The nature of the self-focusing singularity

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(Submitted October 4, 1974)
Zh. Eksp. Teor. Fiz. 68, 940–947 (March 1975)

We propose a simple energy derivation for the way the amplitude of light waves behaves near the focus, when there is self-focusing. We compare the result with the data of a numerical experiment. We give an analytical description of a pulsing waveguide in a medium with a weakly saturating nonlinearity. We discuss the nature of the self-focusing singularity in a medium of arbitrary dimensionality which has an arbitrary power-law nonlinearity.

1. The phenomenon of the self-focusing of beams of waves in non-linear media has recently been studied extensively (see the review articles [1,2]). In particular, it has been shown (by Vlasov, Talanov, and Petrishchev [5]) that when self-focusing occurs in a medium with a cubic non-linearity the wave field becomes singular—the amplitude of the field becomes infinite in one point (the focus). This effect has, in the form of a steep increase in the amplitude near the focus, been observed several times in laboratory and numerical experiments. [1,4-6]

The study of the wave field near the singularity is of great interest as a matter of principle. In our earlier papers [7,8] we have made a statement that the amplitude of the field near the singularity must turn to infinity as $(z_0 - z)^{-2/3}$. In the present paper we give a simple proof of the correctness of this expression and discuss the results of a numerical experiment. Moreover, we analyze the problem of the self-focusing for an arbitrary power-law non-linearity in a space of arbitrary dimensionality and also the problem of the self-focusing in a medium with a weakly saturating non-linearity.

2. The stationary self-focusing of an axially symmetric beam of waves in a medium with a cubic non-linearity is described by the equation

$$2i\varphi_z+\frac{1}{r}\frac{\partial}{\partial r}r\varphi_r+|\varphi|^2\varphi=0. \qquad (1)$$

Equation (1) has the integrals of motion

$$I_1=\int_0^\infty r|\varphi|^2\,dr,\quad I_2=\int_0^\infty\left(|\varphi_r|^2-\frac{1}{2}|\varphi|^4\right)r\,dr \qquad (2)$$

and admits of a set of exact solutions

$$\varphi_n(r,z,\lambda)=\exp(i\lambda^2 z/2)\lambda R_n(\lambda r). \qquad (3)$$

The function $R_n(\xi)$ satisfies the equation

$$\frac{d^2}{d\xi^2}R_n+\frac{1}{\xi}\frac{d}{d\xi}R_n+R_n^3-R_n=0 \qquad (4)$$

and has n zeroes in the range $0<\xi<\infty$. We shall, in what follows, write $R_0(\xi) = R(\xi)$. The integral I_2 vanishes for all φ_n. The magnitude of the integral I_1 is independent of λ for these functions. The solutions (3) describe stationary self-focusing. The quantity

$$I_0=2\pi\int_0^\infty|\varphi_0(r)|^2 r\,dr$$

is the critical power of the self-focusing.

From Eq. (1) we get the following relation [3]

$$\frac{\partial^2}{\partial z^2}\int_0^\infty r^3|\varphi|^2\,dr=I_2, \qquad (5)$$

which demonstrates the fact that when $I_2 < 0$ there occurs at a finite distance from the point where the beam enters the non-linear medium a singularity of the amplitude φ. The study of the nature of this singularity is just the subject of the present paper.[1)]

3. To establish a heuristic description of the nature of the singularity we applied a numerical experiment. We solved the Cauchy problem for Eq. (1) with the boundary condition

$$\varphi(r,0)=a_0\exp(-r^2/l^2),$$

where the parameters a_0 and l were chosen such that the integral I_1 was 5 and 13.5 times larger than I_0. We used a variable mesh (which decreased towards the center) in r and an automatically changed mesh for z and controlled the accuracy of the calculation by the conservation of the integrals I_1 and I_2. The scheme used enabled us without loss of accuracy to obtain values of the amplitude $|\varphi|$ which were two orders of magnitude larger than the initial magnitude a_0. For those amplitudes the nature of the singularity exhibited itself rather clearly. For both values of the initial power the radial behavior of the amplitude $\varphi(r, z)$ near the singularity has the form (see Fig. 1) of a flat plateau with on its background a beam which narrows in a self-similar manner. The energy of the narrowing beam is conserved and about equal to I_0. As $z \to z_0$ (z_0 is the singularity) the height of the plateau remained unchanged, while the maximum of the beam amplitude as function of the distance from the singularity started to show with a good accuracy the asymptotic behavior

