Instability of waveguides and solitons in nonlinear media

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The instability is demonstrated of stationary waveguides that cause self-focusing of waves in homogeneous noninertial nonlinear media, or in media with relaxing or striction nonlinearities. Estimates are made of the characteristic instability increments. The dimensions of the wave field "clusters" formed as a result of development of the instabilities are presented. Instability of a plasma soliton and wave train at the surface of a deep liquid is deduced.

INTRODUCTION

It is well known [1–3] that in homogeneous nonlinear media there can exist waveguide configurations that realize energy propagation without diffraction divergence namely stationary self-focusing of waves. The stability of stationary self-focusing is of great interest and this problem is the subject of the present article.

From the formal point of view, the problem of determining the shape of the waveguide in which the self-focusing is produced is very similar to the problem of the soliton, which is a solitary wave that propagates without distortion in a nonlinear medium with dispersion. In one case, that of a flat waveguide and a solitary envelope wave (soliton filled with a monochromatic carrier [4,5]), these problems are mathematically identical. Problems of the stability of waveguide propagation and the stability of a soliton also have much in common and are considered in the present paper jointly. They reduce to an analysis of the spectral analysis of certain non-self-adjoint linear differential operators of high order (of fourth order for waveguide propagation). This analysis can be carried out because we know the eigenfunctions corresponding to the neutrally-stable perturbations of the configuration tested for stability (waveguide or soliton) and respond to small variations of its shape. Knowledge of these neutrally-stable eigenfunctions makes it also possible to obtain, by one method or another, information on the unstable eigenfunctions (or to establish that they do not exist). This idea seems to have been used first by Zel'dovich and Barenblat to solve the problem of the stability of a combustion wave [6], and was used to calculate the oscillation spectrum of a vortical filament in a non-ideal Bose gas [7], to study the stability of moving domains in the Gunn effect [8], and to investigate the stability of a soliton within the framework of the nonlinear Klein-Gordon equation [9].

One of us [10] has previously employed, in the problem of the stability of the stationary waveguide, a variational principle, in combination with the concept of neutrally-stable modes. In this manner it was possible to prove the instability of a waveguide in a non-inertial medium with positive dispersion ($\omega'' > 0$) and the instability of a flat waveguide with respect to small perturbations of its shape in the stationary problem. In the present paper we use a different method, based on expansion in terms of the small deviation of the long-wave unstable mode from the neutrally-stable one[1)]. This makes it possible to study the stability of waveguides in envelope solitons in media with dispersion of arbitrary sign and also in media with inertial and striction nonlinearities.

The results of this investigation lead to rather pessimistic conclusions concerning the possibility of realizing stable stationary self-focusing. In all the cases considered by us, the waveguide that produces the self-focusing turns out to be unstable. The strongest instability takes place in a medium with a noninertial nonlinearity, since perturbations with a longitudinal dimension on the order of the transverse dimension of the waveguide develop in this medium regardless of the sign of the dispersion, thus destroying the waveguide completely. The instability has an increment on the order of the linear frequency shift $\Delta\omega_n$ in the center of the waveguide. The situation is somewhat more favorable in media with a nonlinearity inertia. The instability produced in these media has an increment equal to the smaller of the quantities $\Delta\omega_n$ or $1/\tau$ (τ is the nonlinearity relaxation time), and then the waveguide breaks up into regions with respective dimensions $\lambda\omega\tau$ and $\lambda\omega/\Delta\omega_n$ that can greatly exceed the transverse dimension of the waveguide. The most favorable situation is obtained in a medium with striction nonlinearity where, if the nonlinearity level is not too high, stable existence of a waveguide is possible, with a ratio of length to transverse dimension not exceeding the ratio of the speed of light to the speed of sound. It appears that this is precisely the case realized in the successful experiments on self-focusing of electromagnetic waves in a plasma [11].

We also solve in this paper the problem of instability of a plasma soliton, which is of interest for the theory of plasma turbulence; from the result on the instability of a flat waveguide in a noninertial medium, obtained in Sec. 1 of the present paper, it follows that a stationary train of waves on the surface of a deep liquid is unstable [11].

1. INSTABILITY OF FLAT WAVEGUIDES AND ENVELOPE SOLITONS

The propagation of a quasimonochromatic wave in an isotropic transparent nonlinear medium is described by the equation for its complex envelope Ψ [4]

$$i(\Psi_t + u\Psi_z) + \frac{u}{2k}\Delta_\perp\Psi + \frac{\omega''}{2}\Psi_{zz} + kuf(|\Psi|^2)\Psi = 0. \tag{1}$$

In (1), z is the wave propagation direction, and $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$. It is assumed that the medium has a nonlinear dispersion law:

$$\omega = \omega_k\left(1 + \frac{ku}{\omega_k}f(|\Psi|^2)\right);$$

$u = \partial\omega/\partial k$ is the group velocity of the wave.

Changing over to a reference frame that moves with the group velocity and introducing dimensionless coordinates and time, we obtain from (1)

$$i\Psi_t + {}^1/_2\Delta_\perp\Psi + {}^1/_2\alpha\Psi_{zz} + f(|\Psi|^2)\Psi = 0, \tag{2}$$

where $\alpha = k\omega''/u$.

A nonlinear medium is self-focusing if $f'(\xi) > 0$, and is defocusing if $f'(\xi) < 0$. In both cases we assume that $f' > 0$. In a self-focusing medium, Eq. (2) has a solution in the form of a stationary flat waveguide:

$$\Psi = \exp(i\lambda^2 t)\varphi(x).$$

Here $\varphi(x)$ is real and represents the solution of the equation

$$-\lambda^2\varphi + {}^1/_2\varphi_{xx} + f(\varphi^2)\varphi = 0 \tag{3}$$

with boundary conditions

$$\varphi'|_{x=0} = \varphi(\infty) = 0$$

and decreases monotonically at $0 \le x \le \infty$.

