ONSET OF TURBULENCE DURING PARAMETRIC EXCITATION OF WAVES

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The influence of thermal noise on the nonlinear periodic regimes arising in a continuous medium with a parametric instability under the action of a temporally periodic field is studied in this work. It is shown that two cases can occur: either a "smeared-out" periodic regime and the formation of a finite spectral width, or the conservation on the background of the fluctuations of the periodic component, which is coherent within the system. In the latter case the loss of stability in a continuous medium can be compared with a phase transition of the second kind.

INTRODUCTION

We consider in the present paper phenomena occurring upon parametric excitation of waves in a nonlinear medium in the presence of thermal noise. We assume, however, that this investigation has a more general physical meaning and can be regarded as an example of an investigation of the influence of fluctuations on the onset of turbulence in a nonlinear medium.

The problem of onset of turbulence arose in hydrodynamics. In the study of the nonlinear stage of development of certain hydrodynamic instabilities it was established that a periodic motion sets in, with amplitudes proportional to $\sqrt{R - R_0}$, where R is the Reynolds number and R_0 is its critical value (see, for example[1]).

L. D. Landau[2] advanced the hypothesis that further increase of R is accompanied by successive occurrence of a large number of such periodic motions, which ultimately form a completely disordered motion—developed turbulence.

It was subsequently established that the occurrence of periodic regimes following loss of stability is a widespread phenomenon in the physics of continuous media. It occurs, for example, in magnetohydrodynamics and in many problems of plasma physics (see the review[3]). In each concrete case, the role of the Reynolds number is played by some parameter characterizing the degree of instability of the system.

Thermal fluctuations are present in all real systems, and great interest attaches to an investigation of the behavior of the fluctuations as $R \to R_0$. Obviously, as $R \to R_0$ the level of the fluctuations in the system increases. When $R > R_0$, there are at least two cases possible: either the fluctuations in the spectral region close to the frequency of the periodic motion increase to a macroscopic level, in which case a quasiperiodic regime with a certain line width sets in, or else a strictly periodic regime is established as before, and is coherent over the dimension of the entire system, where the fluctuations remain "frozen" at a certain level.

Such an occurrence of a coherent periodic regime can be compared with a second-order phase transition, for example, with condensation of a Bose gas or with a transition to the superconducting state. The phase transition is a realignment of the state of the system, due to the instability of this state at $T < T_0$, where T_0 is the transition temperature. With this, long-range order is established in the system, and the ordering parameter depends on the temperature like $\sqrt{T_0 - T}$.

In establishing an analogy with a phase transition, we can set the existence of a coherent periodic regime in correspondence with long-range order, the amplitude of the periodic motion with the ordering parameter, and the temperature with the Reynolds number or with an equivalent parameter. Continuing the comparison, we note that near the phase transition, as $T \to T_0$, the fluctuations also increase strongly, and it is precisely the behavior of the fluctuations at $T < T_0$ which determines whether a phase transition does or does not occur.

The study of the onset of turbulence in a weakly nonlinear medium (weak turbulence) is much simpler than in an incompressible liquid. Weakly-damped waves exist in such a medium, a continuous distribution of the waves in **k** space is excited in the case of developed turbulence, and the phases of the individual waves can be regarded as random with a high degree of accuracy. Such a turbulence arises, for example, in parametric excitation of waves by applying to the medium a homogeneous field that is periodic in time.

Parametric excitation of waves is widely used for the generation of spin waves in ferromagnets (see, for example, the review[4]). In general form, the problem of the nonlinear stage of parametric instability of waves was investigated by the authors together with S. S. Starobinets in[5], henceforth cited as I.

In I, no account was taken of the thermal noise, and use was made of a definite simplification of the wave-interaction Hamiltonian, reminiscent of the BCS approximation in superconductivity theory. Under these assumptions, we traced the sequence of nonlinear periodic regimes that arise at different values of the external-field amplitude. All the regimes constituted singular distributions of waves in **k**-space, concentrated at points, on a line, or on a surface. We shall show below that in most cases, when account is taken of the thermal noise, and also of the discarded terms of the Hamiltonian, these distributions become "smeared out," but under certain rather stringent limitations a phenomenon of the phase-transition type takes place.

1. FUNDAMENTAL EQUATIONS

Just as in I, we shall describe the medium within the framework of the canonical Hamiltonian formalism. We choose the Hamiltonian of the medium in the form

$$H=\sum \omega_k a_k a_k^* + H_p + H_i. \tag{1}$$

Here a_k is the canonical variable—the complex amplitude of the traveling wave, ω_k is the dispersion law, H_p is the Hamiltonian of the interaction with the external field (with the pump):

$$H_p=\frac{1}{2}\sum [h^*(t)V_k^* a_k a_{-k} + \text{c.c.}], \tag{2a}$$

$h(t) = h\exp(i\omega_p t)$ is the pump field, and H_i is the wave interaction Hamiltonian:

$$H_i=\frac{1}{2}\sum_{12,34} T_{12,34} a_1^* a_2^* a_3 a_4 \Delta(k_1+k_2-k_3-k_4) \tag{2b}$$

($\Delta(\mathbf{k})$ is the Kronecker symbol, $\Delta = 1$ at $\mathbf{k} = 0$). The function $T_{1,2,3,4}$ is assumed to be real and continuous.