$$\varphi(0,z)\to c/(z_0-z)^\nu,\quad \nu={}^2/_3\pm0.05. \qquad (6)$$

Let us show that the result $\nu = 2/3$ can be obtained analytically. To do this we write $\varphi = Ae^{i\Phi}$ and in accordance with the numerical result we write $A(r, z)$ in the form

$$A=\frac{1}{f(z_0-z)}R\left(\frac{r}{f(z_0-z)}\right)+A_0. \qquad (7)$$

Here z_0 is the singularity, A_0 a constant, and $R(\xi)$ is the first of the stationary waveguides. In terms of the variables A and Φ, the integral I_2 takes the form

$$I_2=\int_0^\infty(A^2\Phi_r^2+A_r^2-{}^1/_2A^4)r\,dr. \qquad (8)$$

For a stationary waveguide $I_2 = 0$, so that

$$\int_0^\infty r(R_r^2-{}^1/_2R^4)\,dr=0. \qquad (9)$$

It follows from Eq. (1) that

$$\frac{\partial}{\partial z}A^2+\frac{1}{r}\frac{\partial}{\partial r}rA^2\Phi_r=0. \qquad (10)$$

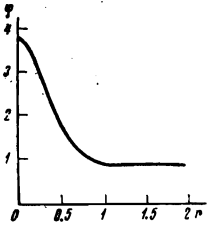

FIG. 1. Field amplitude in the near-axial region for z = 4.52 in the case where $a_0 = 1$, $l = 6$.

Substituting (7) into (10) and neglecting the term A_0 as $f \to 0$, we find

$$\Phi_r = f'r/f. \tag{11}$$

Substituting (7) and (11) into (8) and using Eq. (9) we find

$$I_2 = I_2^{(0)} + \ldots + I_2^{(3)},$$
$$I_2^{(0)} = \int_0^\infty \left\{ \frac{f'^2}{f^4} R^2\left(\frac{r}{f}\right) r^2 - \frac{2A_0}{f^3} R^3\left(\frac{r}{f}\right) \right\} r\,dr. \tag{12}$$

The expression $I_2^{(0)}$ contains the main terms which grow as $1/f$ when $z \to z_0$. The other terms in I_2 remain constant as $z \to z_0$. This is also true for the term

$$I_2^{(3)} = \int_0^\infty (A_0^2\Phi_r^2 - {}^1/_2 A_0^4) r\,dr,$$

which is formally divergent, if it is cut off at the external diameter of the beam. The conservation of the complete integral I_2 and the fact that the lower terms in I_2 are constant as $z \to z_0$ enables us to reach the conclusion that near the singularity

$$I_2^{(0)} = E = \text{const.}$$

Here E is the "internal" value of the integral I_2. We can now find from (12) an equation for $f(z)$:

$$\alpha f_z'^2 - 2A_0\beta/f = E;$$
$$\alpha = \int_0^\infty R^2(\xi)\xi^3\,d\xi, \quad \beta = \int_0^\infty R^3(\xi)\xi\,d\xi. \tag{13}$$

Here α and β are constants which can be evaluated in terms of $R(\xi)$.

Expression (13) is the energy integral for the motion of a non-relativistic particle of mass $1/2\alpha$ in an attractive Coulomb field. In the one-dimensional case any motion of a particle in such a field starts or ends with the singularity. Near the singularity the quantity E can be neglected. We then have

$$f(z) = (9\beta A_0/2\alpha)^{1/3}(z_0 - z)^{2/3}, \tag{14}$$

which explains the result of the numerical experiment. When $E > 0$ the beam diverges either as $z \to +\infty$ or as $z \to -\infty$. When $E < 0$ the beam is "closed"—in that case it starts and ends in foci. The power incident on the focus is exactly equal to the critical power.

4. Let now the non-linearity be saturating. The equation describing the non-linear medium has in that case the form

$$2i\varphi_z + \frac{1}{r}\frac{\partial}{\partial r} r\varphi_r + F(|\varphi|^2)\varphi = 0,$$
$$F(\xi) = \xi - \gamma\xi^2.$$

Repeating the previous reasoning we easily get the analog of Eq. (13), taking into account the corrections caused by the saturation of the non-linearity. This equation has the form

$$\alpha f_z'^2 - \frac{2A_0\beta}{f} + \frac{q\gamma}{f^4} = E, \quad q = \int_0^\infty rR^6(r)\,dr. \tag{15}$$