We shall henceforth use the notation

$$I(\lambda) = \int_{-\infty}^{\infty} \varphi^2\, dx.$$

In a medium with cubic nonlinearity, for which $f(\xi) = a\xi$, we have

$$\varphi(\lambda, x) = \sqrt{\frac{2}{a}}\frac{\lambda}{\operatorname{ch}(\sqrt{2}\lambda x)}, \quad I(\lambda) = \frac{2\sqrt{2}\lambda}{a}. \tag{3a}$$

In a medium with a power-law nonlinearity, in which $f(\xi) \propto \xi^S$,

$$\varphi(\lambda, x) = \lambda^{1/s}\varphi_0(\lambda, x), \quad I(\lambda) \propto \lambda^{2/s-1}.$$

If $\alpha f(\xi) > 0$, then Eq. (1) admits of a solution in the form of a stationary wave packet propagating with group velocity, namely an envelope soliton:

$$\Psi = \exp(i\lambda^2 z)\varphi(z), \quad -\lambda^2\varphi + {}^1/_2\alpha\varphi_{zz} + f(\varphi^2)\varphi = 0.$$

The envelope soliton can exist at $\alpha < 0$ also in a defocusing medium.

We consider the problem of the stability of a flat waveguide relative to the development of modulation along the z axis. We rewrite the solution of (2) in the form

$$\Psi = (\varphi + u + iv)\exp(i\lambda^2 t) \quad (u, v \ll \Psi) \tag{4}$$

and linearize this equation. We assume further that $u, v \sim \exp(i\kappa z - i\Omega t)$. We obtain

$$\Omega u = (L_0 + {}^1/_2\alpha\varkappa^2)v, \quad \Omega v = (L_1 + {}^1/_2\alpha\varkappa^2)u, \tag{5}$$

where

$$L_0 = -{}^1/_2 d^2/dx^2 + \lambda^2 - f(\varphi^2),$$
$$L_1 = -{}^1/_2 d^2/dx^2 + \lambda^2 - f(\varphi^2) - 2\varphi^2 f'(\varphi^2) \tag{6}$$

are self-adjoint linear operators.

From (5) we obtain

$$(L_0 + {}^1/_2\alpha\varkappa^2)(L_1 + {}^1/_2\alpha\varkappa^2)u = \Omega^2 u. \tag{7}$$

We introduce the notation

$$u_0^- = \varphi_x, \quad u_0^+ = -\partial\varphi/\partial\lambda^2, \quad v_0^- = -x\varphi, \quad v_0^+ = \varphi. \tag{8}$$

It follows directly from (3) that

$$L_0 v_0^+ = 0. \tag{9}$$

Differentiating Eqs. (3) with respect to x and with respect to λ^2, we get

$$L_1 u_0^- = 0, \quad L_1 u_0^+ = v_0^+. \tag{10}$$

It is also easy to verify the relation

$$L_0 v_0^- = u_0^-. \tag{11}$$

Formulas (9)–(11) yield the solution of the equation

$$L_0 L_1 u_0 = 0, \tag{12}$$

which is obtained from (7) by putting $\kappa^2 = 0$ and $\Omega = 0$, and the conjugate equation

$$L_1 L_0 v_0 = 0. \tag{13}$$

Obviously

$$u_0 = C_1 u_0^+ + C_2 u_0^-, \quad v_0 = C_1' v_0^+ + C_2' v_0^-, \tag{14}$$

where C_1, C_2, C_1', and C_2' are arbitrary constants. The functions u_0^+ and v_0^+ are even while u_0^- and v_0^- are odd with respect to the replacement of x by −x.

Scalar products of functions that differ in their parity are equal to zero. For functions of like parity we have

$$\langle v_0^+|v_0^+\rangle = I, \quad \langle v_0^+|u_0^+\rangle = -{}^1/_2 dI/d\lambda^2, \quad \langle v_0^-|u_0^-\rangle = {}^1/_2 I. \tag{15}$$

Relations (12) and (14) show that the operator L_0L_1 has two neutrally-stable loads: even u_0^+ and odd u_0^-. In a recent paper, Kolokolov and Vakhitov [14] have shown that the operator L_0L_1 has, in addition, an even mode $\tilde{u}_0^+$, which is stable at $dI/d\lambda^2 > 0$ and is unstable at $dI/d\lambda^2 < 0$. At $dI/d\lambda^2 = 0$, the even neutrally-stable mode is doubly degenerate. The even and odd neutrally-stable modes correspond at $\kappa \ne 0$ to two branches of the spectrum of the operator $(L_0 + {}^1\!/_2\,\alpha\kappa^2)(L_1 + {}^1\!/_2\,\alpha\kappa^2)$: even $u^+(x), \Omega^{2+}(\kappa)$ and odd $u^-(x), \Omega^{2-}(\kappa)$.