The equations of motion of the medium are

$$\frac{\partial a_k}{\partial t}+\gamma_k a_k = -i\frac{\delta H}{\delta a_k^*}+f_k(t), \tag{3}$$

where γ_k is the damping of the waves. Parametric excitation of the waves takes place if

$$\max(hV_k/\gamma_k) > 1. \tag{4}$$

In (3) we have introduced phenomenologically the term $f_k(t)$, which represents a Langevin random force with a correlator

$$\langle f_k(t) f_{k'}^*(t')\rangle = 2\gamma_k n_0(\omega_k)\delta(\mathbf{k}-\mathbf{k}')\delta(t-t'),$$

where $n_0(\omega) = (1 + e^{h\omega/T})^{-1}$ is the equilibrium distribution function.

The main approximation assumed in I consisted of replacing the interaction Hamiltonian (2b) by the approximate Hamiltonian

$$\tilde{H}_i=\sum_{k,k'}\left[T_{kk'}a_k a_k^* a_{k'} a_{k'}^* + \frac{1}{2}S_{kk'}a_k^* a_{-k}^* a_{k'} a_{-k'}\right] \tag{5}$$

$$-\frac{1}{2}\sum_k (T_{kk}a_k a_k^* a_k a_k^* + 2S_{kk}a_k^* a_{-k}^* a_k a_{-k}),$$

$$T_{kk'}=T_{kk',kk'},\quad S_{kk'}=T_{k,-k;k',-k'}.$$

The Hamiltonian (5) is diagonal in pairs of waves with equal and opposite wave vectors. Within the framework of the Hamiltonian (5) it is possible to obtain exact equations for the correlation functions $n_k = \langle a_k a_k^*\rangle$ and $\sigma_k = \langle a_k a_{-k}\exp(-i\omega_p t)\rangle$:

$$\frac{1}{2}\frac{dn_k}{dt}+\gamma_k(n_k-n_0)=\operatorname{Im}P_k^*\sigma_k, \tag{6}$$

$$\frac{1}{2}\frac{d\sigma_k}{dt}+\gamma_k\sigma_k = in_k P_k - i\tilde{\omega}_k\sigma_k,$$

where

$$\tilde{\omega}_k=\omega_k-\frac{\omega_p}{2}+2\sum_{k'}T_{kk'}n_{k'}-(T_{kk}+2S_{kk})n_k, \tag{7}$$

$$P_k = hV_k+\sum_{k'}S_{kk'}\sigma_{k'}.$$

From (6) follows the relation

$$\left(\frac{1}{4\gamma_k}\frac{d}{dt}+1\right)(|\sigma_k|^2-n_k^2)+n_k n_0=0.$$

In I it was assumed that $n_0 = 0$. Then $|\sigma_k|$ relaxes to n_k within times $\sim 1/\gamma$ and in the stationary state we have $|\sigma_k| = n_k$.

The approximation in which we use the diagonal Hamiltonian (5) and put $n_0 = 0$ will be called the S model. The solution of the problem of stationary distributions of n_k within the framework of the S model leads to singular distributions in **k**-space and is strongly ambiguous. The ambiguity is eliminated by requiring that the distribution be "externally stable," i.e., that it be stable against the appearance of new pairs of waves. The external-stability condition causes the distribution of n_k to be concentrated on the surface

$$\tilde{\omega}_k = 0 \tag{8}$$

and to fill this entire surface at sufficiently large amplitudes of the external field.

At smaller amplitudes, the distribution of the waves on the surface (8) depends essentially on the structure of the coefficient V_k.

If V_k is maximal at the point $\mathbf{k} = \mathbf{k}_1$ (we assume for simplicity $\gamma_k = \gamma$ = const), then near the threshold of the instability there arises at this point a pair of waves whose amplitude n_1 and phase Ψ_1 are given by the formulas

$$n_1=\frac{\sqrt{(hV)^2-\gamma^2}}{S_{k_1k_2}},\quad \sin\Psi_1=\frac{\gamma}{hV} \tag{9}$$

(V is the value of V_k at the maximum point). With further increase of the amplitude h, there appear new pairs, which at a certain $h = h_S$ fill the entire surface.

If V_k is maximal on a line (for example, in the case of axial symmetry), then at small excesses above the instability threshold only this line is filled, followed by the filling of new lines which gradually "cover" the entire surface. Finally, if V_k = const, then the entire surface $\tilde{\omega}_k = 0$ becomes filled immediately beyond the threshold.

All the foregoing have pertained to the case of a sufficiently "good" function $S_{kk'}$. If this function, regarded as the kernel of an integral operator, turns out to be degenerate, then there is either a "collapse" of the stationary regime, or a loss of its uniqueness. Thus, at $S_{kk'} = S$ = const there exists no stationary regime at $V_k \neq$ const, $hV > \sqrt{2}\gamma$, the distribution is arbitrary and satisfies the condition

$$\sum_k n_k=\frac{\sqrt{(hV)^2-\gamma^2}}{S}\quad\text{for}\quad V_k=V=\text{const}.$$

A similar picture occurs also in a physically two-dimensional medium, in which pair distributions are possible in the form of individual points or lines in the **k**-plane.