Equation (15) describes the motion of a particle in a potential which is a combination of an attractive Coulomb potential and a repulsive short-range potential. The particle will be repelled from the center in this field and this leads to a minimum beam radius equal to

$$r_{min} = (q\gamma/2A_0\beta)^{1/3}. \tag{16}$$

If $E > 0$ the beam, focusing up to the size r_{min} will then again diverge. If $E < 0$ we get a pulsating waveguide—the beam radius oscillates between r_{min} and r_{max}:

$$r_{max} \sim 2A_0\beta/|E|.$$

When the leading terms in the expansion (7) are taken into account in the equation for f there must arise terms containing higher derivatives of f with respect to z which take into consideration the "radiation damping" of the waveguide oscillations caused by the emission of part of the energy from the waveguide regime. If such radiation occurs the beam drops to the bottom of the potential well (15) reaching ultimately an equilibrium value of the radius

$$R_{\text{equil}} = (2q\gamma/\beta A_0)^{1/3} = 4^{1/3} R_{min}. \tag{17}$$

Equation (17) gives the expression for the radius of a self-channelling beam in a medium with a weak saturation of the non-linearity.

Many authors [5,6] have observed waveguide oscillations in media with saturation, using computers. We show in Fig. 2 a typical form of the field on the axis for this case, as obtained in our numerical experiments. We also note that in real experiments it is difficult to obtain a stationary (oscillating or equilibrium) waveguide due to the fact that its longitudinal instability is easily excited (see [10]).

We found earlier the behavior of the wave field near the singularity z_0. The problem of the determination of the point z_0 from the initial beam profile is also of interest. For a number of profiles (Gaussian, parabolic) this distance has been evaluated using certain approximations (geometric optics, self-focusing without aberration, and so on; see the review [11]). These calculations have, however, only a limited value as the quantity z_0 turns out to be very sensitive to the structure of the initial beam profile. We show in Fig. 3 the behavior of the field amplitude on the axis for two beams which have the same total intensity but which differ somewhat in form (maximum difference in intensity 10%). The initial profiles of the beams are shown in Fig. 4. It is clear that the singularity is reached for appreciably lower values of z_0 for the "distorted" beam. This result is explained by the development of the modulation instability of a large amplitude beam.

5. In a number of physical problems it becomes

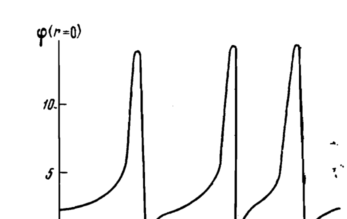

FIG. 2. Field amplitude on the axis for a medium with saturation ($a_0 = 1$, $l = 10$).

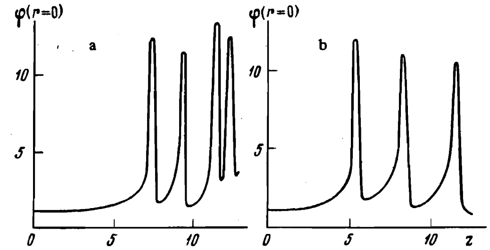

FIG. 3. Field amplitude on the axis (a) for an "undistorted" beam, and (b) for a "distorted" beam.

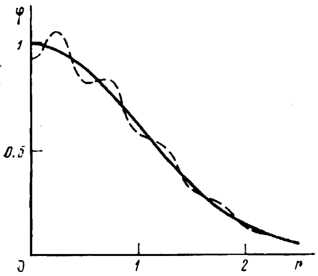

FIG. 4. Initial profiles of the field amplitude: full-drawn line: "undistorted" beam; dashed line; "distorted" beam.

necessary to study self-focusing not only in two dimensions (as we did here) but also in one- or three-dimensional media. The non-linearity is also, in general, not cubic. It is reasonable to generalize the results obtained above to the case of a medium of arbitrary dimensionality and with an arbitrary power-law non-linearity. For the sake of generality we shall also include in our discussion non-integral values of the dimensionality. Let $F(\xi) = \xi^a$ and let the medium have the dimensionality $1 + b$. The analog of Eq. (1) in such a medium has the form

$$2i\varphi_z + \frac{1}{r^b}\frac{\partial}{\partial r} r^b \frac{\partial\varphi}{\partial r} + |\varphi|^{2a}\varphi = 0. \tag{18}$$

The conservation laws then have the form

$$I_1 = \int_0^\infty r^b|\varphi|^2\,dr, \quad I_2 = \int_0^\infty r^b\left(|\varphi_r|^2 - \frac{1}{1+a}|\varphi(r)|^{2a+2}\right)dr. \tag{19}$$

Equation (18) has, like Eq. (1), a set of exact stationary solutions

$$\varphi(r, z, \lambda) = \lambda^{1/a}\exp(i\lambda^2 z/2)R(r\lambda). \tag{20}$$

For these solutions

$$I_1(\lambda) = I_1(0)\lambda^{2/a-b-1}.$$

The value $a = 2/(1 + b)$ is a limit one: when $a > 2/(1 + b)$, $\partial I_1/\partial\lambda < 0$; when $a < 2/(1 + b)$, $\partial I_1/\partial\lambda > 0$. When $a = 2/(1 + b)$ the magnitude of the integral I_1 is independent of λ.