At values of $\alpha\kappa^2$ that are small in comparison with λ^2 we have

$$u^\pm(x) = u_0^\pm + u_1^\pm + u_2^\pm + \ldots, \quad \Omega^{2\pm}(\varkappa) = \xi_1^\pm + \xi_2^\pm + \ldots \tag{16}$$

Substituting (16) in (7) we obtain in first order in $\alpha\kappa^2$:

$$L_0 L_1 u_1^+ = \xi_1^+ u_0^+ - {}^1/_2\alpha\varkappa^2(L_0 + L_1)u_0^+. \tag{17}$$

The condition for the solvability of Eq. (17) is the orthogonality of its right-hand side to the solutions of the conjugate equation (13). It obviously suffices to verify the orthogonality to the even solution v_0^+. Multiplying (17) scalarly by v_0^+, we obtain, with allowance for (9), (10), and (15)

$$\xi_1^+ = \frac{1}{2}\alpha\varkappa^2\frac{\langle v_0^+|L_0 + L_1|u_0^+\rangle}{\langle v_0^+|u_0^+\rangle} = -\alpha\varkappa^2\frac{I}{dI/d\lambda^2}. \tag{18}$$

Analogously, for the odd mode

$$\xi_1^- = \frac{\alpha\varkappa^2}{2}\frac{\langle v_0^-|L_0 + L_1|u_0^-\rangle}{\langle v_0^-|u_0^-\rangle} = \alpha\varkappa^2\frac{\langle u_0^{-2}\rangle}{I} = \alpha\varkappa^2\frac{\langle\varphi_x^2\rangle}{I}. \tag{19}$$

In a medium with cubic nonlinearity, using (3a), we obtain

$$\Omega^{2+}(\varkappa) \approx \xi_1^+ = -2\alpha\varkappa^2\lambda^2, \quad \Omega^{2-}(\varkappa) \approx \xi_1^- = {}^2/_3\alpha\varkappa^2\lambda^2. \tag{20}$$

In a medium with nonlinearity of general form we get in order of magnitude

$$\Omega^{2+} \sim -\alpha\varkappa^2\lambda^2, \quad \Omega^{2-} \sim \alpha\varkappa^2\lambda^2. \tag{20a}$$

It follows from (20) and (20a) that stability takes place regardless of the sign of the dispersion α. At $\alpha > 0$ the symmetrical mode is unstable, and at $\alpha < 0$ the antisymmetrical mode is unstable It follows directly from (7) that $\Omega^2 \sim \alpha^2\kappa^4$ at $\alpha\kappa^2 \ge \lambda^2$. Therefore the instability is limited at $\alpha\kappa^2 \sim \lambda^2$, and for the maximum increment of both modes we have the estimate

$$\gamma_{max} \sim \lambda^2 \sim \Delta\omega_n.$$

The increment is of the order of the frequency shift of the monochromatic wave as a result of the nonlinearity, and is of the same order as the increment of the self-modulation instability of the monochromatic wave. At $\alpha \sim 1$, the characteristic length of the perturbation, corresponding to the maximum increment, is of the same order as the transverse dimension of the waveguide. Symmetrical instability leads to modulation of the amplitude of the waveguide in the longitudinal direction (instability of the neck type), whereas antisymmetric instability leads to a bending of the waveguide in the transverse direction (instability of the "snake" type)[2)].

Formulas (20) were obtained under the assumption that

$$dI/d\lambda^2 \sim I/\lambda^2.$$

As $dI/d\lambda^2 \to 0$, the increment of the asymmetrical mode becomes formally infinite, and at $dI/d\lambda^2 < 0$ the instability vanishes. In fact, of course, at

$$\frac{dI}{d\lambda^2} \sim \frac{\alpha\varkappa^2}{\lambda^2}\frac{I}{\lambda^2}$$

formula (18) no longer holds. This case, which is particularly important for cylindrical geometry, will be analyzed in the next section.

We note also that it can be easily verified that the operator L_0 can be represented in the form

$$L_0 = -\frac{1}{2\varphi}\frac{d}{dx}\varphi^2\frac{d}{dx}\frac{1}{\varphi}. \tag{21}$$

In the planar case the function $\varphi(x)$ has no zeroes, and the operator L_0 is non-negative: $\langle uL_0u\rangle \geq 0$. It follows therefore that at $\alpha > 0$ the operator $L_0 + \alpha\kappa^2/2$ is positive, and we can apply to Eq. (7) a variational method based on the relation

$$\Omega^2 = \min\frac{\langle u|L_1 + {}^1/_2\alpha\varkappa^2|u\rangle}{\langle u|(L_0 + {}^1/_2\alpha\varkappa^2)^{-1}|u\rangle}. \tag{22}$$

It was shown earlier [10] with the aid of the variational method (22) that instability of the stationary waveguide sets in at $\alpha > 0$, regardless of the form of the nonlinearity. At $dI/d\lambda^2 > 0$ this is the symmetrical instability obtained by us, and at $dI/d\lambda^2 < 0$ we get the instability obtained by Kolokolov and Valhitov [14]; its increment is finite as $\alpha\kappa^2 \to 0$, so that the existence of this instability does not depend on the sign of α. At $\alpha < 0$ we have, for any type of nonlinearity, an instability in the antisymmetrical mode. We note also that in planar geometry the case $dI/d\lambda^2 = 0$ is realized for a medium with a nonlinearity $f(\varphi^2) \propto \varphi^4$.

To calculate the remaining terms of the expansions (10) it must be borne in mine that the quantity $\xi_n^\pm$ is the result of the solvability of the equation for $u_n^\pm$ and its calculation calls for knowledge of the quantity $u_{n-1}^\pm$. To find $u_{n-1}^\pm$ it is necessary, generally speaking, to invert the operator L_0L_1. We note that it follows from (10) that the operator L_1 can be represented in the form

$$L_1 = -\frac{1}{2\varphi_x}\frac{d}{dx}\varphi_x^2\frac{d}{dx}\frac{1}{\varphi_x}.$$

This formula, in combination with (21), demonstrates the feasibility in principle of inverting the operator L_0L_1 and of calculating any term of the series (16) in quadratures.