In I the regime in which one or a finite number of pairs exists was called dynamic, and the regime in which there exists a distribution of pairs on a line or on the surface was called stochastic, under the assumption that the individual phases of the pairs are random in this case. In the present paper greater importance attaches to the dimensionality δ of the space in which the problem is considered (δ = 2, 3), and also to the dimensionality δ_S of the stationary distribution in the S model (δ_S = 0, 1, 2). We shall show in the next section that the influence of thermal noise on the distribution depends essentially on the difference of these numbers.

2. INFLUENCE OF THERMAL NOISE ON THE STATIONARY REGIME

Let us consider the possible physical causes of "smearing" of a distribution that is singular in **k**-space in the presence of thermal noise.

The resultant singular distribution of the waves changes their damping decrement. For the effective decrement we have within the framework of the S model (see I)

$$\gamma_{eff} = \gamma - |P|,$$

where P is given by (7).

The principle of external stability requires that γ_{eff} be positive in all of space with the exception of the points at which the distribution is concentrated; at these points $\gamma_{eff} = 0$. Near these points, at a distance κ from them, we have

$$\gamma_{eff} \sim \alpha\kappa^2.$$

The amplitude of the thermal noise increases by a factor γ/γ_{eff} compared with its stationary value, so that

$$n_k = n_0\gamma / \gamma_{eff}.$$

"Smearing" of the singular distribution occurs in the case when the singularity is nonintegrable. It occurs if $\delta - \delta_S = 1$ (the distribution is concentrated on a surface in three-dimensional space or on a line in two-dimensional space)—in this case the divergence is proportional to $1/\kappa$—and also in the case $\delta - \delta_S = 2$ (line in three-dimensional space or a pair in two-dimensional space)—in this case the divergence is logarithmic. The singularity is integrable only if $\delta - \delta_S = 3$ (a pair in three-dimensional space), and only in this case can one hope to get a "phase transition"—the occurrence of a singular regime in the presence of noise.

We now analyze quantitatively the smearing of the singular distribution. In the stationary case we write Eqs. (6) with allowance for

$$|\sigma_k|^2 = n_k(n_k - n_0), \tag{10}$$

in the form

$$\begin{aligned} \gamma_k(n_k - n_0) &= |\sigma_k|\,[\sin\Psi_k(hV_k + A_k) - B_k\cos\Psi_k], \\ -\omega_k(n_k - n_0) &= |\sigma_k|\,[B_k\sin\Psi_k + (hV_k + A_k)\cos\Psi_k]. \end{aligned} \tag{11}$$

We have introduced the quantities

$$A_k = \sum_{k'} S_{kk'}|\sigma_{k'}|\cos\Psi_{k'}, \qquad B_k = \sum_{k'} S_{kk'}|\sigma_{k'}|\sin\Psi_{k'},$$

in terms of which the solution of (11) is expressed:

$$\frac{n_k - n_0}{n_0} = \frac{|P_k|^2}{\tilde\omega_k^2 + \nu_k^2} = \frac{|P_k|^2}{v_\Omega^2\kappa_\parallel^2 + \nu_k^2}, \tag{12}$$

where

$$\nu_k^2 = \gamma_k^2 - |P_k|^2, \quad |P_k|^2 = (hV_k + A_k)^2 + B_k^2, \tag{13}$$

$\kappa_\parallel$ is the deviation from the surface $\tilde\omega(\mathbf{k}) = 0$ in the normal direction, $v_\Omega = \partial\omega/\partial\kappa_\parallel$, and for A_k and B_k we obtain the system of integral equations:

$$\begin{aligned} A_k &= \sum_{k'} S_{kk'}n_0(k')\frac{-\gamma_{k'}B_{k'} + \tilde\omega_{k'}(hV_{k'} + A_{k'})}{\tilde\omega_{k'}^2 + \nu_{k'}^2} \\ B_k &= \sum_{k'} S_{kk'}n_0(k')\frac{\gamma_{k'}(hV_{k'} + A_{k'}) + \tilde\omega_{k'}B_{k'}}{\tilde\omega_{k'}^2 + \nu_{k'}^2}. \end{aligned} \tag{14}$$

Assuming that the quantities γ_k, hV_k, and $S_{kk'}$ depend only on the angle variables Ω, we find that the quantities A_k, B_k, P_k, and ν_k are also functions only of Ω. It follows therefore that the distribution function n_k has a Lorentz form with a maximum at $\tilde\omega_k = 0$ and parameters that depend on Ω. This result is in accord with the consequence of the "external stability" condition in the S model.

