Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method

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A method is proposed for deriving and classifying relativistically invariant integrable systems that are sufficiently general to encompass all presently known two-dimensional solvable models, and for the construction of a few new ones. The concept of "gauge equivalence" introduced in this paper allows one to clarify the relation between several different models of classical field theory, such as the n-field, the sine-Gordon equation, and the Thirring model. We study the model of the principal chiral field for the group $SU(N)$. It is shown that at $N=3$ this model exhibits nontrivial interactions: decay, fusion and resonant scattering of solitons. New chiral models are proposed with fields taking values in homogeneous spaces of Lie groups and exhibiting a high degree of symmetry. We prove the integrability of these models when the homogeneous space is a Grassmann manifold.

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INTRODUCTION

Over the ten years of existence of the inverse scattering problem method[1] the exact integrability of about thirty different nonlinear partial differential or difference systems was established, systems which have applications in theoretical physics (cf. the review[2]). Among these systems there are several which are relativistically invariant. Foremost among these are the sine-Gordon (SG) equation,[3] the massive Thirring model,[4] and the system of Pohlmeyer-Getmanov-Regge-Lund.[5-6] Several attempts have been made at considering relativistically invariant systems from a more general point of view. We have in mind the papers of Pohlmeyer,[5] Budagov and Takhtadzhyan[7] (cf. also [4]). Pohlmeyer has discovered infinite series of conservation laws (and in fact L–A pairs) for an important class of relativistically invariant models of classical field theory: the **n**-fields on spheres of arbitrary dimension. At the present time the quantum theory of these **n**-fields as well as of the SG equation[8] are well advanced. In Ref. 7 Budagov and Takhtadzhyan have demonstrated the integrability of a special system of equations (the U-V system) which generalizes the sine-Gordon equation to the group $SO(N)$.

There is no need to prove how important a matter of principle it is to find completely integrable models of relativistically invariant field theories even in two-dimensional space-time. Therefore it seems quite appropriate to pose the question of constructing a regular procedure for finding relativistically invariant systems and for their classification.

In the present paper we propose such a procedure. We show that relativistically invariant models represent a quite natural class of integrable systems and that all cases studied to date do not exhaust by far even the simplest variant of integrable relativistically invariant systems in its general formulation. More precisely, we show that each relativistically invariant system is characterized by a definite rational $N\times N$ matrix function of a complex parameter λ. All presently known models belong to the case when this function has a single simple pole.

After formulating the general system (§ 1) we investigate only the simplest case of the classification (corresponding to one pole), which, as we show, encompasses a large number of integrable systems. These systems admit a natural geometric interpretation and exhibit a high degree of symmetry. In the general formulation these are the principal chiral fields (free fields on Lie groups). The models of principal chiral fields on $SU(N)$ groups are distinguished in principle from all previously discussed relativistically invariant models by the fact that they exhibit nontrivial interactions: decay, fusion, and resonance scattering of solitons (cf.[9]).

An important special case of principal chiral fields are the simpler objects represented by chiral fields on homogeneous spaces of Lie groups. In the present paper we study only one example of such fields: chiral fields on Grassmann manifolds. Among these fields are included also the usual **n**-fields on spheres. We also show that chiral fields are naturally related to the U-V systems, and in a certain sense they represent the same object. In the whole paper we stress three facts: a procedure for calculating integrable systems, a general scheme for their integration, and the simplest method of calculating soliton solutions.

§ 1. RELATIVISTICALLY INVARIANT INTEGRABLE SYSTEMS IN TWO-DIMENSIONAL SPACE-TIME

At the present time there exists no method allowing one to determine algorithmically whether a given system of equations is integrable or not. However, there exist several methods of constructing in a regular way manifestly integrable systems. In this section we describe one of these methods, general enough to encompass all presently known two-dimensional relativistic-

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ally invariant models of classical field theory, to construct a multitude of new models, and to clarify the relation between some models which are in fact related but at a first glance are different. The method described here is based on a paper by Shabat and one of the present authors.[10]

Let $2\xi=t-x$, $2\eta=t+x$ denote the "light-cone" variables in two-dimensional space-time, and let Ψ, U, V be some complex-valued $N\times N$ matrix functions of ξ and η. We assume that the overdetermined system of partial differential equations for the matrix Ψ:

$$i\Psi_\xi=U\Psi,\quad i\Psi_\eta=V\Psi \tag{1.1}$$

has a compatible fundamental matrix of solutions, denoted also by Ψ. Then (1.1) implies the relation

$$U_\eta-V_\xi-i[U,V]=0. \tag{1.2}$$

As long as one does not impose additional restrictions on U and V the equation (1.2) is trivial: its general solution has the form

$$U=i\Psi_\xi\Psi^{-1},\quad V=i\Psi_\eta\Psi^{-1},$$

where Ψ is an arbitrary matrix function of ξ and η. We now impose on U and V the additional condition that U and V should be rational functions of the complex parameter λ. Then the relation (1.2) becomes a nontrivial nonlinear system of equations.

Assume that the matrix U has N_1 poles at the points $\lambda_1, \lambda_2, \ldots, \lambda_{N_1}$, that the matrix V has N_2 poles at the points $\mu_1, \mu_2, \ldots, \mu_{N_2}$, and that all these poles are simple and do not depend on ξ and η. By means of an appropriate linear-fractional transformation of λ one can realize that neither of the poles is at the point $\lambda=\infty$. Expanding U and V into simple fractions; we have:

$$U=U_0+\sum_{n=1}^{N_1}\frac{U_n}{\lambda-\lambda_n},\quad V=V_0+\sum_{n=1}^{N_2}\frac{V_n}{\lambda-\mu_n}. \tag{1.3}$$

These functions are characterized by the set of N_1+N_2+2 matrix function parameters U_k and V_k. We substitute (1.3) in (1.2). The left-hand side of (1.2) is a rational function with a total number of poles (counting their multiplicities if some of the numbers λ_i and μ_i coincide) N_1+N_2. To satisfy (1.2) it suffices to equate to zero all the residues as well as the value of the left-hand side at the point $\lambda=\infty$. The result is a system of N_1+N_2+1 equations for the functions U_k and V_k.

The system is underdetermined (the number of equations is one less than the number of unknowns). This indeterminacy is due to the "gauge invariance" of the system (1.2). Let U, V be a solution of the system (1.2) and Ψ the appropriate solution of the system (1.1). We choose f, an arbitrary singular matrix function of ξ and η and consider the matrices

$$\tilde U=fUf^{-1}+if_\xi f^{-1},\quad \tilde V=fVf^{-1}+if_\eta f^{-1}. \tag{1.4}$$

It is easy to check that the system (1.1) is satisfied for the simultaneous substitution $U\to\tilde U$, $V\to\tilde V$, $\Psi\to f\Psi$, and consequently $\tilde U$, $\tilde V$ is a new solution of the system (1.2).