The quantities $\xi_2^\pm$ can be calculated relatively simply for a cubic medium. We note first, that in a cubic medium we have

$$u_0^+ = -\frac{\partial\varphi}{\partial\lambda^2} = -\frac{1}{2\lambda^2}(\varphi + x\varphi_x). \tag{23}$$

Substituting (23) and ξ_1^+ from (20) in (17) and using the representation of the operator L_0 in the form (21), we find that relation (17) can be rewritten in the form

$$L_0L_1u_1^+ = \alpha\varkappa^2L_0(-{}^1/_2x^2\varphi + \partial\varphi/\partial\lambda^2),$$

whence

$$L_1u_1^+ = \alpha\varkappa^2(-{}^1/_2x^2\varphi + \partial\varphi/\partial\lambda^2).$$

Multiplying the second-approximation equation

$$L_0L_1u_2^+ = (\xi_2^+ - {}^1/_4\alpha^2\varkappa^4)u_0^+ + \xi_1^+u_1^+ - {}^1/_2\alpha\varkappa^2(L_0 + L_1)u_1^+$$

scalarly by $v_0^+ = \varphi$ and using the relation $\varphi = L_1u_0^+$, we obtain after simple transformations

$$\xi_2^+ - \frac{\alpha^2\varkappa^4}{4} = \frac{2\lambda^2\alpha^2\varkappa^4}{I}\left\langle x^2\varphi^2 + \frac{1}{\lambda^2}\left(-\frac{\varphi^2}{4} + x^2\varphi_x^2\right)\right\rangle.$$

Integrating, we obtain for the symmetrical mode in a cubic medium:

$$\Omega^{2+}(\varkappa) = -2\alpha\varkappa^2\lambda^2 + {}^5/_{12}(1 + {}^1/_3\pi^2)\alpha^2\varkappa^4. \tag{24a}$$

We can similarly perform the calculations for the antisymmetrical mode:

$$\Omega^{2-}(\varkappa) = {}^2/_3\alpha\varkappa^2\lambda^2 + {}^1/_9(1 + {}^2/_3\pi^2)\alpha^2\varkappa^4. \tag{24b}$$

Formulas (24) show that the increments reach a maximum and are bounded at $\alpha\kappa^2 \sim \lambda^2$. In this region, however, all the terms of the series are approximately equal, and formulas (24) yield only the order of magnitude.

We have considered so far the instability of a planar waveguide against the onset of modulation along the z axis. It is easy to generalize these results by taking into account the possibility of modulation along the y axis. Putting for the perturbations

$$u \sim v \sim \exp[-i\Omega t + \varkappa(\cos\theta z + \sin\theta y)],$$

we arrive at the same formulas as before, in which, however, it is necessary to replace α by $\alpha_{\text{eff}} = \alpha\cos^2\theta + \sin^2\theta$. Here θ is the angle between the wave vectors of the perturbation and the initial wave. We see therefore that in a medium with $\alpha > 0$ (medium with positive dispersion) symmetrical instability takes place at all perturbation propagation angles. In fact, it leads to a breakdown of the planar waveguide into three-dimensional bunches, within which amplitude singularities in the form of wave collapses appear after a finite time.[15] These collapsing bunches propagate at the group velocity. In a medium with negative dispersion ($\alpha < 0$) we have antisymmetrical instability inside the cone $\tan^2\theta < \alpha$, and symmetrical instability ouside the cone. Their combination leads to a subdivision of the planar waveguide into a flexing cylindrical waveguide and to a subsequent energy scattering through considerable angles. The question of the possible existence of singularities at $\alpha < 0$ remains open.

These results can be directly transferred to envelope solitons. In a self-focusing medium, the envelope soliton breaks up into collapsing bunches, and in a defocusing medium it becomes unstable against the appearance of inflections of its plane.

We now stop in greater detail on the importance case of envelope solitons on the surface of a deep liquid. Let the shape of the surface $\eta(x, y)$, referred to the plane $z = 0$, be given by

$$\eta(x, y) = {}^1/_2[\Psi \exp(ikx - i\omega_k t) + \text{c.c.}].$$

Here $\omega_k = (gk)^{1/2}$ is the wave dispersion law, Ψ is a slow function of the coordinates and of the time, and $u = \omega_k/2k$ is the group velocity of the wave. We have previously obtained [16] for the function Ψ an equation that can be conveniently written in the form

$$i(\Psi_t + u\Psi_x) + {}^1/_2\frac{u}{k}(\Psi_{yy} - {}^1/_2\Psi_{xx}) - {}^1/_4 k^2\omega_k|\Psi|^2\Psi = 0. \quad (25)$$

This equation belongs to type (2), where $f(\xi) \propto -\xi$, $\alpha = -\frac{1}{2}$. Equation (25) describes the envelope soliton

$$\Psi = \frac{\sqrt{2}\lambda}{k}\frac{\exp(-i\omega_k\lambda^2 t/4)}{\operatorname{ch}[\sqrt{2}\lambda k(x-ut)]} \quad (\lambda \ll 1). \quad (26)$$

The parameter λ has the meaning of the characteristic slope of the soliton. From the results obtained above it follows that the soliton (26) is unstable against perturbations of the type $\exp(\gamma t + ipy)$, which bend its front. The instability increment γ is given by formula (26), in which it is necessary to put

$$\Omega = 4\gamma/\omega_k, \quad \varkappa^2 = 2p^2/k^2, \quad \alpha = -1.$$

The maximum increment $\gamma_{max} \sim \lambda^2\omega_k$ is reached for perturbations with length on the order of $1/k\lambda$. Thus, the soliton (26) can propagate stably only in channels whose width does not exceed greatly the length of the soliton.