The width ν_Ω and the intensity N_Ω of the Lorentz function, as follows from (12), are connected by the relation

$$N_\Omega = \frac{1}{2\pi}\int(n_{\kappa,\Omega} - n_0)\,d\kappa, \quad N_\Omega = \frac{\gamma^2 - \nu_\Omega^2}{2v_\Omega\nu_\Omega}n_0. \tag{15}$$

Integrating (11) with respect to $\kappa_\parallel$, we obtain

$$\begin{aligned} A_\Omega &= \frac{n_0}{2}\sum_{\Omega'}\frac{S_{\Omega\Omega'}}{v_{\Omega'}}\left[-\frac{\gamma B_{\Omega'}}{\nu_{\Omega'}} + \alpha_{\Omega'}(hv_{\Omega'} + A_{\Omega'})\right], \\ B_\Omega &= \frac{n_0}{2}\sum_{\Omega'}\frac{S_{\Omega\Omega'}}{v_{\Omega'}}\left[\frac{\gamma(hv_{\Omega'} + A_{\Omega'})}{\nu_{\Omega'}} + \alpha_{\Omega'}B_{\Omega'}\right]. \end{aligned} \tag{16}$$

Here the parameter α_Ω is of the order of unity; it can have either sign and arises when the integral $\int\omega d\omega/(\omega^2 + \nu^2)$ is cut off in the region of low frequencies as a result of the decrease of the density of states and in the region of high frequencies as a result of the decrease of the thermal-noise amplitude.

In the limit as $n_0 \to 0$ it follows from (13) and (15) that the width of the packet ν_Ω tends to zero, and $n_{\kappa,\Omega} \to N_\Omega\delta(\kappa)$, corresponding to a transition to the S model. As already noted, the influence of the thermal noise depends on the value of $\delta - \delta_S$. Let us consider the case $\delta - \delta_S = 1$. For simplicity we confine ourselves to the case when V_k and $S_{kk'}$ are constants, and ω_k depends only on $|\mathbf{k}|$. We then obtain from (11) a biquadratic equation for the width of the packet ν:

$$\nu^4(1 - \alpha\xi_\delta)^2 + \nu^2[(hV)^2 - \gamma^2(1 - \alpha\xi_\delta)^2 + \gamma^2\xi_\delta^2] - \gamma^4\xi_\delta^4 = 0;$$

here the small parameter ξ_δ, namely

$$\xi_3 = n_0k_0^2S / 2\pi v, \quad \xi_2 = k_0S / 2v,$$

depends on the level of the thermal noise n_0 and the nonlinearity of the medium S. For parametric excitation of spin waves in ferromagnets in the usual experimental situation we have

$$\xi_3 \approx \frac{T}{h\omega}(ak_0)^3 \sim ak_0\frac{T}{T_c} \approx 10^{-4} - 10^{-7},$$

where a is the lattice constant, k_0 is the characteristic wave vector of the pairs ($\tilde\omega_{k_0} = 0$), T is the crystal temperature, and T_C is the Curie temperature. For a hypothetical two-dimensional exchange ferromagnet

$$\xi_2 \approx T / T_c.$$

The expression for the total amplitude is obtained by integrating (12):

$$SN = (hV)^2\frac{\xi\nu}{\nu^2(1 - \alpha\gamma\xi) + (\gamma\xi)^2}. \tag{17}$$

We note that the case V_k = const, $S_{kk'}$ = const is strongly indeterminate in the S model. On the other hand, the distribution obtained by us for n_k is a unique solution of Eqs. (11). Thus, the introduction of the thermal noise lifts the indeterminacy of the distribution in the S model.

The concept of instability threshold loses its rigorous meaning when account is taken of thermal noise, and can be defined accurate to $\xi^2\gamma$. We introduce nevertheless the notation $h_cV = \gamma(1-\alpha\xi)$. For $V(h_c - h) < \gamma\xi^2$ we then get

$$\nu^2 = \gamma^2\left(1-\frac{h^2}{h_c^2}\right)\left(1+\xi^2\frac{\gamma^4}{(h_c^2-h^2)^2V^4}\right),$$
$$SN = \frac{\xi h^2 V}{\gamma\sqrt{h_c^2-h^2}}\left(1-\xi^2\frac{\gamma^4}{2(h_c^2-h^2)^2V^4}\right).$$

Above the "threshold," when $V(h-h_c) > \gamma\xi^2$, we have

$$\nu^2 = \frac{\xi^2\gamma^4}{V^2(h^2-h_c^2)}\left[1-\xi^2\frac{\gamma^4}{2V^4(h_c^2-h^2)^2}\right], \qquad (18)$$
$$SN = V\sqrt{h^2-h_c^2}\left[1+\xi^2\frac{\gamma^4}{2V^4(h_c^2-h^2)^2}\right].$$

It is seen from these formulas that the influence of the thermal noise on the parametric excitation of the waves has led, first, to a shift of the threshold amplitude of the field by an amount that is linear in the small parameter of the problem ξ, second, to the appearance of a width $\nu \sim \gamma\xi$, and third, to the appearance of increments that are quadratic in ξ to the dependence of SN on $h^2 - h_c^2$, compared with the result obtained by the S model.