The system (1.2) becomes determined if one imposes an additional condition that fixes the choice of the matrix f. To different choices of f correspond different systems of equations; we shall call such systems gauge-equivalent. It is obvious that the solutions of two gauge-equivalent systems differ only by a transformation (1.4) with some matrix f.

Assume, for instance, that the matrix U has a single pole at the point $\lambda=-1$, and that the matrix V has a single pole at the point $\lambda=1$. Then

$$U=U_0+\frac{U_1}{\lambda+1},\quad V=V_0+\frac{V_1}{\lambda-1}.$$

The system of equations (1.2) has the form

$$U_{0\eta}-V_{0\xi}-i[U_0,V_0]=0, \tag{1.5}$$

$$U_{1\eta}-i[U_1,V_0-{}^1/_2V_1]=0, \tag{1.6}$$

$$V_{1\xi}+i[U_0+{}^1/_2U_1,V_1]=0. \tag{1.7}$$

We perform the transformation (1.4) assuming that after the transformation $\tilde U_0=\tilde V_0=0$. This can be done by determining f from the conditions

$$U_0=-if^{-1}f_\xi,\quad V_0=-if^{-1}f_\eta. \tag{1.8}$$

(On account of (1.5), the conditions (1.8) are compatible.) We are led to the system of equations

$$A_\eta={}^1/_2i[A,B],\quad B_\xi=-{}^1/_2i[A,B]. \tag{1.9}$$

Here $A=\tilde U_1$, $B=-\tilde V_1$. The equations (1.9) are the compatibility conditions for the system of equations

$$i\Psi_\xi=\frac{A}{\lambda+1}\Psi,\quad i\Psi_\eta=-\frac{B}{\lambda-1}\Psi. \tag{1.10}$$

A study of the system (1.9) is the main content of the present paper. From (1.4) it follows that $\tilde U_1=fU_1f^{-1}$. We choose f in such a manner that $\tilde U_1=A_0$ becomes diagonal. Then it follows directly from (1.6) that $A_{0\eta}=0$, $A_0=A_0(\xi)$, and the equation (1.6) can be solved. We set

$$U_0=-A_0/2+C,\quad \Gamma=V_1=2(V_0-S),\quad (\lambda-1)/(\lambda+1)=z,$$

where S is some diagonal matrix. Now the system (1.1) takes the form

$$\begin{aligned}&i\Psi_\xi+C\Psi+{}^1/_2zA_0\Psi=0,\\&i\Psi_\eta+S\Psi+{}^1/_2z^{-1}\Gamma\Psi=0,\end{aligned} \tag{1.11}$$

and the equation (1.5) becomes

$$iC_\eta-iS_\xi={}^1/_4[A_0,\Gamma]-[S,C].$$

Setting $C=C_\perp+C_\parallel$, where $C_\parallel$ is a diagonal matrix, we note that

$$C_{\parallel\eta}-S_\xi=0,$$

or

$$C_\parallel=R_\xi,\quad S=R_\eta.$$

Finally, the system (1.5) – (1.7) takes the form

$$iC_{\perp\eta}=\tfrac{1}{4}[A_0,\Gamma]-[R_\eta,C_\perp],\quad i\Gamma_\xi=-[C_\perp+R_\xi,\Gamma]. \tag{1.12}$$

The system of equations (1.12) (the U-V system) was considered in Refs. 3 and 7. In a particular case the system (1.11) reduces to the SG equation. It is obvious that the systems (1.9) and (1.12) are gauge-equivalent.

We now discuss the problem of relativistic invariance of the systems under consideration. The system (1.2) becomes manifestly invariant with respect to the special Lorentz group if one assumes that U, V transform under Lorentz transformations like the components of a vector (cf.[4]). Indeed, for a transformation with parameter γ we have $\xi\to\gamma\xi$, $\eta\to\gamma^{-1}\eta$ and for invariance it suffices to set $U\to\gamma^{-1}U, V\to\gamma V$. Under reflections of the coordinate x we have the interchange $\xi\leftrightarrow\eta$, and therefore for invariance under the full Lorentz group it suffices to require the existence of a discrete involution (including a linear-fractional transformation of the λ plane) which realizes the interchange $U\to V, V\to U$.

The system (1.5) – (1.7) considered by us is invariant with respect to the full Lorentz group, since the involution $U_0\leftrightarrow V_0$, $U_1\leftrightarrow V_1$, $\lambda\to-\lambda$ realizes the transformation $U\leftrightarrow V$. A more general example of a system invariant under the full Lorentz group is given by the choice $\mu_n=-\lambda_n$ in (1.3):

$$iV_{n\xi}=[\Phi_n,V_n],\quad iU_{n\eta}=[\Omega_n,U_n],$$
$$\Phi_n=\sum_k\frac{U_k}{\lambda_n+\lambda_k}-U_0,\quad \Omega_n=\sum_k\frac{V_k}{\lambda_n+\lambda_k}+V_0, \tag{1.13}$$

with $\lambda_i+\lambda_j\neq 0$ for arbitrary i,j.

It presents no difficulty to write the system (1.2) also in the case when the poles are multiple and zeroes of the denominators of Φ_n and Ω_n are admitted.

We also remark that the matrix Ψ in (1.1) can be considered as an element of a certain group of matrices G. Then U, V turn out to be elements of the corresponding Lie algebra. It is obvious that the equation (1.2) does not take us out of this algebra.

§ 2. PRINCIPAL CHIRAL FIELDS

The equations (1.9) are related to one of the classical field-theory models having a geometric interpretation: the principal chiral field. Assume that at each point of the (ξ,η) plane an element $g(\xi,\eta)$ of a Lie group G is specified, and that the group G is realized as a subgroup of the group of nonsingular complex matrices. We consider the Lagrangian density

$$L=\frac{1}{2}\,\mathrm{Sp}\left(\frac{\partial}{\partial\xi}g\frac{\partial}{\partial\eta}g^{-1}\right) \tag{2.1}$$

and the corresponding action $S=\int d\xi d\eta L$. The expression (2.1) is remarkable because it is a bi-invariant form on G. Indeed, carrying out a left translation in G by means of the constant matrix h: $g\to hg$, $h\in G$, and a cyclic permutation in (2.1), we verify the invariance of L. L is also obviously invariant with respect to right translations $g\to gh$. The Euler equation of the Lagrangian (2.1) is

$$g_{\xi\eta}=\tfrac{1}{2}(g_\xi g^{-1}g_\eta+g_\eta g^{-1}g_\xi). \tag{2.2}$$

The equations (2.2) are called the equations of the principal chiral field on the group G. They can be brought to another form. We introduce the fields A and B with values in the Lie algebra of the group G:

$$A=ig_\xi g^{-1},\quad B=ig_\eta g^{-1}. \tag{2.3}$$

They obviously satisfy the relation

$$A_\eta-B_\xi-i[A,B]=0. \tag{2.4}$$

From (2.2) follows also the easily verified relation

$$A_\eta+B_\xi=0. \tag{2.5}$$

The system of equations (2.4), (2.5) is obviously equivalent to the system (1.9). The quantities A and B can be interpreted as the currents engendered by the symmetry of the Lagrangian with respect to translations on the group, and the equation (2.5) can be viewed as the conservation law for these currents.