Our results enable us also to establish the fact that a planar waveguide is spatially unstable. We put in Eq. (2), expressed in the immobile coordinate frame, $\Psi = \exp(i\lambda^2 t)$, and neglect the term $\alpha\Psi_{zz}$. This gives rise to the well-known self-focusing equation

$$i\Psi_z + {}^1/_2\Delta_\perp\Psi - \lambda^2\Psi + f(|\Psi|^2)\Psi = 0.$$

A stationary planar waveguide constitutes the solution of this equation in the form $\Psi = \varphi(x)$.

Let us consider the closely related stationary solution

$$\Psi = \varphi(x) + (u + iv)\exp(i\varkappa y - ipz)$$

and let us determine $p(\varkappa)$. Obviously, this problem is identical with the problem of symmetrical instability in a self-focusing medium at $\alpha = 1$, and formula (24a) holds true for the increment. The spatial instability causes the planar waveguide to break up into cylindrical beams that collapse to form pointlike foci [21].

2. INSTABILITY OF CYLINDRICAL WAVEGUIDE IN A NONINERTIAL MEDIUM

Stationary cylindrical waveguides constitute the solution of Eq. (2) in the form

$$\Psi = \exp(i\lambda^2 t)\varphi(r).$$

The function $\varphi(r)$ is real and satisfies the equation

$$-\lambda^2\varphi + \frac{1}{2r}\frac{d}{dr}r\varphi_r + f(\varphi^2)\varphi = 0 \quad (27)$$

with the boundary conditions

$$\varphi_r|_{r=0} = 0, \quad \varphi(\infty) = 0.$$

Equation (27) describes an infinite set of stationary waveguides $\varphi_n(r)$, $n = (0, 1, \ldots)$, the n-th of which has exactly n zeroes on the semiaxis $0 \le r \le \infty$. The monotonically decreasing waveguide $\varphi_0(r)$ will be called a simple waveguide.

In the case of a power-law nonlinearity $f(\xi^2) \propto \xi^s$ we have as before

$$\varphi(\lambda, r) = \lambda^{1/s}\varphi_0(\lambda r).$$

Further

$$I(\lambda) = \int_0^\infty r\varphi^2\,dr \sim \lambda^{2/s-2}.$$

In particular, for a cubic medium we have $s = 1$ and $I(\lambda)$ is constant, i.e., the waveguide power does not depend on λ. It is easy to show that for a medium close to cubic we have $dI/d\lambda^2 > 0$ if $f''(\xi) < 0$ (saturating nonlinearity) and, to the contrary, $dI/d\lambda^2 < 0$ if $f''(\xi) > 0$.

The problem of the instability of a stationary waveguide against the onset of modulation along the z axis leads to Eq. (7), where the operators L_0 and L_1 take the form

$$L_0 = \frac{1}{2r}\frac{d}{dr}r\frac{d}{dr} + \lambda^2 + \frac{m^2}{2r^2} - f(\varphi^2),$$
$$L_1 = -\frac{1}{2r}\frac{d}{dr}r\frac{d}{dr} + \lambda^2 + \frac{m^2}{2r^2} - f(\varphi^2) - 2\varphi^2 f'(\varphi^2). \quad (28)$$

It is assumed here that the perturbation has an angular dependence $e^{im\theta}$, where r and θ are the polar coordinates in the (x, y) plane.

We consider first the case $m = 1$. Introducing the notation $v_0^- = -r\varphi$ and $u_0^- = \varphi_r$, we verify, by differentiating (27), that relations (10) and (11) remain valid as before. It follows therefore that in the cylindrical case the instability at the $m = 0$ mode is perfectly analogous to the plane antisymmetrical instability. For the instability increment we have from formula (19)

$$\Omega^2 \approx \alpha\varkappa^2\langle\varphi_r^2\rangle/I \sim \alpha\varkappa^2\lambda^2. \quad (29)$$

The instability takes place in media with negative dispersion ($\alpha < 0$) and leads to the spontaneous bending of the waveguide.

In a cubic medium, Eq. (27) takes the form

$$-\lambda^2\varphi + \frac{1}{r}\frac{d}{dr}r\varphi_r + \varphi^3 = 0. \quad (30)$$

Multiplying (30) by $r\varphi$ and integrating, we get

$$-\lambda^2\langle\varphi^2\rangle - {}^1/_2\langle\varphi_r^2\rangle + \langle\varphi^4\rangle = 0. \quad (31)$$

Further, multiplying (30) by $r^2\varphi_r$ and integrating, we obtain

$$2\lambda^2\langle\varphi^2\rangle = \langle\varphi^4\rangle. \quad (32)$$

From (31) and (32) we obtain $\langle\varphi_r^2\rangle = 2\lambda^2 I$, and consequently

$$\Omega^2 = 2\alpha\varkappa^2\lambda^2. \quad (33)$$

We consider now the centrally-symmetrical mode ($m = 0$). It is impossible to employ here the results of the planar problem directly, since $dI/d\lambda^2 = 0$ in the most interesting case of a cubic medium and formula (18) no longer holds. Only in a far-from-cubic medium with a strong saturation of the nonlinearity do we have

$$0 < dI/d\lambda^2 \sim I/\lambda^2,$$

and the "ordinary" symmetrical instability takes place.

