Multidimensional method of the inverse scattering problem and duality equations for the Yang-Mills field

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A general method is proposed for solving nonlinear equations with arbitrary number of variables with the aid of the method of the inverse scattering problem. The method is applicable to nonlinear equations that can be obtained as the conditions for the compatibility of systems of linear equations. By way of application of the method, the most general—instanton— duality-equation solutions are obtained for Yang-Mills fields.

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1. So far, the applicability of the method of the inverse scattering problem was restricted only to integration of equations containing two or three independent variables.[1,2] In the present paper we show that there exists a variant of the inverse-scattering-problem method which makes it possible to integrate equations of arbitrary dimensionality. The simplest example of integrable equations of this kind are the duality equations for the Yang-Mills field[3]

$$F_{\mu\nu} = \alpha F^*_{\mu\nu} = \frac{\alpha}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \qquad \alpha = \pm 1, \tag{1}$$

where

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu], \quad \mu,\nu = 1,\ldots 4, \qquad \nabla_\mu = \frac{\partial}{\partial x_\mu} + A_\mu .$$

Equations (1) are considered in four-dimensional Euclidean space. Of physical meaning are their bounded solutions that tend as $r^2 = \Sigma x_i^2 \to \infty$ to free fields

$$A_\mu \to -\frac{\partial g}{\partial x_\mu} g^{-1} \qquad F_{\mu\nu} \to 0 . \tag{2}$$

These solutions constitute classical subbarrier trajectories that arise in the quasiclassical calculation of the quantum amplitudes of the transition between non-equivalent vacuums.[4] Their analysis can play a fundamental role in the problem of non-escape of quarks. The asymptotic state g is characterized by a topological charge

$$n = \frac{1}{8\pi^2}\int \epsilon_{\alpha\beta\gamma}\mathrm{Sp}\, l_\alpha l_\beta l_\gamma d^2x , \qquad l_\alpha = g^{-1}\frac{\partial g}{\partial x_\alpha} .$$

The integration is carried out here over an infinitely remote three-dimensional sphere, n is an integer, and its sign coincides with the sign of α. As shown by Shwartz,[10] the manifold of these solutions is finite-dimensional and has dimensionality $8|n|$. Bounded solutions with $n = \pm 1$ are called instantons,[3] and solutions with $|n|>1$ are called n-instanton solutions. Individual series of n-

instanton solutions (with dimensionality not greater than $5n$) were obtained in[4–7].

2. Nonlinear equations that are integrable by the method of the inverse scattering problem come into play as the condition for the compatibility of two linear equations containing the spectral parameter λ or differentiation $\partial/\partial\xi$ with respect to an additional variable. The case when this dependence is rational has already been considered earlier.[8,9] We shall show that the case of a rational dependence admits a multidimensional generalization.

We consider a pair of compatible differential equations of first order

$$L_1\Psi = 0 \qquad\qquad L_2\Psi = 0$$

$$L_1 = \sum_{n=1}^{N_1} \frac{\nabla_n}{\lambda - \lambda_n} + \nabla_0; \qquad L_2 = \sum_{n=1}^{N_2} \frac{\widetilde{\nabla}_n}{\lambda - \lambda_n} + \widetilde{\nabla}_0 . \tag{3}$$

Here

$$\nabla_n = \frac{\partial}{\partial z_n} + U_n, \quad \widetilde{\nabla}_n = \frac{\partial}{\partial \widetilde{z}_n} + \widetilde{U}_n, \; U_n, \; \widetilde{U}_n ,$$

while U_n and $\widetilde{U}_n$ are matrices of nth order that depend on z_n and $\widetilde{z}_n$. All the variables z_n and $\widetilde{z}_n$ are generally speaking independent. The condition for the compatibility of Eqs. (3) is of the form

$$[L_1, L_2] = 0 \tag{4}$$

and constitute a set of N_1+N_2+1 equations for the N_1+N_2+2 quantities U_n and $\widetilde{U}_n$. The indeterminacy of this system is due to its gauge invariance with respect to the transformation

$$U_n \to g^{-1}\frac{\partial g}{\partial z_n} + g^{-1}U_n g . \tag{5}$$

3. We put $L = \lambda\nabla_1 + \nabla_2^+$ and $L_2 = \lambda\nabla_2^+ - \nabla_1$, where

$$\nabla_1 = \frac{\partial}{\partial z_1} + B_1;$$

$$\nabla_2 = \frac{\partial}{\partial z_2} + B_2; \quad \nabla_1^+ = \frac{\partial}{\partial \bar{z}_1} - B_1^+; \quad \nabla_2^+ = \frac{\partial}{\partial \bar{z}_2} - B_2^+; \quad z_1 = \frac{1}{2}(x + ix_2);$$

$$z_2 = \frac{1}{2}(x_3 + ix_4); \quad B_1 = A_2 - iA_1; \quad B_2 = A_4 - iA_3.$$