In terms of the variables (2.3) the Lagrangian (2.1) has the form

$$L=\tfrac{1}{2}\mathrm{Sp}\,AB, \tag{2.6}$$

and Eq. (2.5) is the corresponding Euler equation taking into account the constraint (2.4).

Explicitly solving the equation (2.5) we obtain

$$A=\Phi_\xi,\quad B=-\Phi_\eta,$$

and then (2.4) takes the form

$$\Phi_{\xi\eta}+\tfrac{1}{2}i[\Phi_\xi,\Phi_\eta]=0. \tag{2.7}$$

It can be obtained from the Lagrangian

$$L=\tfrac{1}{2}\mathrm{Sp}\,\Phi_\xi\Phi_\eta+\tfrac{1}{3}i\,\mathrm{Sp}(\Phi[\Phi_\xi,\Phi_\eta]). \tag{2.8}$$

As was shown by Regge and Lund[6] in terms of rank-two Hermitian matrices, with the additional conditions

$$\mathrm{Sp}\,A=\mathrm{Sp}\,B=0,\quad \mathrm{Sp}\,A^2=\mathrm{Sp}\,B^2=1$$

the equation (2.7) describes in four-dimensional space-time the motion of a relativistic string placed in a massless scalar field. In general (1.9) implies

$$\frac{\partial}{\partial\eta}\mathrm{Sp}\,A^n=0,\quad \frac{\partial}{\partial\xi}\mathrm{Sp}\,B^n=0 \tag{2.9}$$

for arbitrary power n. It also follows from (1.9) that the Jordan normal form A_0 of the matrix A does not depend on η and the normal form B_0 of B does not depend on ξ.

Thus, $A_0=A_0(\xi)$ and $B_0=B_0(\eta)$. Representing A and B in the form

$$A=f_1A_0f_1^{-1},\quad B=f_2B_0f_2^{-1},$$

we see that the matrices f_1, f_2 are determined accurate to multiplication from the right by arbitrary matrices f_1^0, f_2^0 which commute with A_0, B_0, respectively.

Let $G=SU(N)$. Then the matrices A and B are Hermitian with $\mathrm{Tr}A=\mathrm{Tr}B=0$. The matrices A_0 and B_0 are diagonal and the matrices f_1 and f_2 can be chosen as elements of the group $SU(N)$. If the eigenvalues of the matrices A and B are different, the matrices f_1, f_2 are determined accurate to multiplication by an arbitrary diagonal unitary matrix. Thus, one may consider the matrices f_1 and f_2 as defined on the manifold M which arises when the group $SU(N)$ is factored with respect to the subgroup H of diagonal matrices, i.e., on the "flag manifold" $SU(N)/H$. For fixed A_0, B_0 the matrices A, B are also defined on this manifold. If among the eigenvalues of A_0 and B_0 there are coinciding ones, the corresponding manifold M is a "degenerate flag manifold" of lower dimension.

Going over in the system (2.7) from the light-cone variables ξ, η to the physical variables $x_0=t=\eta+\xi$, $x_1=x=\eta-\xi$ we calculate the energy-momentum tensor:

$$T^{\mu\nu}=\mathrm{Sp}\left(\Phi^\mu\frac{\partial L}{\partial\Phi_\nu}\right)-g^{\mu\nu}L,$$

with components

$$T^{00}=T^{11}=\tfrac{1}{2}\mathrm{Sp}(\Phi_t^2+\Phi_x^2)=\tfrac{1}{2}\mathrm{Sp}(\Phi_\eta^2+\Phi_\xi^2),$$
$$T^{01}=T^{10}=\mathrm{Sp}\Phi_t\Phi_x=\mathrm{Sp}(\Phi_\eta^2-\Phi_\xi^2). \qquad (2.10)$$

This yields the following expression for the Hamiltonian of the system (2.8)

$$H=\int T^{00}\,dx=\tfrac{1}{2}\int \mathrm{Sp}(\Phi_t^2+\Phi_x^2)\,dx=\tfrac{1}{2}\int \mathrm{Sp}(A_0^2(\xi)+B_0^2(\eta))\,dx. \qquad (2.11)$$

It can be seen from (2.11) that the Hamiltonian of the system (2.7) is a quantity independent of the form of A and B. When the constant part is subtracted the Hamiltonian vanishes. This is a characteristic of dynamical systems with constraints (cf., e.g., [11]) and can be explained by the fact that the relation (2.9) can be treated as a constraint imposed upon the system (2.7). A similar result, vanishing of the Hamiltonian, can be derived directly for the system (2.2) and for the system (2.4), (2.5).

In order to avoid the difficulties due to a vanishing Hamiltonian (e.g., the difficulty of correctly defining the energy), one must solve the constraints (2.9) at fixed matrices A_0 and B_0 by introducing coordinates on the manifold M. This leads to a new difficulty: we must show that the equations obtained in this way are Lagrangian (i.e., follow from some principle of least action). Following Pohlmeyer we show how this difficulty can be avoided for $G=SU(2)$. For simplicity we set

$$\tfrac{1}{2}\mathrm{Sp}\,A^2=\tfrac{1}{2}\,\mathrm{Sp}\,B^2=1 \qquad (2.12)$$

and expand the matrices A, B in terms of the Pauli matrices

$$A=\boldsymbol{\sigma}\mathbf{A},\quad B=\boldsymbol{\sigma}\mathbf{B}.$$

The vectors $\mathbf{A}, \mathbf{B}$ have unit length.