To consider the case of a medium close to cubic, we multiply (7) by φ and discard terms of second order in $\alpha\varkappa^2$. We obtain

$$ {}^1/_2\alpha\varkappa^2\langle\varphi L_1 u\rangle = \Omega^2\langle\varphi u\rangle. \quad (34)$$

In the left-hand side of (34) we can put $u \approx u_0^+$, whereas in the right-hand side it is necessary to retain the term of next order. Putting $u = u_0^+ + u_1^+$, we assume in the calculation of u_1^+ that the medium is cubic and put $\alpha\kappa^2 = 0$. We then obtain for u_1^+ the equation

$$L_0 L_1 u_1{}^+ = \Omega^2 u_0{}^+. \tag{35}$$

In a cubic medium we have

$$u_0{}^+ = -\frac{\partial\varphi}{\partial\lambda^2} = -\frac{1}{2\lambda^2}\frac{d}{dr}r\varphi.$$

In addition, in the cylindrical case

$$L_0 u = -\frac{1}{2r\varphi}\frac{d}{dr}r\varphi^2\frac{d}{dr}\frac{u}{\varphi},$$

and Eq. (35) is integrated twice. We obtain

$$L_1 u_1{}^+ = \Omega^2 r^2 \varphi / 4\lambda^2. \tag{36}$$

Next, recognizing that $\varphi = L_1 u_0^+$, we have

$${}^1/_2 \alpha\varkappa^2 I = \Omega^2 (\langle \varphi u_0{}^+ \rangle + \langle u_0{}^+ L_1 u_1{}^+ \rangle).$$

Taking (36) into account, we obtain ultimately

$$\Omega^2(\Omega^2 - \Omega_0{}^2) = \Omega_1{}^4,$$
$$\Omega_0{}^2 = \frac{4\lambda^4}{\langle r^2\varphi^2\rangle}\frac{dI}{d\lambda^2}, \quad \Omega_1{}^4 = \frac{4\alpha\varkappa^2\lambda^4 I}{\langle r^2\varphi^2\rangle}. \tag{37}$$

At $\kappa = 0$, Eq. (37) describes two modes, a neutrally stable mode u_0^+ and a "Kolokolov mode" $\tilde{u}_0^+$, which is stable at $dI/d\lambda^2 < 0$, but is unstable at $dI/d\lambda^2 < 0$. In fact, the existence of an unstable mode at $\kappa = 0$ means "spatial" instability of the cylindrical waveguide. Although the increment of the spatial instability vanishes for a cubic medium, nevertheless, as shown by a numerical experiment [17], a nonlinear spatial instability takes place and leads to the formation of foci [21]. We note that this result is valid for both simple and complicated waveguides.

At $\kappa \neq 0$ and $dI/d\lambda^2 < 0$, the instability takes place regardless of the sign of α. The same holds in a cubic medium. In this case $\Omega_0^2 = 0$ and

$$\Omega^2 \approx \pm 2\left(\frac{\alpha\varkappa^2\lambda^4 I}{\langle r^2\varphi^2\rangle}\right)^{1/2} \sim \pm\lambda^2(\lambda^2\alpha\varkappa^2)^{1/2}. \tag{38}$$

In a weakly-saturating medium, the instability (38) takes place only for not too small κ, at

$$\left(\frac{\alpha\varkappa^2}{\lambda^2}\right)^{1/2} \gg \frac{\lambda^2 dI}{I\,d\lambda^2}.$$

At small κ and $\alpha > 0$ we have an instability with an increment

$$\Omega^2 \sim \frac{\Omega_1{}^4}{\Omega_0{}^2} \sim \alpha\varkappa^2\frac{I}{dI/d\lambda^2},$$

similar to the "ordinary" symmetrical instability. The symmetrical instability of a cylindrical waveguide reached its maximum increment $\gamma \sim \lambda^2 \sim \Delta\omega_n$, as before, at $\alpha\kappa^2 \sim \lambda^2$.

3. INSTABILITY OF THE WAVEGUIDE IN A MEDIUM WITH INERTIAL NONLINEARITY

Equation (1) presupposes that the nonlinear medium is inertialess, i.e., the nonlinearity "follows" instantaneously the wave field. In many physically important situations the nonlinearity has a finite relaxation time, which is connected with the inertia of the processes that occur in the medium under the influence of the wave field. In this case Eq. (2) must be replaced by a pair of equations, which in the laboratory system take the form

$$i(\Psi_t + \Psi_z) + {}^1/_2\Delta_\perp\Psi + {}^1/_2\alpha\Psi_{zz} + p\Psi = 0, \quad \hat{A}p = f(|\Psi|^2). \tag{39}$$

Here $\hat{A}$ is a linear operator which generally speaking is nonlocal in the coordinates and takes into account the delay of the nonlinearity. If $|\Psi|^2$ does not depend on the time, and the operator $\hat{A} = 1$, then the system (39) has the same stationary solutions as Eq. (2).

Let us compare the effects of the inertia of the nonlinearity and dispersion. Let the time of the inertia of the nonlinearity be τ. The inertialess instability of the waveguide has the largest increment $\gamma \sim \Delta\omega_n$ at $\alpha\kappa^2 \sim \Delta\omega_n$, i.e., at $\kappa \sim (\Delta\omega_n/\omega'')^{1/2}$. A perturbation of this scale drifts away over a length on the order of its dimension within a reciprocal time

$$\varkappa u \approx \omega\left(\frac{\Delta\omega_n}{k^2\omega_k''}\right)^{1/2} \gg \Delta\omega_n.$$

Obviously, the inertia of the nonlinearity can be neglected if $1/\tau \gg \kappa u$, i.e., if

$$\frac{\Delta\omega_n}{\omega} < \frac{\omega'' k^2}{\omega}\frac{1}{\omega^2\tau^2}.$$

At $\omega'' k^2/\omega \sim 1$ we have

$$\Delta\omega_n / \omega \ll 1 / (\omega\tau)^2.$$

This rather stringent condition is usually not satisfied in laser experiments. It is therefore natural to consider the opposite case

$$\Delta\omega_n / \omega \gg 1 / \omega^2\tau^2,$$

when the decisive factor is the inertia of the nonlinearity, and neglect the dispersion term $\alpha\Psi_{zz}$.

