |E|^2 E$. In this case $a = 0$, $b = 2\epsilon_2$, and $P = Q$. For this mechanism, waves with different polarizations behave in the same manner.

Another important nonlinearity mechanism is the low-frequency Kerr effect, for which $D_{nl} = (1/4)\epsilon_2 [3E^*(E \cdot E) + E(E \cdot E^*)]$,

$$a = {}^3\!/_4\varepsilon_2, \quad b = {}^1\!/_2\varepsilon_2, \quad \text{т. е. } Q = \varepsilon_2/4, \quad P = {}^7\!/_4\varepsilon_2.$$

For the Kerr effect, the nonlinearity index for circular polarization ($Q = \epsilon_2/4$) is smaller by a factor of 4 than the nonlinearity index for linear polarization ($(P + Q)/2 = \epsilon_2$), as was already noted earlier in[1,3,6]. In this connection, a hypothesis was advanced in a number of papers[5,11] that in the case of the Kerr effect a wave with circular polarization breaks up into waves with linear polarization. However, in order to form a linearly polarized wave from a circularly-polarized wave it is necessary to have waves with opposite circular polarizations, which is impossible by virtue of the conservation of the quantities in (11).

We note that owing to the inertia of the Kerr effect, Eqs. (8) are suitable only for an analysis of modulation with sufficiently large longitudinal dimensions $l \gg v\tau_{rel}$, where τ_{rel} is the nonlinearity relaxation time.

Equations (8) describe also stationary beams. In this case

$$\frac{\partial}{\partial t} = 0, \quad \frac{\omega_k''}{2}\frac{\partial^2}{\partial x^2} \ll v\frac{\partial}{\partial x},$$

so that the equations take the form

$$\frac{\partial a^+}{\partial x} + \frac{1}{2k}\Delta_\perp a^+ = \frac{1}{v}[Q|a^+|^2 + P|a^-|^2]a^+,$$

$$\frac{\partial a^-}{\partial x}+\frac{1}{2k}\Delta_\perp a^-=\frac{1}{v}[Q|a^-|^2+P|a^+|^2]a^-. \qquad (12)$$

In view of the similarity of this system to the system (8) (x plays the role of the time), all the results obtained by us below can be transferred to the stationary case. In particular, from (12) we get the relations

$$\frac{\partial}{\partial x}\int a^+a^{+*}\,dr_\perp=0,\quad \frac{\partial}{\partial x}\int a^-a^{-*}\,dr_\perp=0,$$

which express the conservation of the number of quanta of each polarization along the x axis.

3. STABILITY OF MONOCHROMATIC WAVE

The problem of the stability of electromagnetic waves in nonlinear media was investigated in[4,7-9]. We confine ourselves to an investigation of the stability of a linearly-polarized and circularly-polarized wave with amplitude A_0. Waves with circular polarizations are described by the equation

$$i\left(\frac{\partial a}{\partial t}+v\frac{\partial a}{\partial x}\right)+\hat{\mathscr{L}}a=-Q|a|^2a. \qquad (13)$$

We choose a perturbation in the form $\delta a\sim e^{-i\Omega t+ipr}$. Then an investigation of the stability of Eq. (13) leads to the dispersion equation

$$\Omega_{1,2}=-vp\pm[(Lp^2)^2+2Lp^2QA_0^2]^{1/2}.$$

Here

$$L=\frac{1}{2}\xi_\alpha\xi_\beta\frac{\partial^2\omega}{\partial k_\alpha\partial k_\beta},\quad \boldsymbol{\xi}=\frac{\mathbf{p}}{|\mathbf{p}|}.$$

For $\boldsymbol{\xi}\parallel\mathbf{k}_0$ we have $L=(1/2)\partial^2\omega/\partial k^2$, and for $\boldsymbol{\xi}\perp\mathbf{k}_0$ we have $L=(1/2k)\partial\omega/\partial k>0$. The wave is unstable if $LQ<0$, which corresponds in the one-dimensional case to Lighthill's theorem[7] concerning the connection of the sign of the frequency shift and of the sign of the second derivative ω_k'' with the instability of the wave. In this instability, the waves excited have the same polarization as the initial wave.