We introduce $\cos\alpha=\mathbf{A}\cdot\mathbf{B}$ and calculate the derivatives $\mathbf{A}_\xi$ and $\mathbf{B}_\eta$:

$$\mathbf{A}_\xi=[\mathbf{A}\times a_1\mathbf{B}+a_2[\mathbf{A}\times\mathbf{B}]],$$
$$\mathbf{B}_\eta=[\mathbf{B}\times b_1\mathbf{A}+b_2[\mathbf{A}\times\mathbf{B}]], \qquad (2.13)$$

where a_1, a_2, b_1, b_2 are numerical functions. Calculating $\partial_\xi\mathbf{A}\cdot\mathbf{B}$, $\partial_\eta\mathbf{A}\cdot\mathbf{B}$ we obtain

$$a_2=\frac{\alpha_\xi}{\sin\alpha},\quad b_2=\frac{\alpha_\eta}{\sin\alpha}. \qquad (2.14)$$

We also introduce

$$a_1=u/\sin\alpha,\quad b_1=v/\sin\alpha,$$
$$u=[\mathbf{A}_\xi\times\mathbf{B}_\xi]/\sin\alpha,\quad v=[\mathbf{A}_\eta\times\mathbf{B}_\eta]/\sin\alpha. \qquad (2.15)$$

Differentiating α, u, v, and using (2.13)–(2.15) we are led to the following system of equations

$$\alpha_{\xi\eta}+\sin\alpha+uv/\sin\alpha=0, \qquad (2.16)$$
$$u_\eta=\alpha_\xi v/\sin\alpha, \qquad (2.17)$$
$$v_\xi=\alpha_\eta u/\sin\alpha. \qquad (2.18)$$

It follows from (2.17), (2.18) that

$$\partial_\eta(u\,\mathrm{ctg}(\alpha/2))+\partial_\xi(v\,\mathrm{ctg}(\alpha/2))=0,$$

so that

$$\beta_\xi=u\,\mathrm{ctg}(\alpha/2),\quad \beta_\eta=-v\,\mathrm{ctg}(\alpha/2).$$

Finally, the quantities α and β satisfy the equations

$$\beta_{\xi\eta}+\frac{1}{\sin\alpha}(\alpha_\eta\beta_\xi+\alpha_\xi\beta_\eta)=0, \qquad (2.19)$$
$$\alpha_{\xi\eta}+\sin\alpha-\frac{\sin(\alpha/2)}{2\cos^3(\alpha/2)}\beta_\xi\beta_\eta=0. \qquad (2.20)$$

The equations (2.19), (2.20) are the Euler equations for the Lagrangian

$$L=\tfrac{1}{2}\alpha_\xi\alpha_\eta+\mathrm{tg}^2(\alpha/2)\beta_\xi\beta_\eta+1-\cos\alpha. \qquad (2.21)$$

The components of the appropriate energy-momentum tesnor have the form

$$T^{00}=\tfrac{1}{2}\partial^\mu\alpha\partial^\mu\alpha+\tfrac{1}{2}\,\mathrm{tg}^2(\alpha/2)\partial^\mu\beta\,\partial^\mu\beta+\cos\alpha-1, \qquad (2.22)$$
$$T^{01}=\partial^0\alpha\,\partial^1\alpha+\mathrm{tg}^2(\alpha/2)\partial^0\beta\,\partial^1\beta. \qquad (2.23)$$

The energy-momentum vector for the fields α and β can be expressed in terms of the original fields A and B:

$$E=\int dx\left[\frac{(AB)_\eta^2+(AB)_\xi^2+(A_\eta B_\eta)^2+(A_\xi B_\xi)^2}{4(1-(AB)^2)}+1-(AB)\right], \qquad (2.24)$$
$$P=\int dx\frac{(AB)_\eta^2-(AB)_\xi^2+(A_\eta B_\eta)^2-(A_\xi B_\xi)^2}{4(1-(AB)^2)}. \qquad (2.25)$$

In the case under consideration here

$$A_0=B_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

More generally

$$A_0=a(\xi)\sigma_3,\quad B_0=b(\eta)\sigma_3.$$

However, setting

$$A(\eta,\xi)=a(\xi)\tilde{A}(\tilde{\eta},\tilde{\xi}),\quad B(\eta,\xi)=b(\eta)\tilde{B}(\tilde{\eta},\tilde{\xi}),$$
$$\tilde{\eta}=\int b(\eta)\,d\eta,\quad \tilde{\xi}=\int a(\xi)\,d\xi, \qquad (2.26)$$

we return in the variables $\tilde{A}, \tilde{B}, \tilde{\xi}, \tilde{\eta}$ to the case considered above. We note that the substitution (2.26) is bijective only if the functions $a(\xi)$ and $b(\xi)$ have no zeroes.

We note that the proposed procedure of solving the constraints extends to the case of the group $SU(N)$ and is essentially a transition from the original system (1.9) to the system (1.12). Indeed, in the derivation of the system (1.12) a partial integration has already been performed: the introduction of the arbitrary diagonal matrix R. We reduce the system (1.12) to a single equation of second order. For this we note that the matrix Γ can be represented in the form

$$\Gamma=\varphi B_0\varphi^+, \quad \varphi=f_1f_2^+.$$

The matrix φ is unitary and belongs to the "flag manifold" $SU(N)/H$. The equations (1.12) imply

$$[i\varphi^+\varphi_\xi+\varphi^+C_\perp\varphi, B_0]=0,$$

or

$$R_\xi+C_\perp=-i\varphi_\xi\varphi^++\varphi W\varphi^+,$$

where W is a diagonal matrix commuting with B_0 and has elements

$$W_{ik}=\delta_{ik}W_i.$$

The quantities W_i can be determined from the vanishing of the diagonal elements of the matrix $C_\perp$, i.e., from the system of equations

$$\sum_s W_s|\varphi_{\alpha s}|^2=i(\varphi_\xi\varphi^+)_{\alpha\alpha}+\partial_\xi R_\alpha. \tag{2.27}$$

Now the function φ is subject to the equations

$$-i\partial_\eta(\varphi_\xi\varphi^+)+\partial_\eta(\varphi W\varphi^+)+{}^1/_4[\varphi B_0\varphi^+, A_0]+[R_\eta, i\varphi_\xi\varphi^++\varphi W\varphi^+]=0. \tag{2.28}$$

Setting [for the group $SU(2)$]

$$\varphi=\begin{bmatrix} \cos(\alpha/2) & \sin(\alpha/2)e^{-i\omega} \\ -\sin(\alpha/2)e^{i\omega} & \cos(\alpha/2) \end{bmatrix}$$

$$R_\eta=R_\xi=0, \tag{2.29}$$

$$\omega_\eta=\frac{\beta_\eta}{2\cos^2(\alpha/2)}, \quad \omega_\xi=\frac{\beta_\xi\cos\alpha}{2\cos^2(\alpha/2)}$$

we obtain for α and β the system (2.19), (2.20).