Taking this circumstance into account, linearization of the system (39) against the background of the stationary waveguide leads to dispersion equations that are equivalent to each other

$$L_0(L_1 + \delta L)u = \Omega^2 u, \tag{40}$$

$$(L_1 + \delta L)L_0 v = \Omega^2 v, \tag{41}$$

where

$$\delta L = 2\varphi(A^{-1}(\Omega, \varkappa) - 1)f'(\varphi^2)\varphi;$$

at $\Omega = 0$ we have $A^{-1}(0, \kappa) = 1$ and $\delta L = 0$.

As shown in Secs. 1 and 2, a flat waveguide, or else a cylindrical waveguide in a medium close to cubic, experiences a "spatial" instability (at $\Omega = 0$) which is preserved for media with any relaxation time. We therefore confine ourselves to the case of a cylindrical waveguide in a medium with nonlinearity saturation, in which there is no spatial instability and in which we have stable propagation in the stationary problem [17].

Multiplying (40) from the left by φ, we observe that the symmetrical mode is neutrally stable. Multiplying (41) from the left by φ_r and putting $v \approx v_0^+ = -r\varphi$, we obtain after simple transformations

$$(\Omega - \varkappa)^2 = -4\langle \varphi\varphi_r | A^{-1}(\Omega, \varkappa) - 1 | \varphi\varphi_r f'(\varphi^2)\rangle / I. \tag{42}$$

The criterion for the applicability of this formula is the condition

$$\Omega - \varkappa \ll \Delta\omega_n.$$

We consider the case of a medium with nonlinearity that relaxes in accordance with the law

$$\tau\frac{\partial p}{\partial t} + p = f(|\Psi|^2). \tag{43}$$

In this case $I = 1 - i\Omega\tau$, and Eq. (42) in the dimensional variables takes the form

$$(\Omega-\varkappa u)^2=-\frac{4i\Omega\tau}{1-i\Omega\tau}\Delta\omega_n^2 c(\lambda), \tag{44}$$

where

$$c(\lambda)=\langle\varphi^2\varphi_r^2 f'(\varphi^2)\rangle/I\lambda^4>0$$

is a dimensionless structure factor.

At $\kappa=0$, Eq. (44) has a neutrally stable solution $\Omega=0$, which generates at $\kappa\neq 0$ an unstable branch of the spectrum. At $\Omega\tau\ll 1$ we have

$$\Omega=\varkappa u+2i\tau(\Delta\omega_n)^2 c(\lambda)\left[\sqrt{1+\frac{i\varkappa u}{\tau(\Delta\omega_n)^2 c(\lambda)}}-1\right]. \tag{45}$$

Expanding the radical at small κu, we obtain

$$\Omega=\frac{i}{4}\frac{(\varkappa u)^2}{(\Delta\omega_n)^2 c(\lambda)\tau}. \tag{46}$$

The positive sign of the imaginary part in (46) corresponds to instability.

If the inertia is large $\Delta\omega_n\tau\gg 1$, formula (46) is valid up to

$$\varkappa\sim\frac{\Delta\omega_n}{u}\sim k_0\frac{\Delta\omega_n}{\omega},$$

where the instability increment reaches a maximum value $\gamma\sim 1/\tau$. In this case only waveguides of lengths $L<\lambda\omega/\Delta\omega_n$ are stable (λ is the wavelength). In the opposite limiting case of small relaxation times ($\Delta\omega_n\tau\ll 1$) one should neglect unity under the radical in formula (45). We then have for the increment

$$\gamma=2(\varkappa u\tau\Delta\omega_n^2 c(\lambda))^{1/2}.$$

The maximum increment is reached at $\kappa\sim 1/u\tau$ and is of the order of $\gamma\sim\Delta\omega_n$, the same as the inertial medium. The maximum length of the stable waveguide is of the order of $L\sim\lambda\omega\tau$.

4. INSTABILITY OF A WAVEGUIDE IN A MEDIUM WITH STRICTION NONLINEARITY

Considerable physical interest attaches to self-focusing in a medium with striction nonlinearity, in which the connection between p and Ψ is given by the wave equation

$$-s^{-2}p_{tt}+\Delta p=\Delta f(|\Psi|^2).$$

To investigate the instability of the waveguide in a medium with striction, we used the results of the preceding section. Obviously, the operator A is given by

$$A=1+\Omega^2 s^{-2}\Delta^{-1}. \tag{47}$$

At sufficiently small Ω ($\Omega^2\ll k^2s^2\Delta\omega_n/\omega$) we can put

$$A^{-1}-1\approx-\Omega^2 s^{-2}\Delta_\perp^{-1}. \tag{48}$$

Substituting (48) in (42), we obtain in dimensional variables

$$(\Omega-\varkappa u)^2=-\frac{\Omega^2u^2}{s^2}c_1(\lambda),$$
$$c_1(\lambda)=-4\frac{\langle\varphi\varphi_r|\Delta_\perp^{-1}|f'(\varphi^2)\varphi_r\varphi\rangle}{I}>0. \tag{49}$$

Here $c_1(\lambda)$ is a dimensionless structure factor of the order of $\Delta\omega_n/\omega$. Equation (49) has an unstable root

$$\Omega=\frac{\varkappa u[1+ius^{-1}c_1^{1/2}(\lambda)]}{1+u^2s^{-2}c_1(\lambda)}. \tag{50}$$