We now consider the case $\delta - \delta_S = 2$. Assuming as before that $S_{\mathbf{kk}'}$ = const, $\omega_{\mathbf{k}}$ depends only on $|\mathbf{k}|$, and $V_{\mathbf{k}}$ has a maximum in the region where the solution is concentrated in the S model and decreases quadratically with increasing distance from the maximum, namely $V_{\mathbf{k}} = V(1-\beta^2\theta^2)$, $\beta \sim 1$, we obtain formula (17) for the total amplitude, where

$$\frac{1}{\nu} = \int_0^1 \frac{d\theta}{\nu_\theta} \approx \int_0^1 \frac{d\theta}{\sqrt{\nu_0^2+\gamma^2\theta^2}} \approx \frac{1}{\gamma}\ln\frac{\gamma}{\nu_0} \qquad (19)$$

$$\nu^2\nu_0^2\frac{(h_cV)^2}{\gamma^2} + \nu^2V^2(h^2-h_c^2) + \nu_0^2\gamma^2\xi^2 - \gamma^4\xi^2 = 0.$$

Solving these equations, we obtain above the "threshold" at

$$V(h-h_c) > \gamma\exp(-\sqrt{h^2-h_c^2}/\xi h_c)$$

the expressions

$$SN = V\sqrt{h^2-h_c^2}\,[1+(hV)^2\exp(-\sqrt{h^2-h_c^2}/\xi h_c)],$$
$$\nu_0 = \gamma\exp(-\sqrt{h^2-h_c^2}/\xi h_c),\quad \nu_\theta^2 = \nu_0^2+\gamma^2\beta^2\theta^2.$$

We see therefore that when $\delta - \delta_S = 2$ the threshold amplitude shifts in the same manner as when $\delta - \delta_S = 1$; however, the influence of the thermal noise on the amplitude N has an exponential smallness.

3. INFLUENCE OF THE "INTRINSIC" NOISE

We now examine the role played by the interaction-Hamiltonian terms discarded in the S model. As noted in I, within the framework of the S model the sum of the phases of the waves making up the pair is fixed, whereas their difference can be arbitrary. Actually, in the derivation of the S-model equations, it was assumed that these differences (and with them also the individual phases of the waves) are random. Physically this is justified by the fact that the phase difference in the S model has no stability margin and can be randomized by an arbitrarily small external action, say by thermal noise.

That part of the interaction Hamiltonian (2) which is not diagonal in the pairs depends on the individual phases, and on this basis it was disregarded in the S model. It leads, however, to a certain correlation between the individual phases, and can be taken into account by a perturbation theory that uses the S model as the zeroth approximation. To do this, it is necessary to carry out a decoupling of sixth-order correlations with allowance for the correlation of the phases of the waves in the pair.

The result is a system of kinetic equations for the correlation functions in n and σ and containing terms that are cubic in these variables. The solution of this system of equations turns out to be a much more complicated problem than the solution of the corresponding equations for thermal noise, and we shall therefore disregard in this paper the influence of the nondiagonal terms of the Hamiltonian in the order-of-magnitude estimates. In this case we can neglect in the cubic terms the correlations of the phases within the pair, (in spite of the fact that $|\sigma| \sim n$), and use the known expression for the collision term, which takes into account four-wave interactions (see, for example,[6]),

$$\left(\frac{\partial n_1}{\partial t}\right)_{st} = 2\pi\sum |T_{12,34}|^2\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)$$
$$\times\,[n_2n_3n_4+n_1n_3n_4-n_1n_2n_3-n_1n_2n_4]. \qquad (20)$$

The collision term (20) has terms of two types: a stochastic external force with spectral density

$$\gamma n_i(\mathbf{k}) = 2\pi\sum |T_{12,34}|^2\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)\,n_2n_3n_4, \qquad (21)$$

the introduction of which is equivalent to introduction of an "intrinsic" noise with a distribution $n_i(\mathbf{k})$, and an effective nonlinear damping of the wave due to the four-wave interactions

$$2\gamma^{nl} = 2\pi\sum |T_{12,34}|^2\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4).$$
$$\times\,(n_2n_3+n_2n_4-n_3n_4).$$

At not too large amplitudes,

$$SN/\gamma \ll (\omega/\gamma)^{1/2}, \qquad (22)$$

to which we shall henceforth confine ourselves, the linear damping γ is much larger than the nonlinear one, and the latter can be neglected. To the contrary, the "intrinsic" noise n_i may turn out to be much larger than the thermal noise n_0, so that the intrinsic noise must be taken into account.

Let us examine the influence of the intrinsic noise in the case (22), when it is sufficiently small. The integral intensity of the excited waves will be described as before by the S-model formulas. However, the intrinsic noise affects the magnitude and form of the broadening of the singular distribution.

1. We start with the case $\delta - \delta_S = 1$. Then the distribution is concentrated, as before, near the surface (or line) $\tilde{\omega}(\mathbf{k}) = 0$ and has a certain width ν (with respect to the frequencies). Obviously, the distribution of $n_{\mathbf{k}}$ near the surface will no longer be Lorentzian, since the intensity of the intrinsic noise itself depends strongly on $\mathbf{k}$ and is concentrated in a narrow layer, the width of which is also of the order of ν.