For a linearly-polarized wave, the dispersion equation is given by

$$(\Omega+vp)^2-2QLp^2A_0^2-(Lp^2)^2=\pm2|P|Lp^2A_0^2.$$

This equation has two pairs of roots

$$\Omega_{1,2}=-vp\pm\sqrt{(Lp^2)^2+2Lp^2(Q+P)A_0^2},$$
$$\Omega_{3,4}=-vp\pm\sqrt{(Lp^2)^2+2Lp^2(Q-P)A_0^2}.$$

The first pair of roots corresponds to excitation of waves having the same polarization as the initial wave. Lighthill's theorem holds for this instability, too. The second pair of roots corresponds to "decay" of a linearly-polarized wave into two waves with different circular polarizations. When $|P|>|Q|$, one of the instabilities must occur.

If the nonlinearity mechanism is scalar, only instability of the first type is possible, and a wave with linear polarization behaves in the same manner as a circularly-polarized wave. In the Kerr effect ($P>Q>0$), the character of the instability depends on the direction of the vector $\boldsymbol{\xi}$. When $\boldsymbol{\xi}\perp\mathbf{k}_0$ we have $L=(1/2k)\partial\omega/\partial k>0$ and instability of the first type takes place. When $\boldsymbol{\xi}\parallel\mathbf{k}_0$, instability of the first type occurs if $\omega''>0$, and instability of the second type if $\omega''<0$.

In the case of a combination of a scalar defocusing nonlinearity (e.g., thermal defocusing) and the Kerr effect, cases are possible in which instability with decay into two circularly polarized waves occurs at all directions ξ.

4. STEADY-STATE WAVES AND SELF-FOCUSING

We change over in (8) to real variables $a^+=A_1\exp(i\Phi_1)$ and $a^-=A_2\exp(i\Phi_2)$. We obtain (for $L_{\alpha\beta}=\partial^2\omega/\partial k_\alpha\partial k_\beta$)

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)A_{1,2}^2+L_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\left(A_{1,2}\frac{\partial\Phi_{1,2}}{\partial x_\beta}\right)=0,$$
$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)\Phi_{1,2}+\frac{1}{2}L_{\alpha\beta}\frac{\partial\Phi_{1,2}}{\partial x_\alpha}\frac{\partial\Phi_{1,2}}{\partial x_\beta}$$
$$=QA_{1,2}+PA_{2,1}+\frac{1}{2A_{1,2}}L_{\alpha\beta}\frac{\partial^2A_{1,2}}{\partial x_\alpha\partial x_\beta}. \qquad (14)$$

We seek the solutions of these equations in the form

$$A_{1,2}=A_{1,2}(\zeta),$$
$$\Phi_{1,2}=s_{1,2}t+\frac{(\mathbf{v}\boldsymbol{\xi})^2-c^2}{2L}t-\frac{c-(\mathbf{v}\boldsymbol{\xi})}{L}z+\tilde{\Phi}_{1,2}(\zeta), \qquad (15)$$

where

$$\zeta=z-ct,\quad z=\xi_\alpha x_\alpha,\quad |\xi_\alpha|=1.$$

For these values we obtain the equations

$$\frac{\partial}{\partial\zeta}A_1^2\frac{\partial}{\partial\zeta}\tilde{\Phi}_1=0,\quad \frac{\partial}{\partial\zeta}A_2^2\frac{\partial}{\partial\zeta}\tilde{\Phi}_2=0, \qquad (16)$$

$$s_1+\frac{1}{2}\left(\frac{\partial\tilde{\Phi}_1}{\partial\zeta}\right)^2=QA_1^2+PA_2^2+\frac{L}{2A_1}\frac{\partial^2A_1}{\partial\zeta^2},$$
$$s_2+\frac{1}{2}\left(\frac{\partial\Phi_2}{\partial\zeta}\right)^2=QA_2^2+PA_1^2+\frac{L}{2A_2}\frac{\partial^2A_2}{\partial\zeta^2}. \qquad (17)$$

Solutions of this type constitute steady-state plane envelope waves propagating in the direction of the vector $\boldsymbol{\xi}$ with velocity c. When $(\boldsymbol{\xi}\cdot\mathbf{v})=0$ and $c=0$, these solutions constitute beams with stationary envelopes, modulated in a transverse direction, including plane self-focused beams. For self-focused beams, the phase is constant in all of space, $\tilde{\Phi}_{1,2}=\text{const}$, and the system (17) reduces to

$$\frac{\partial^2A_1}{\partial\zeta^2}=-\frac{\partial U}{\partial A_1},\quad \frac{\partial^2A_2}{\partial\zeta^2}=-\frac{\partial U}{\partial A_2}, \qquad (18)$$

where

$$U(A_1,A_2)=\frac{1}{L}\left\{-\frac{s_1A_1^2}{2}-\frac{s_2A_2^2}{2}+\frac{1}{4}Q(A_1^4+A_2^4)+\frac{P}{2}A_1^2A_2^2\right\}.$$