We note that if φ is a real matrix, then $(\varphi_\xi\varphi^+)_{\alpha\alpha}=0$ and by choosing $R=0$ one may set $W=0$. The corresponding system

$$\partial_\eta(\varphi_\xi\varphi')+{}^1/_4[\varphi B_0\varphi', A_0] \tag{2.30}$$

for $B_0=A_0$ was considered before,[7] where for the group $SU(3)$ it was possible to bring it to a Lagrangian form.

We remark that although the system (2.30) is defined on the group of orthogonal matrices it by no means coincides with the principal chiral field of the orthogonal group. Indeed, the real flag manifold of the group $SU(N)$ consists of the whole group of orthogonal matrices, whereas the flag manifold of the orthogonal group is of lower dimension. In particular, equation (2.30) on the group $SO(2)$ is the sine-Gordon equation, whereas the principal chiral field on $SO(2)$, in view of its commutativity, is described by a linear equation.

§ 3. CHIRAL FIELDS AND THE REDUCTIONS PROBLEM

In spite of the fact that we have proposed a quite general method of constructing two-dimensional relativistically invariant systems which are integrable, this is insufficient for physical applications. The reason is that the systems constructed by us represent equations for a very large number of interacting fields. The general system (1.13) in the case of n poles on $SU(N)$ contains $nN(N-1)/2$ complex fields and only for the first few values of this number may one hope to find any usefulness for physics. Therefore it is of great importance to solve the problem of reducing the number of interacting fields, what we call the reduction problem. The reduction can be achieved by imposing on the general system additional constraints. Assume we have found such (algebraic or differential) constraints on the fields making up our system, constraints which are compatible with the field equations. Then we can effectively reduce the number of equations and go over to a new system containing a smaller number of fields.

Let us illustrate this with an example. We consider the principal chiral field g on the group G [in the sequal we shall assume $G=SU(N)$ or $G+SO(N)$] and assume that it satisfies the additional condition

$$g^2=1. \tag{3.1}$$

We show that this condition is compatible with the equations of motion of the principal chiral field. Under the condition (3.1) these equations have the form

$$g_{\xi\eta}={}^1/_2(g_\xi g g_\eta)+{}^1/_2(g_\eta g g_\xi). \tag{3.2}$$

We differentiate (3.1) with respect to ξ and η, obtaining

$$g_{\xi\eta}g+gg_{\eta\xi}+g_\xi g_\eta+g_\eta g_\xi=0. \tag{3.3}$$

We substitute (3.2) in (3.3) and make use of the relations $g_\xi g=-gg_\xi$, $g_\eta g=-gg_\eta$ which follow from (3.1). We then verify that (3.3) is satisfied identically, which proves the compatibility of Eq. (3.2) with the condition (3.1).

It is remarkable that the condition (3.1) has a geometrical meaning. Represent g in the form

$$g=1-2P, \tag{3.4}$$

then (3.1) implies that $P^2=P$, i.e., P is a projection operator acting on the space of the fundamental representation of the group G. The condition (3.1) determines in the group G, considered as a Riemannian manifold, a submanifold $\tilde{G}$. The manifold $\tilde{G}$ is not connected. Each projection operator P_k is characterized by the dimension k of its image (the space onto which it projects). It is obvious that throughout the evolution *the number* k cannot change, therefore one may assume in (3.4)

$$g=1-2P_k.$$

Let the dimension of the space of the fundamental representation of the group G be N. The element g of the group G is represented in the form $g = -1 + 2\bar{P}$, where $\bar{P} = 1 - P$ is the projector complementary to P. Therefore, without loss of generality, we may assume that $k \leq [N/2]$ (the largest integer contained in $N/2$—Transl. note).

Thus the equation (3.2) constitutes a set of equations

$$[\partial_\xi \partial_\eta P_k, P_k] = 0, \quad k = 1, 2, \dots, [N/2]. \tag{3.5}$$

The corresponding espression for the action has the form

$$S_k = {}^1/_2 \int \mathrm{Sp}(P_{k\eta} P_{k\xi})\, d\eta\, d\xi. \tag{3.6}$$

Each of the equations (3.5) represents a chiral field defined on the space of projection operators of given dimension k. In the real case this space is called the real Grassmann manifold $\Gamma^R_{N,k}$, in the complex case it is called the complex Grassmann manifold $\Gamma^C_{N,k}$. Grassmann manifolds are homogeneous spaces of the groups $SO(N)$ or $SU(N)$, in the real and complex cases, respectively, and are the result of factorization of these groups with respect to lower-dimensional subgroups:

$$\Gamma^R_{N,k} = \frac{SO(N)}{SO(k)SO(N-k)}, \quad \Gamma^C_{N,k} = \frac{SU(N)}{SU(k)SU(N-k)}. \tag{3.7}$$

Grassmann manifolds are collections of hyperplanes of dimension k in real or complex N-dimensional space; the groups $SO(N)$ or $SU(N)$ act on these planes by rotation.

In the special case $k = 1$ Grassmann manifolds are just the projective spaces (real or complex) of dimension N^{-1} and are denoted by RP^{N-1} or CP^{N-1}, respectively. In this case the projector P_1 can be represented in the form

$$(P_1)_{ik} = n_i \bar{n}_k, \qquad \sum_{i=1}^{N} \bar{n}_i n_i = (\mathbf{n}, \mathbf{n}) = 1. \tag{3.8}$$

where $(\mathbf{n}, \mathbf{n})$ denotes the inner product in N-space. Substituting (3.8) into the action (3.6) we obtain

$$S_1 = \int [(\mathbf{n}_\xi \mathbf{n}_\eta) + (\mathbf{n}_\eta \mathbf{n}_\xi) + 2(\mathbf{n}\mathbf{n}_\xi)(\mathbf{n}\mathbf{n}_\eta)]\, d\xi\, d\eta. \tag{3.9}$$

In the real case the corresponding equation of motion is

$$\mathbf{n}_{\xi\eta} + (\mathbf{n}_\xi \mathbf{n}_\eta)\mathbf{n} = 0, \tag{3.10}$$

and in the complex case

$$\begin{aligned} &\mathbf{n}_{\xi\eta} + {}^1/_2((\mathbf{n}_\xi \mathbf{n}_\eta) + (\mathbf{n}_\eta \mathbf{n}_\xi) + (\mathbf{n}_{\xi\eta}\mathbf{n}) - (\mathbf{n}\mathbf{n}_{\xi\eta}))\mathbf{n} \\ &- (\mathbf{n}\mathbf{n}_\xi)\mathbf{n}_\eta - (\mathbf{n}\mathbf{n}_\eta)\mathbf{n}_\xi + 2(\mathbf{n}\mathbf{n}_\eta)(\mathbf{n}\mathbf{n}_\xi)\mathbf{n} = 0. \end{aligned} \tag{3.11}$$

(The commas have been omitted from the inner products—Transl. note.)