Just as before, the condition for the applicability of (50) is

$$\Omega-\varkappa u\ll\Delta\omega_n. \tag{51}$$

Two limiting cases can be separated. At a low nonlinearity level $u^2\Delta\omega_n/s^2\omega\ll 1$ we have

$$\Omega-\varkappa u\approx ius^{-1}c_1^{1/2}(\lambda)\varkappa u.$$

The maximum instability increment $\gamma\sim\Delta\omega_n$ is reached at $\kappa\sim ksu^{-1}(\Delta\omega_n/\omega)^{1/2}$. As a result of the instability, the waveguide breaks up into elongated bunches with $L_\parallel\sim uL_\perp/s\gg L_\perp$, which are particularly long in the case of striction self-focusing of light: $L_\parallel\sim(c/s)L_\perp$. It is easy to verify that the waveguide is stable with respect to shorter perturbations. In the case of large nonlinearity $u^2\Delta\omega_n/s^2\omega>1$, the instability becomes periodic:

$$\Omega\approx i\varkappa sc_1^{-1/2}(\lambda).$$

The maximum increment is reached at $\kappa\sim k\Delta\omega_n/\omega$ and is equal to

$$\gamma\sim\omega_k\frac{s}{u}\left(\frac{\Delta\omega_n}{\omega}\right)^{1/2}.$$

5. INSTABILITY OF PLASMA SOLITON

In a collisionless plasma there can exist localized regions of intense Langmuir oscillations—plasma solitons, standing or moving. In these regions the plasma density is lower, and they constitute in essence resonators for plasma waves. We shall examine one-dimensional plasma solitons, which were apparently observed by Gurevich and Pitaevskiĭ in 1963 [18]. Such formations and the effects associated with them were recently investigated in a number of papers [19,20].

Plasma solitons and their stability are best investigated within the framework of an equation for the complex envelope of a high-frequency electrostatic potential

$$\varphi={}^1/_2(\Psi\exp(i\omega_p t)+\text{c.c.}).$$

It was shown in [19] that for not too large amplitudes $|\nabla\varphi|^2/8\pi nT<m/M$) and neglecting dissipation effects, the function Ψ satisfies the equation

$$\Delta\left(i\Psi_t+\frac{3}{2}\omega_p r_d^2\Delta\Psi\right)+\frac{e^2}{8\pi mT}\operatorname{div}|\nabla\Psi|^2\nabla\Psi. \tag{52}$$

Introducing the natural dimensionless variables, we rewrite (52) in the form

$$\Delta(i\Psi_t+\Delta\Psi)+\operatorname{div}|\nabla\Psi|^2\nabla\Psi=0. \tag{53}$$

In one-dimensional geometry Eq. (53) reduces to the form

$$iE_t+{}^1/_2E_{xx}+|E|^2E=0 \tag{54}$$

and has solutions of the type

$$E=\exp[ikx-i(k^2-\lambda^2)t]\varphi(x-kt), \tag{55}$$

where $\varphi(\xi)$ satisfies the equation

$$-\lambda^2\varphi+{}^1/_2\varphi_{\xi\xi}+|\varphi|^2\varphi=0. \tag{56}$$

It is seen from (54)–(56 that it is possible to separate two limiting cases: $k\gg\lambda$ and $k\ll\lambda$. In the first case the solution (55) is an ordinary envelope soliton. Putting $\Psi\sim e^{ikx}$ and expanding in λ/k, we obtain Eq. (1) with $f(\xi)\propto\xi$, which we investigated in Sec. 1. The soliton experiences a symmetrical instability and is destroyed within a time $\tau\sim(\omega_p|\nabla\varphi|^2/8\pi nT)^{1/2}$, breaking up into collapsing bunches [15].

In the present section we focus our attention on the opposite case, $k\ll\lambda$. We investigate the stability of the solutions (55) with $k=0$ relative to perturbations proportional to $\exp(-i\Omega t+ik_y y)$. Linearizing (52) we obtain a system of equations close to (5):

$$\Omega u=(L_0+L')v, \quad \Omega v=(L_1+L')u,$$

where the operators L_0 and L_1 coincide with those introduced in Sec. 1 for the case of cubic medium, and the action of the operator L' on the function u is determined in the following manner:

$$\frac{d}{dx}L'u=k_y^2\left(u_x+(\varphi^2-\lambda^2)\int_{-\infty}^{x}u\,dx\right).$$

Assuming k_y to be sufficiently small and applying the results of Sec. 1, we investigate the stability of a soliton relative to excitation of symmetrical and antisymmetrical modes. It is easy to verify that the odd mode is stable. For symmetrical perturbations, we obtain the following dispersion equation:

$$\Omega^2=-2k_y\frac{\langle\varphi L'\varphi\rangle}{dI/d\lambda^2}.$$

Substituting the explicit form of φ from (3a) and performing rather laborious integration, we obtain

$$\Omega^2=-\lambda^2k_y^2(12-7\zeta(3))\approx 3.6\lambda^2k_y^2,$$

where $\zeta(x)$ is the Riemann zeta function.

The maximum increment $\gamma\sim\lambda^2$, or in terms of dimensional variables

$$\gamma\sim\omega_p|\nabla\varphi|^2/8\pi nT,$$

is reached at $k_y\sim\lambda$. As a result of the instability the soliton breaks up into collapsing bunches with longitudinal dimension on the order of the transverse one. We note that in all the previously considered cases there occurred a collapse of the envelopes of the bunches with a monochromatic carrier [15]. In this case, however, it is the Langmuir waves proper that collapse [19].

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[1)]Analogous considerations were used by Kadomtsev and Petviashvili in a study of soliton instability in media with weak dispersions [11].

[2)]The terms "neck" and "snake" are borrowed from the terminology used in the description of the instability of the pinch effect in a plasma

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102