Equations (18) are formally analogous to the equations of two-dimensional motion of the particle in a field with potential $U(A_1,A_2)$, and ζ plays the role of the time. Self-focused beams having limited energy correspond to solutions that go off at the "instant of time" $\zeta=-\infty$ from the point $A_1=A_2=0$ and return to this point at $\zeta\to\infty$.

We note that in the two-dimensional case, specification of the condition $A_1=A_2=0$ as $\zeta\to-\infty$ does not define the solution uniquely. It is necessary also to specify the "direction" in the (A_1, A_2) plane, along which the particle begins its motion. The subsequent character of motion will depend on this direction, in particular, the particle will return to the initial point as $\zeta\to+\infty$ only if it has a definite initial direction of motion. As applied to our problem, this means that the self-focused beam will have in each concrete case (at each set of values of the constants s_1 and s_2) a perfectly defined elliptical polarization on its periphery. The polarization of the beam will then, generally speaking, depend on the transverse coordinate ζ.

Self-focusing is possible for not all values of P and Q. When $P = Q > 0$, self-focusing of waves with arbitrary elliptical polarization is possible (in this case $s_1 = s_2$). When $P \neq Q$, $P > 0$, and $Q > 0$, self-focusing of waves with arbitrary polarization on the "wings" is also possible, but now the self-focusing depends on the ratio of s_1 and s_2. When $Q > 0$ and $P < 0$, only waves whose polarization is circular or sufficiently close to circular becomes self-focused. To the contrary, when $Q < 0$ and $P > 0$, only waves with linear polarization or sufficiently close to linear polarization are self-focused. Finally, when $P < 0$ and $Q < 0$, there is no self-focusing at all. We note that in the case $0 > Q > P$ instability of a linearly polarized wave is possible with respect to decay into two circularly-polarized waves, but no focusing is possible (see Sec. 3). We see that this instability differs in principle from the ordinary instability of plane waves, which leads to the formation of self-focused channels.

If the vector $\boldsymbol{\xi}$ in the solutions (15) is parallel to the direction of the initial wave, then these solutions constitute longitudinal modulation waves, and the self-focused solutions correspond to single pulses[4] propagating without waveform distortion. All the statements made above concerning self-focusing pertain also to these waves, the only difference being that P and Q must be replaced everywhere by P/L and Q/L, since in this case $L = \partial^2\omega/\partial k^2$ can have different signs.

As already noted, a particle emerging at the instant $\zeta = -\infty$ from the point $A_1 = A_2 = 0$ does not necessarily return to this point. In order for it to return, as already mentioned, it must move at $\zeta \to -\infty$ in a perfectly defined "direction" in the (A_1, A_2) plane. At any direction, the particle gets "entangled" in the potential field and never returns to the initial point, although from time to time it will come sufficiently close to it (see[10]). In this case the solution consists of regions of irregular oscillations, separated by more or less large interval, in which the solution is close to zero. A detailed investigation of such solutions does not have much sense, since such solutions are expected to be unstable in the self-focusing case (see[4]).

Greater interest attaches to the anti-self-focusing case. (We consider concretely longitudinal envelope waves and put $P/L < 0$ and $Q/L < 0$.) In this case we can no longer confine ourselves to solutions with stationary phase, so that it is necessary to start from the general equations (16) and (17). Integrating Eqs. (16), we obtain

$$\frac{\partial\Phi_1}{\partial\zeta} = \frac{M_1}{A_1^2}, \qquad \frac{\partial\Phi_2}{\partial\zeta} = \frac{M_2}{A_2^2},$$

where M_1 and M_2 are constants.