In (3.8), for the real case, the vector $\mathbf{n}$ is determined only up to sign, i.e., a chiral field in real projective space corresponds only to those solutions of Eq. (3.10) for which the vector $\mathbf{n}$ moves on one hemisphere S^{N-1}. The equation (3.10) remains valid also if this restriction is removed. In this case it is the equation of the $\mathbf{n}$-field on the sphere S^{N-1}. The local agreement between the chiral fields on RP^{N-1} and S^{N-1} is explained by the fact that projective space RP^{N-1} is the result of factoring the sphere S^{N-1} with respect to the discrete group Z_2 (i.e., identification of antipodal points on S^{N-1} determines the same manifold RP^{N-1} as the lines through the origin of R^N; Transl. note).

The real and complex spheres are also homogeneous spaces of the groups $SO(N)$ and $SU(N)$, respectively and are the results of factorizations of these groups with respect to the subgroups $SO(N-1)$ and $SU(N-1)$ respectively. These spheres are special cases of more general homogeneous subspaces: the Stiefel manifolds $\mathfrak{S}^R_{N,k}$ and $\mathfrak{S}^C_{N,k}$ of orthonormal frames of dimension k in real or complex N-space:

$$\mathfrak{S}^R_{N,k} = \frac{SO(N)}{SO(N-k)}, \quad \mathfrak{S}^C_{N,k} = \frac{SU(N)}{SU(N-k)}.$$

Chiral fields occur naturally on Stiefel manifolds and can be interpreted as systems of k orthogonal $\mathbf{n}$-fields. The action and equations of motion of these fields have the form

$$S = {}^1/_2 \int \sum_{\alpha=1}^{k} ((\mathbf{n}_\eta{}^\alpha \mathbf{n}_\xi{}^\alpha) + (\mathbf{n}_\xi{}^\alpha \mathbf{n}_\eta{}^\alpha))\, d\eta\, d\xi, \qquad (\mathbf{n}^\alpha \mathbf{n}^\beta) = \delta^{\alpha\beta},$$

$$\mathbf{n}_{\xi\eta}{}^\beta + \sum_{\alpha=1}^{k} ((\mathbf{n}_\eta{}^\alpha \mathbf{n}_\xi{}^\beta) + (\mathbf{n}_\xi{}^\alpha \mathbf{n}_\eta{}^\beta))\mathbf{n}^\alpha = 0.$$

We note that in the complex case even in the case $k = 1$ (the complex $\mathbf{n}$-field) we obtain an equation differing from (3.11). This is explained by the fact that CP^N differs from the complex N-sphere by a factorization with respect to the continuous group $U(1)$.

Chiral fields both on Grassmann and on Stiefel manifolds are natural models for field theories which have a geometric meaning. We have shown above that the equations for chiral fields on Grassmann manifolds are a reduction of principal chiral fields. Below we will use the same fact to show the integrability of fields on Grassmann manifolds. For chiral fields on Stiefel manifolds (except for the case of the real sphere which reduces to the Grassmann case) the fact of reduction and integrability has not yet been proved. Nevertheless one may conjecture that reduction and integrability exist in this case. This hypothesis can be stated in a more general form: it seems likely that chiral fields on all homogeneous spaces of the groups $SO(N)$ or $SU(N)$ are integrable and that all the corresponding equations can be obtained as a result of the reduction of equations of principal chiral fields. It also seems likely that this exhausts all the reductions of equations of principal chiral fields.

§ 4. THE METHOD OF THE INVERSE SCATTERING PROBLEM

We now describe a procedure for integrating the system (1.2). It is based on a representation of this system as compatibility conditions for the linear equations (1.1) and is essentially a procedure of proliferation of solutions. Let us assume that we know a particular solution U_0, V_0 of the system (1.2) and the corresponding solution Ψ_0 of the system (1.1). We show that with the help of U_0, V_0 one can construct new solutions of the system (1.2). We note that the solution U_0, V_0 can always be found explicitly, setting in (1.7), for instance,

$$U_0=\Phi_{0\xi}, \quad V_0=\Phi_{0\eta}, \qquad \Psi_0=\exp(-i\Phi_0), \tag{4.1}$$

where Φ_0 is a diagonal matrix. We also show that our procedure of proliferation in simple cases is equivalent to the usual method of the inverse scattering problem and allows one to find all the solutions of the system (1.2) with given asymptotic behavior at infinity. For the construction of the integration procedure we will need some facts about the Riemann problem for matrices.

Assume that in the complex plane of the variable λ there is given a contour Γ and on it an $N\times N$ matrix-function $G(\lambda)$ without singularities, but which in general does not admit an analytic continuation off the contour. We are required to find two matrix functions $\Psi_1(\lambda)$, analytic outside the contour Γ, and $\Psi_2(\lambda)$, analytic inside the contour, such that on the contour

$$\Psi_1(\lambda)\Psi_2(\lambda)=G(\lambda). \tag{4.2}$$

If the problem is posed this way the solution of the Riemann problem is manifestly nonunique, since it allows the transformation $\Psi_1\to\Psi_1 g$, $\Psi_2\to g^{-1}\Psi_2$ with arbitrary nonsingular matrix g. In order to remove the nonuniqueness we normalize the Riemann problem by fixing the value of one of the functions Ψ_1 or Ψ_2 at an arbitrary point. We call the normalization canonical if $\Psi_2(\infty)=1$. We call the Riemann problem regular if det $(\Psi_{1,2})\neq 0$ in the domains of analyticity of the functions Ψ_1 and Ψ_2. If a regular Riemann problem has a solution then this solution is unique. If the determinants of the functions Ψ_1 and Ψ_2 have finite numbers of zeroes in their domains of analyticity we say that we have a Riemann problem with zeroes.

We show how to multiply the solutions of the system (1.2) by means of the solution of the Riemann problem. We first consider the case of a regular Riemann problem. Consider an arbitrarily given contour Γ and on it a function $G_0(\lambda)$. We form the new function

$$G(\lambda,\eta,\xi)=\Psi_0(\lambda,\eta,\xi)G_0(\lambda)\Psi_0^{-1}(\lambda,\eta,\xi); \tag{4.3}$$

where Ψ_0 is a known particular solution of the system (1.1), e.g., the solution (4.1).