We shall obtain a self-consistent estimate of the width ν for the simplest situation $V_k = V = \text{const}$, $T_{12,31} = S = \text{const}$. Estimating (21), we note that if $n_k \sim \delta(\omega_k - \omega_0)$, then also $\gamma n_i(k) \sim \delta(\omega_k - \omega_0)$, so that expression (21) contains a large factor ω/ν. Taking this into account, we have

$$\gamma n_i \sim \frac{\omega}{\nu}\frac{(SN)^2}{\omega}n(k_0) \approx \frac{S^2N^2}{\nu}n(k_0).$$

This intrinsic-noise intensity corresponds to a dimensionless parameter $\xi_{eff} \sim (SN)^3/\omega\gamma\nu$. Assuming that the condition $V(h - h_c) \gg \gamma\xi^2$ is satisfied, we shall use formula (18) $\nu \approx \xi\gamma/SN$, from which we get

$$\nu \sim \left(\frac{\gamma}{\omega}\right)^{1/2} SN, \qquad \xi_{eff} \sim \left(\frac{\gamma}{\omega}\right)^{1/2}\left(\frac{SN}{\gamma}\right)^2. \tag{23}$$

The quantity $\gamma\xi^2_{eff}/(hV - \gamma)$ remains small if $SN/\gamma \ll (\omega/\gamma)^{1/2}$. It follows from (18) that the correction to the total wave amplitude, $\delta N/N \sim \gamma/\omega$, is small at all excesses above threshold. As seen from (23), the width of the ν distribution, due to the intrinsic noise, increases with increasing amplitude of the waves, whereas, to the contrary, the width due to the thermal noise decreases. These effects become comparable at an amplitude

$$SN/\gamma \sim (\omega/\gamma)^{1/4}\xi^{1/2}.$$

For ferromagnets, this corresponds to a rather small excess above threshold:

$$(h - h_c)/h_c \ll 10^{-2};$$

at larger amplitudes, the influence of the thermal noise can be neglected.

2. We consider now the influence of the intrinsic noise in the case $\delta - \delta_S = 2$, when in the S model there is realized a distribution concentrated on a line in three-dimensional **k**-space. Just as before, we confine ourselves to consideration of an axially symmetrical problem in which V_k has a maximum on the equator and decreases in accordance with the formula $V_\theta = V(1 - \alpha^2\theta^2)$, where $\pi/2 - \theta$ is the azimuthal angle, $\alpha \sim 1$. We also put $S'_{kk} = S = \text{const}$.

We note that the intrinsic noise leads to a different character of the distribution of n_k than the thermal noise. The thermal noise, uniformly distributed over all of space, leads to a weak localization of the distribution near the equator. If ν is the width of the distribution with respect to frequency, then the intensity of the distribution decreases along the surface like $1/\theta$ at $\theta > \nu/\gamma$. Unlike thermal noise, intrinsic noise is strongly localized near the equator in the angle region $\theta \sim \nu/\gamma$, corresponding to the interval of the values of the transverse (with respect to $\mathbf{k}_0$) component of the wave vector $\delta k_\perp \sim k_0\nu/\gamma$. By comparison with the width along the wave vector $\delta k_\parallel \sim \nu/V \sim \nu k_0/\omega$, we get $\delta k_\parallel/\delta k_\perp \sim \gamma/\omega$. The distribution along the wave vector $\mathbf{k}_0$ (across the surface) is thus better localized, by a factor γ/ω, than along the surface.

When estimating expression (21) it must be borne in mind that if $n_k \sim \delta(\omega_k - \omega_0)\delta(\mathbf{k}_\perp)$, then also $\gamma n_i \sim \delta(\omega_k - \omega_0)\delta(\mathbf{k}_\perp)$, and therefore expression (21) contains the large factor $(\omega/\nu)(k_0/\delta k_\perp)$. Taking this into account, we have

$$\gamma n_i \sim \frac{\omega}{\nu}\frac{k_0}{\delta k_\perp}\frac{(SN)^2}{\omega}n(k_0) \approx \frac{\gamma}{\nu^2}(SN)^2 n(k_0). \tag{24}$$

Specifying a certain distribution $n_i(\theta)$ of the intrinsic noise with respect to the angle, we substitute it in (11). Integrating, we obtain

$$A + \xi_{eff}B = 0, \quad B - \xi_{eff}(hV + A) = 0, \quad SN = \frac{\xi_{eff}P^2}{\gamma}, \tag{25}$$

where

$$\xi_{eff} = \frac{Sk_0^2}{2\pi V}\int_0^1 \frac{n_i(\theta)\,d\theta}{\nu(\theta)}. \tag{26}$$

Solving Eq. (25), we substitute A and B in the formula for P^2, from which (assuming $\nu \ll \gamma$) we determine ξ_{eff}:

$$\xi_{eff} = \frac{(hV)^2 - \gamma^2}{\gamma^2}. \tag{27}$$

Hence $SN = \sqrt{(hV)^2 - \gamma^2}$ (accurate to terms of order $(\nu/\gamma)^2$).

Using formulas (24) and (25), we obtain the estimate

$$\xi_{eff} = (SN)^3/\omega\nu^2.$$

Comparing with (27), we have

$$\nu \sim (\gamma/\omega)^{1/2}SN.$$

For the width ν of the distribution over the frequencies, we obtain the same estimate as in the case $\delta - \delta S = 1$. Estimating the value of ν/γ, we verify that it remains small when $SN/\gamma \ll (\omega/\gamma)^{1/2}$, i.e., in the entire region of validity of our analysis.