Substituting in (4), we obtain the system of equations (1) for $\alpha = -1$ (the anti-duality equation). Analogously, putting $L = \lambda\nabla_1 + \nabla_2$ and $L_2 = \lambda\nabla_2^+ - \nabla_1^+$, we obtain the duality equation ($\alpha = 1$). To integrate Eq. (4), we can use the Riemann-problem method developed in[8,9]. In addition, we can obtain in elementary fashion particular (n-soliton) solutions by stipulating that the matrix function

Ψ be rational in λ. To find the solutions, we note that

$$U_o + \Sigma \frac{U_n}{\lambda - \lambda_n} = - L_o^{(1)} \Psi\Psi^{-1}; \quad L_o^{(1)} = \sum_{n=1}^{N} \frac{\frac{\partial}{\partial z_n}}{\lambda - \lambda_n} + \frac{\partial}{\partial z_o}.$$

A similar equation follows from $L_2\Psi = 0$. We stipulate that the expression $L_0^{(1)}\Psi\Psi^{-1}$ has poles only at the points $\lambda = \lambda_n$, while $L_0^{(2)}\Psi\Psi^{-1}$ has poles at the points $\lambda = \tilde{\lambda}_n$. These requirements impose a finite number of conditions on the residues of the matrix function at its poles, and obviously yield an exact solution of the system (4). This method can be used to obtain n-soliton solutions in all equations that can be integrated by the method of the inverse scattering problem.

4. In the course of the construction of the n-instanton solutions it turned out that the condition that the solution be bounded for all x_i imposes very stringent requirements on the form of the dependence of λ. It turns that an n-instanton solution corresponds to a function Ψ that has poles of nth order at $\lambda = 0$ and $\lambda = \infty$. We stipulate that the function satisfy the condition

$$\Psi^+\left(-\frac{1}{\lambda}, \bar{z}_i, z_i\right) = \Psi^{-1}(\lambda, z_i, \bar{z}_i) \tag{6}$$

in which case we have for $n = -1$

$$\Psi = \sqrt{1 + |A|^2} + \lambda f A + \frac{1}{\lambda} f A^+, \quad \text{where } A^2 = 0 \tag{7}$$

and $AA + AA = 1$. From the condition that the expressions

$$\left(\lambda \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right)\Psi\Psi^{-1}$$

have no poles at $\lambda = 0$ and have no poles of higher order at $\lambda = \infty$ we obtain

$$A = \frac{1}{|a|^2 + |b|^2}\begin{bmatrix} ab & a^2 \\ -b^2 & -ab \end{bmatrix},$$

where $a(\bar{z}_1, \bar{z}_2)$ and $b(\bar{z}_1, \bar{z}_2)$ are arbitrary analytic functions of $\bar{z}_1$ and $\bar{z}_2$. For f we have

$$\frac{\sqrt{1 + |f|^2}}{f} = \frac{z_1\left(a\frac{\partial b}{\partial \bar{z}_2} - b\frac{\partial a}{\partial \bar{z}_2}\right) - z_2\left(a\frac{\partial b}{\partial \bar{z}_1} - b\frac{\partial a}{\partial \bar{z}_1}\right) + c(\bar{z}_1, \bar{z}_2)}{|a|^2 + |b|^2}, \tag{8}$$

where $c(\bar{z}_1, \bar{z}_2)$ is also an arbitrary function of $\bar{z}_1$ and $\bar{z}_2$. The condition that the algebraic equation (8) has a solution makes it possible to lower the leeway from three functions of two variables to eight constants. In the general case we have

$$a = a_{11}(\bar{z}_1 - a_o) + a_{12}(\bar{z}_2 - b_o),$$

$$b = a_{21}(\bar{z}_1 - a_o) + a_{22}(\bar{z}_2 - b_o),$$

$$c = \kappa^2 - \bar{a}_o(\bar{z}_1 - a_o) - \bar{b}_o(\bar{z}_2 - b_o).$$

The constant κ corresponds to the dimension of the instanton, while the constants a_0 and b_0 correspond to its position and the unitary matrix a_{ij} corresponds to rotation of the instanton relative to the coordinate system.

To calculate the n-instanton solutions we have used a recurrence procedure, which makes it possible to determine a Ψ_{N+1}-instanton matrix from an N-instanton matrix Ψ_N by putting $\Psi_{N+1} = \Psi\Psi_N$, where Ψ is a matrix of the type (7), which is determined as before accurate to three arbitrary functions of complex variables. Explicit calculations for $N=1$ and 2 have shown that in this case, too, when account is taken of the bounded character of the solution, the matrix Ψ depends on eight independent constants. With allowance for the already mentioned Shwartz theorem, this gives grounds for hoping that we have obtained the complete n-instanton solutions.

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