For the potential $U(A_1, A_2)$ we now have the expression

$$U(A_1, A_2) = \frac{1}{L}\left\{-\frac{s_1A_1^2}{2} - \frac{s_2A_2^2}{2} + \frac{1}{4}Q(A_1^4 + A_2^4) + \frac{1}{2}PA_1^2A_2^2 + \frac{1}{2}\frac{M_1^2}{A_1^2} + \frac{1}{2}\frac{M_2^2}{A_2^2}\right\}. \quad (19)$$

Now the amplitudes can no longer vanish. However, when $s_1, s_2 < 0$ there exists on the (A_1, A_2) plane a saddle point defined as a solution of the system of equations $\partial U/\partial A_1 = 0$ and $\partial U/\partial A_2 = 0$. The saddle point corresponds to a wave with a perfectly defined elliptical polarization (depending on the constants s_1 and s_2). Interest attaches to the solutions that emerge from this saddle point at $\zeta \to -\infty$. For waves with constant polarization, such solutions again return to the initial point at $\zeta \to +\infty$, corresponding to propagation of a solitary rarefaction pulse along the initial wave. A similar solution can exist also in our case for two polarizations, although it is necessary to choose for it a definite direction in the (A_1, A_2) plane near the saddle point. There can exist also solutions that do not return to the saddle point, but become "entangled" in the potential well. With appropriate choice of the initial direction, such a solution can execute an arbitrarily large number of oscillations before it comes sufficiently close to the initial point.

"Envelope shock waves," which are produced in a medium with a dissipative nonlinearity (concretely, when account is taken of the delay of the nonlinearity), were considered earlier in[12]. The stationary envelope wave described by us above can be interpreted as a completely nondissipative shock wave. Indeed, as $\zeta \to -\infty$, the amplitudes $A_1, A_2 \to 0$, whereas at large positive ζ there occur irregular oscillations of the amplitudes A_1 and A_2. In view of the invariance of Eqs. (18) against sign reversal, solutions that decrease as $\zeta \to +\infty$ are also possible.

5. ELECTROMAGNETIC WAVES IN A PLASMA

Let us consider now electromagnetic waves in a homogeneous isotropic plasma at zero temperature. The motion of the ions will be neglected and, only purely electronic nonlinearities will be taken into account. The tensor of nonlinear polarizability will be calculated starting from the hydrodynamic system of equations for the electrons:

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v} = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = \frac{e}{m}\mathbf{E} + \frac{e}{mc}[\mathbf{vH}] - \frac{1}{2c^2}\frac{\partial}{\partial t}|\mathbf{v}|^2\mathbf{v}. \quad (20)$$

We have taken into account here relativistic effects in first order in v^2/c^2. We shall assume that there exists in the plasma an electromagnetic wave of arbitrary elliptic polarization, and we calculate the nonlinear plasma polarization produced by this wave. In the calculation we take into account the nonlinearity produced by the relativistic terms, and also the formation of an induced longitudinal plasma wave of frequency 2ω and wave vector **2k**. As a result we obtain

$$\mathbf{D}_{\mathrm{n.l}} = \frac{\omega_p^4}{\omega^4}\frac{1}{4\pi mn_0c^2}\left\{\left(\frac{1}{2} - \frac{2k^2c^2}{s\omega_p^2 + 4k^2c^2}\right)\mathbf{A}^*(\mathbf{AA}) + \mathbf{A}(\mathbf{AA}^*)\right\}, \quad (21)$$

whence, using the expression $\epsilon = 1 - \omega_p^2/\omega^2$, we obtain

$$a = \frac{\omega_p^4}{2\omega^3}\frac{1}{4\pi mn_0c^2}\left(\frac{1}{2} - \frac{2k^2c^2}{3\omega_p^2 + 4k^2c^2}\right) > 0, \quad (22)$$

$$b = \frac{2\omega_p^4}{\omega^3}\frac{1}{4\pi mn_0c^2}. \quad (23)$$

For electromagnetic waves in a plasma, the frequency $\omega_k = \sqrt{\omega_p^2 + c^2k^2}$ and the tensor $\partial^2\omega/\partial k_\alpha\partial k_\beta$ are positive definite. Accordingly, both linearly and circularly polarized waves are unstable. The maximum instability increments are

$$\gamma_{max} = \frac{\omega_p^4}{2\omega^3} \frac{E^2}{4\pi m n_0 c^2}$$

for circularly polarized waves and

$$\gamma_{max} = \frac{\omega_p^4}{2\omega^3} \left(\frac{3}{2} - \frac{2k^2c^2}{3\omega_p^2 + 4k^2c^2} \right) \frac{E^2}{4\pi m n_0 c^2}$$

for linearly polarized waves. We note also that when $\omega \gg \omega_p$ the mechanism of nonlinearity in the plasma becomes purely scalar.

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