From a comparison of ξ_{eff} with the parameter ξ for thermal noise, we find that the intrinsic noise becomes comparable with the thermal noise at an external-field amplitude

$$(h - h_c)/h_c \sim \xi^2,$$

i.e., at much smaller excesses above the instability threshold than in the case $\delta - \delta_S = 1$. This is connected with the fact that in the case $\delta - \delta_S = 2$ the intrinsic noise turns out to be much stronger.

The case when a regime in the form of one pair of waves is realized in the S model must be considered within the framework of the exact Hamiltonian, as will be done in the next section.

4. DYNAMIC REGIME IN THE PRESENCE OF NOISE

In this section we shall show that situations are possible in which the dynamic regime in the form of a single pair in a three-dimensional medium ($\delta - \delta_S = 3$) remains in force in the presence of thermal noise. To this end it is necessary, first, that the growth increment of the noise wave not be positive and, second, that the total intensity of the waves of the background in the stationary state be finite. Accordingly, we obtain, first, the conditions necessary for the stability of the pair within the framework of the exact Hamiltonian, and second, we estimate the integral value of the noise in a stable situation and show that it is small.

Using the Hamiltonian (2), we have written out in[8] linearized equations of motion for perturbation waves with wave vectors $\mathbf{k}_0 \pm \boldsymbol{\kappa}$ and $-(\mathbf{k}_0 \pm \boldsymbol{\kappa})$, and obtained an expression for the instability increment γ_{eff}. For $(\kappa/k_0)^2 \gg SN/\omega$, it reduces to the simple form

$$(\gamma_{eff} + \gamma)^2 = |P_1|^2 - \tilde{\omega}^2, \tag{28}$$

which follows directly from the diagonal Hamiltonian. In I we investigated expression (28) and showed that in order to have $\gamma_{\rm eff} \leq 0$ it is necessary that the wave vector $\mathbf{k}_0$ of the pair satisfy the external-stability condition

$$\omega_{\mathbf{k}_0} - {}^1/_2\omega_p + {}^1/_2 N T_{00} = 0 \qquad (29)$$

and that it be directed such that $V_{\mathbf{k}_0} = V = \max V_{\mathbf{k}}$. The amplitude and phase of the pair are then determined by formula (9).

The condition (29) thus ensures stability of the pair for $(\kappa/k_0)^2 \gg SN/\omega$, and we shall therefore present in the present paper an expression for the increment $\gamma_{\rm eff}$ at small κ/k ($\kappa/k \ll SN/\omega$) for the stationary state of the pair (9)[1)]:

$$(\gamma_{eff}+\gamma)^2-\gamma^2 = -\varkappa v - \gamma^2 W\varkappa^2 - \left(\frac{L\varkappa^2}{2}\right)^2 - \frac{L\varkappa^2}{2}(2S+T)N$$
$$- S(2S+T)N^2 \pm \{(\varkappa v)^2[(L\varkappa^2)^2 + 2L\varkappa^2(2S+T)N + 4S(S+T)N^2]^2 + S^2N^2[L\varkappa^2+(2S+T)N]^2\}^{1/2}. \qquad (30)$$

The coefficients $T_{\alpha\beta\gamma\delta}$ have been replaced here by their limiting values as $\kappa \to 0$, and we have used the expansions

$$\tilde{\omega}_{\mathbf{k}_0\pm\varkappa} = -{}^1/_2 NT \pm \varkappa v + {}^1/_2 \hat{L}\varkappa^2,$$
$$\hat{L}\varkappa^2 = \sum_{\alpha,\beta}\frac{\partial^2\omega_{\mathbf{k}}}{\partial k_\alpha\,\partial k_\beta}\varkappa_\alpha\varkappa_\beta$$

and

$$V_{\mathbf{k}_0\pm\varkappa} = V(1+{}^1/_2 W\varkappa^2),$$
$$\hat{W}\varkappa^2 = \frac{1}{V}\sum_{\alpha,\beta}\frac{\partial^2 V}{\partial k_\alpha \partial k_\beta}\varkappa_\alpha\varkappa_\beta.$$

In[8] we investigated expression (30) in detail and showed that for $\gamma_{\rm eff} \leq 0$ it is necessary to satisfy the following conditions:

$$S(2S+T) > 0 \qquad (31)$$

(the condition of "intrinsic stability" (see I)),

$$ST < 0 \qquad (32)$$

and

$$(TN)^2 \leqslant |S| N\gamma^2 W/L. \qquad (33)$$

For most media $T \sim S$, and then for $Wk_0^2 \sim 1$ and $Lk_0^2 \sim \omega$ the condition (33) is satisfied for small excesses above threshold

$$\frac{SN}{\gamma} \ll \frac{\gamma}{\omega} \quad \text{or} \quad \frac{h-h_c}{h_c} \leqslant \left(\frac{\gamma}{\omega}\right)^2. \qquad (34)$$

Thus, for small excesses above threshold, the pair is stable if the coefficients of the Hamiltonian S and T have opposite signs (32) and $|T| < 2|S|$ (31).