We differentiate the relation (4.3) with respect to ξ. From (1.1), (4.2), and (4.3) it follows that

$$i(\Psi_{1\xi}\Psi_2+\Psi_1\Psi_{2\xi})=U^0\Psi_1\Psi_2-\Psi_1\Psi_2U^0. \tag{4.4}$$

We now define the function U as

$$U=\Psi_1^{-1}(-i\Psi_{1\xi}+U^0\Psi_1)=(i\Psi_{2\xi}+\Psi_2U^0)\Psi_2^{-1}. \tag{4.5}$$

Eq. (4.5) shows that the function U can be analytically continued from the contour Γ onto the whole complex plane of λ. Since in their domains of analyticity the functions Ψ_1 and Ψ_2 are nonsingular (the Riemann problem is regular), the singularities of the function U coincide with the singularities of U^0. Thus, U is a rational function with poles at the points $\lambda=\lambda_n$.

Similarly one can define the function V:

$$V=\Psi_1^{-1}(-i\Psi_{1\eta}+V^0\Psi_1)=(i\Psi_{2\eta}+\Psi_2V^0)\Psi_2^{-1}. \tag{4.6}$$

The poles of the function V coincide with these of V^0. The function Ψ_1 is subject to the equations

$$i\Psi_{1\xi}=U^0\Psi_1-\Psi_1U, \quad i\Psi_{1\eta}=V^0\Psi_1-\Psi_1V. \tag{4.7}$$

Setting $\Psi_1=\Psi_0\chi^{-1}$ we obtain

$$i\chi_\xi=U\chi, \quad i\chi_\eta=V\chi. \tag{4.8}$$

Thus we have obtained a new solution of the system (1.1); consequently U and V satisfy the system (1.2).

Assuming that U^0 and V^0 are given by the expansions (1.3), we obtain the following expressions for the components of the expansions of the matrices $U, V (n=1,2,3,\ldots)$:

$$U_n=q_n^{-1}U_n^0q_n, \quad q_n=\Psi_1|_{\lambda=\lambda_n}, \qquad V_n=p_n^{-1}V_n^0p_n, \quad p_n=\Psi_1|_{\lambda=\mu_n}, \tag{4.9}$$

$$U_0=-iq_0^{-1}q_{0\xi}+q_0^{-1}U_0^0q_0, \quad q_0=\Psi_2^{-1}(\infty), \qquad V_0=-iq_0^{-1}q_{0\eta}+q_0^{-1}V_0^0q_0. \tag{4.10}$$

The equations (4.9) show that a change in normalization of the Riemann problem (a transition from Ψ_1 to $\Psi_1 g$, where g is an arbitrary matrix function of ξ and η) leads to a gauge transformation of the system (1.2) with the matrix g. To different gauges of the system (1.2) correspond different normalizations of the Riemann problem. Thus, the gauge leading to the problem of the principal chiral field (1.9) ($U_0=V_0=0$) corresponds to the canonical normalization $q_0=\Psi_2(\infty)=1$. The gauge leading to the U-V system (1.12) corresponds to the normalization

$$q_1=\Psi_1|_{\lambda=1}=1.$$

We now go over to the consideration of our concrete system (1.9). As the contour Γ we pick the real axis $-\infty<\lambda<\infty$ and set $G(1)=1$. We show that the above-mentioned procedure of proliferation of solutions is equivalent to the traditional inverse scattering problem method.

The solution of the Riemann problem reduces to solving a singular integral equation. We set

$$\Psi_1=1+\frac{1}{\pi i}\int_\Gamma\frac{\rho(\xi,\eta,\lambda')}{\lambda-\lambda'+i0}d\lambda', \qquad \Psi_2^{-1}=1+\frac{1}{\pi i}\int_\Gamma\frac{\rho(\xi,\eta,\lambda')}{\lambda-\lambda'-i0}d\lambda'. \tag{4.11}$$

Substituting into (4.4) we see that the quantity ρ satisfies the equation

$$\rho(\xi,\eta,\lambda)+T(\xi,\eta,\lambda)\left[1+\frac{1}{\pi i}\fint\frac{\rho(\xi,\eta,\lambda')}{\lambda-\lambda'}d\lambda'\right]=0, \tag{4.12}$$

(here $\fint$ denotes the principal value of the integral), where the matrix T is a Rayleigh transform of the matrix G:

$$T=(1-G)(1+G)^{-1}=\Psi_0T_0\Psi_0^{-1}, \qquad T_0=(1-G_0)(1+G_0)^{-1}.$$

As can be seen from the equations (4.9), the quantities A and B do not change if, when the matrix Ψ_1 is multiplied from the left and the matrix Ψ_2 is multiplied from the right by arbitrary matrices, the matrix G_0 which commutes with A^0 and B^0 is multiplied from the left and the right by the same matrices, respectively. Thus, there is an indeterminacy in the definition of the matrix T. In order to lift it we require that the diagonal elements of the matrix T vanish. The remainder of the discussion will be given for the special case when A_0 and B_0 are Hermitian numerical (diagonal) matrices.

In this case

$$A_0=\mathrm{diag}(a_1, a_2, \ldots, a_N), \quad B_0=\mathrm{diag}(b_1, b_2, \ldots, b_N).$$

We order the real numbers a_i, putting

$$a_{i+1}>a_i.$$

As $\xi\to\pm\infty$ the matrix T oscillates rapidly:

$$T_{ik}(\xi, \eta, \lambda)=T_{0ik}(\lambda)\exp[-i(a_i-a_k)\xi/(\lambda+1)+i(b_i-b_k)\eta/(\lambda-1)],$$

therefore the asymptotic values of the matrix ρ as $\xi\to\infty$ are also rapidly oscillating:

$$\rho\to\rho^\pm(\xi,\eta,\lambda),$$
$$\rho_{ik}{}^{\pm}=\rho_{0ik}^{\pm}(\lambda)\exp[-i(a_i-a_k)\xi/(\lambda+1)+i(b_i-b_k)\eta/(\lambda-1)]. \tag{4.13}$$

The matrices $\rho_0^\pm$ can be found by explicitly solving the equation (4.12). For this one must make use of the formula

$$\lim_{|\xi|\to\infty} P\frac{e^{i\lambda\xi}}{\lambda}=i\pi\,\mathrm{sign}\,\xi\,\delta(\lambda), \tag{4.14}$$

which transforms (4.11) into an easily integrable system of singular integral equations.