We shall show that in that case $I_2 = 0$. The function $R(r)$ satisfies the equation

$$\frac{1}{r^b}\frac{\partial}{\partial r} r^b \frac{\partial R}{\partial r} + R^{2a+1} - R = 0. \tag{21}$$

Multiplying Eq. (21) by $r^b R$ and integrating we find

$$-I_1 - X + Y = 0, \quad X = \int_0^\infty r^b\varphi_r^2\,dr, \quad Y = \int_0^\infty r^b\varphi^{2a+2}(r)\,dr. \tag{22}$$

Now, multiplying (21) by $r\varphi_r$ and integrating we find after simple manipulations

$$(1+b)I_1 + (b-1)X - \frac{1+b}{1+a}Y = 0. \tag{23}$$

From (22) and (23) it follows that for $a = 2/(1 + b)$

$$I_2 = X - \frac{1}{1+a}Y.$$

Equation (18) has thus, when $a = 2/(1 + b)$, the same properties as Eq. (1) and we can state that the field near the singularity will also have the form (7). Repeating our earlier discussion we find for $f(z)$ the equation

$$\alpha f_z'^2 - 2A_0\beta/f^{(3-\beta)/2} = E;$$
$$\alpha = \int_0^\infty R^2(\xi)\xi^{1+b}\,d\xi, \quad \beta = \int_0^\infty R^{2a+1}(\xi)\xi^b\,d\xi. \tag{24}$$

It follows from Eq. (24) that

$$f(z) \approx (z_0 - z)^{4/(7-b)}.$$

Apart from the case $b = 1$, the cases $b = 0$ and $b = 2$ are of physical interest. For $b = 0$ the critical non-linearity is $a = 2$, and Eq. (18) has the form

$$2i\varphi_z + \varphi_{xx} + |\varphi|^4\varphi = 0. \tag{25}$$

This equation describes (see [12]) the collapse of a plane plasma soliton. In that case $f(z) \approx (z_0 - z)^{4/7}$. In the case $b = 2$ (three-dimensional medium) the equation with the critical non-linearity has the form

$$2i\varphi_z + \frac{1}{r^2}\frac{\partial}{\partial r} r^2\varphi_r + |\varphi|^{4/3}\varphi = 0, \quad f(z) \approx (z_0 - z)^{4/5}.$$

It is of interest to consider the problem of the nature of the singularity in the case of a stronger non-linearity, $a > 2/(1 + b)$. In that case $\partial I_1/\partial\lambda < 0$, and the stationary solution must "get rid of" part of its energy in order that its amplitude can increase. The behavior of the field near the singularity has now a self-similar substitution in Eq. (18)

$$\varphi(r, z) = \frac{1}{(z_0 - z)^{1/2a}}\psi\left(\frac{r}{(z_0 - z)^{1/2}}\right), \tag{26}$$

we reduce it to the form

$$-i\left(\frac{1}{2a}\psi + \xi\frac{\partial\psi}{\partial\xi}\right) + \frac{1}{\xi^b}\frac{\partial}{\partial\xi}\xi^b\frac{\partial\psi}{\partial\xi} + |\psi|^{2a}\psi = 0. \tag{27}$$

As $\xi \to +\infty$ the main term in Eq. (27) is the first one which vanishes for the function $\varphi(\xi) = c/\xi^{a/2}$. Therefore $\varphi(\xi) \to c/\xi^{a/2}$ as $z \to \infty$. If $z = z_0$, $\xi \to \infty$ for all r; there then occurs in space a power-law singularity $\varphi(r) = c/r^{1/2a}$. As $1/2a > 1 + b$, the integral

$$\int_0^L r^b\varphi^2\,dr$$

diverges at the lower limit and the singularity is an integral one. When there is self-focusing in a three-dimensional medium there occurs a singularity $\varphi \sim c/r$ (see [13,14]). In the case $a < 2/(1 + b)$ no singularities occur and the self-focusing is limited to a finite level.

1)It is not possible to use for the description of the wave field near the singularity the self-similar solutions of Eq. (1) found by Talanov. [9] When these solutions describe the singularity, $I_1 = \infty$, $I_2 = -\infty$, and this deprives these solutions of a physical meaning.

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Translated by D. ter Haar
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