In order for the pair to be stable at $SN \sim \gamma$ (i.e., at $h - h_c \sim h_c$), it is necessary to satisfy the stringent requirement

$$|T/S| \leqslant \sqrt{\gamma/\omega}, \qquad (35)$$

satisfaction of which can be attained when T depends on external conditions—temperature, pressure, magnetic field, pump frequency—and can reverse its sign.

Estimating the summary noise amplitude, we note that in the case when the conditions (31)—(33) are satisfied, the contribution of the region $(\kappa/k)^2 \lesssim SN/\omega$ (where the diagonal-Hamiltonian approximation is not valid) turns out to be small[8].

For $(\kappa/k)^2 \gg SN/\omega$ we obtain from Eqs. (10)—(11) in the approximation quadratic in the noise amplitude

$$\frac{n_{\mathbf{k}}-n_0}{n_0} = \frac{|P_{\mathbf{k}}|^2}{\gamma^2 - |P_{\mathbf{k}}|^2 + \tilde{\omega}_{\mathbf{k}}^2}, \qquad (36)$$

where $P_{\mathbf{k}} = hV_{\mathbf{k}} + S_{\mathbf{k}\mathbf{k}_0} N e^{i\Psi}$, from which it follows that

$$S\sum_{\mathbf{k}}(n_{\mathbf{k}} - n_0) \approx \gamma\xi \ll \gamma. \qquad (37)$$

Using this estimate and the equations of motion for the "condensate" with allowance for the "noise," which follow from the Hamiltonian (2), we can verify[8] that the influence of the noise on the condensate is indeed small; it leads to a shift of the transition point by an amount on the order of $\gamma\xi$ and to corrections of the order of $\gamma\xi^2$ to the amplitude SN.

In the case of instability of the pair, one should expect turbulence to arise with a characteristic scale $l \sim k_0^{-1}(SN/\omega)^{-1/2}$. There are no grounds for assuming that this turbulence will be weak, since the instability increment is $\gamma_{\rm eff} \sim SN \sim L\kappa^2$, i.e., it is of the same order as the characteristic differences of the wave frequencies.

CONCLUSION

Thus of all the variants of the stationary distribution of the pairs in $\mathbf{k}$ space that are possible within the framework of the S model, only one regime, namely an isolated pair in a three-dimensional medium, can retain its coherent character in the presence of noise. It should be noted, however, that the analysis in Sec. 4 cannot be regarded as rigorous proof of realization of a coherent regime even in this case. As was shown, the distribution function of the noise has as $\kappa \to 0$ a singularity $n_\kappa \sim 1/\kappa^2$ and therefore it is incorrect to confine oneself at small values of κ to an approximation that is linear in the noise amplitude. Moreover, near h_c there exists a narrow region of values of the external-field amplitude in which the intensity of the pair is of the order of the summary intensity of the noise in the "nonlinear" region near the singularity. Here our description is generally incorrect. A rigorous determination of n_κ as $\kappa \to 0$ and of the character of the regime as $h \to h_c$ is a very difficult problem, at any rate no less difficult than analogous problems in the theory of second-order phase transitions, and any discussion of these problems is beyond the scope of the present paper. Apparently, the case of a pair in a two-dimensional medium is just as complicated. In this case we cannot advance even qualitative ideas concerning the character of the steady-state regime.

On the other hand, if a distribution concentrated on a line or on a surface is established in the S model, then the influence of the thermal and that of the intrinsic noise can be considered within the framework of our scheme at all external-field amplitudes from zero to $h \sim h_c(\omega/\gamma)^{1/2}$.

In conclusion, the authors consider it their pleasant duty to thank S. S. Starobinets for a discussion of the problems considered in the present paper.

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1)The problem of the stability of the pair in the case when the amplitude is limited by nonlinear damping was investigated by us together with S. S. Starobinets in [7].

[1] A. S. Monin and A. M. Yaglom, Statisticheskaya gidrodinamika (Statistical Hydrodynamics), Vol. 1, Nauka, 1966.
[2] L. D. Landau, Sobranie trudov (Collected Works), Vol. 1, Nauka, 1969, p. 447.
[3] B. B. Kadomtsev, Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 4, Gosatomizdat, 1964.
[4] R. W. Damon, Magnetism, ed. G. T. Rado and H. Suhl, AP, N. Y. and L., **1**, 551, 1963.
[5] V. E. Zakharov, V. S. L'vov and S. S. Starobinets, Zh. Eksp. Teor. Fiz. **59**, 1200 (1970) [Sov. Phys.-JETP **32**, 656 (1971)].
[6] V. E. Zakharov, ibid. **51**, 688 (1966) [**24**, 455 (1967)].
[7] V. E. Zakharov, V. S. L'vov, S. S. Starobinets, Preprint 227, Inst. Nuc. Phys. (IYaF), Novosibirsk, 1968.
[8] V. E. Zakharov and V. S. L'vov, Preprint 69-70, Inst. Nuc. Phys. (IYaF), Novosibirsk, 1970.

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