We denote by $\Psi^\pm$ the limiting values of the function Ψ_1 as $\xi\to\pm\infty$. The matrices $\Psi^\pm$ are defined by the equations (4.11) where $\rho^\pm$ from (4.13) are substituted for ρ. It follows from (4.11) that $\Psi^\pm$ are triangular matrices: for the matrix Ψ^+ all elements below the diagonal vanish, and for the matrix Ψ^- all elements above the diagonal vanish. It is obvious that

$$\Psi^\pm=\exp[-i(A_0\xi+B_0\eta)]S^\pm(\lambda)\exp[i(A_0\xi+B_0\eta)],$$

where $\rho^\pm(\lambda)$ are triangular matrices. The matrix $S=\Psi^{+-1}\Psi^-$ is the usual scattering matrix. Representing it in the form of a product of triangular matrices one can determine $\Psi^\pm$, and reconstruction of T and G in terms of these shows the equivalence of our approach to the usual inverse scattering problem method (cf., e.g.,[12]).

From the condition $G(\pm1)=1$ it follows that $T(1)=T(-1)=0$, therefore $\Psi_1(\lambda=\pm1)\to1$ as $\xi\to\pm\infty$ and consequently $A\to A_0$, $B\to B_0$. Thus, the matrices A_{00} and B_0 are indeed the asymptotic values of the matrices A and B.

We also note that the procedure described above allows one to obtain solutions not only for the system (1.9), but also directly for the chiral field g. Indeed,

$$g=X|_{\lambda=0}=\Psi_1{}^{-1}\Psi_0|_{\lambda=0}. \tag{4.15}$$

Therefore the determination of the function Ψ automatically solves the problem of calculating g.

§ 5. SOLITON SOLUTIONS

The procedure of proliferation of solutions by means of the regular Riemann problem does not make it possible to determine all the solutions of the system (1.2). For a complete description even of this class of solutions, which asymptotically, as $x\to\pm\infty$ go over into A_0, B_0, it is necessary to resort to a Riemann problem with zeroes. Of particular interest is a special Riemann problem with zeroes, for which $G\equiv1$. The solutions of the system (1.2) obtained by means of this Riemann problem will be called soliton solutions. In the case $G=1$ the function $\Psi_2=\Psi_1^{-1}$, where the functions Ψ_1, Ψ_2 are rational. We shall say that Ψ_1 has a simple zero at the point $\lambda=\lambda_0$ if Ψ_2 has a simple pole at that point. Let this zero be the only one, and assume that the function Ψ_2 also has a single zero at the point $\lambda=\mu_0$.

We assume the normalization of the Riemann problem to be canonical; then

$$\Psi_1=1+(\lambda-\mu_0)^{-1}R_1, \quad \Psi_2=1+(\lambda-\lambda_0)^{-1}R_2. \tag{5.1}$$

From the condition $\Psi_1\Psi_2=1$ it follows that

$$R_1=-(\lambda_0-\mu_0)P, \quad R_2=(\lambda_0-\mu_0)P,$$

where $P^2=P$, i.e., P is a projection operator. Finally

$$\Psi_1=1-\frac{\lambda_0-\mu_0}{\lambda-\mu_0}P, \quad \Psi_2=1+\frac{\lambda_0-\mu_0}{\lambda-\lambda_0}P. \tag{5.2}$$

It follows from (4.5) that the function U has additional poles at the points $\lambda=\lambda_0$, $\lambda=\mu_0$. In order to obtain the solution of the system (1.2) we must require that these poles be absent. Setting the residues at the poles $\lambda=\lambda_0$ and $\lambda=\mu_0$ equal to zero we obtain an equation for the projector P:

$$P\left(-i\frac{\partial}{\partial\xi}+U^0|_{\lambda=\lambda_0}\right)(1-P)=0,$$
$$P\left(-i\frac{\partial}{\partial\eta}+V^0|_{\lambda=\lambda_0}\right)(1-P)=0, \tag{5.3}$$

$$(1-P)\left(-i\frac{\partial}{\partial\xi}+U^0|_{\lambda=\mu_0}\right)P=0,$$
$$(1-P)\left(-i\frac{\partial}{\partial\eta}+V^0|_{\lambda=\mu_0}\right)P=0. \tag{5.4}$$

The projection operator P is completely characterized by the two subspaces, its image $\hat{M}=\mathrm{Im}\,P$ and its kernel $\hat{N}=\mathrm{Ker}\,P$, defined by the conditions

$$(1-P)\hat{M}=0, \quad P\hat{N}=0.$$

It is easy to see that the conditions (5.3) and (5.4) are satisfied if one sets

$$\hat{M}=\Psi_0|_{\lambda=\mu_0}\hat{M}_0, \quad \hat{N}=\Psi_0|_{\lambda=\lambda_0}\hat{N}_0.$$

Here $\hat{M}_0$ and $\hat{N}_0$ are fixed constant subspaces having zero intersection and the direct sum of which spans the whole N-dimensional complex space.

For the case of the principal chiral field on $SU(N)$ it follows from (1.10) that the function Ψ satisfies the linear involution

$$\Psi^{+}(\bar{\lambda})=\Psi^{-1}(\lambda). \tag{5.5}$$

From the involution (5.5) it follows that the zeroes of the functions Ψ_1 and Ψ_2 are situated symmetrically with respect to the real axis, $\mu_0=\bar{\lambda}_0$ and in addition that the projector P is Hermitian $P^{+}=P$. Therefore, the subspaces $\hat{N}$ and $\hat{M}$ are orthogonal to one another and it suffices to consider only one of them.

For an explicit construction of the projection operator it suffices to prescribe, e.g., the subspace $\hat{M}_0$ by constructing a base $e_{0j}{}^{i}(i=1,2,\ldots,k$, where k is the dimension of $\hat{M}$; $j=1,2,\ldots,N$ is the vector index). The components of the vectors e_0^i do not depend on ξ and η.

The base in the space M is constructed according to:

$$e^{i}=\Psi_0|_{\lambda=\mu_0}e_0{}^{i}. \tag{5.6}$$

This base is generally not orthonormal. Applying to the vectors e^i the Gram-Schmidt orthogonalization process we are led to the orthonormalized basis n^i. In terms of this basis the projector P has the expression

$$P_{\alpha\beta}=\sum_{i=1}^{k} n_{\alpha}{}^{i}\bar{n}_{\beta}{}^{i}. \tag{5.7}$$

Further, in terms of the formulas (5.2) one reconstructs the functions Ψ_1,Ψ_2, then by means of equations (4.9), (4.10) one reconstructs U and V, and g is obtained by means of (4.13). If the dimension k of the subspace $\hat{M}$ exceeds $N/2$ it is convenient to replace the projector P by the projector $\bar{P}=1-P$. For this it suffices to replace $\hat{M}$ by $\hat{N}$ and the matrix $\Psi_0(\lambda=\mu_0)$ by $\Psi_0(\lambda=\lambda_0)$.

In the general case of a Riemann problem with zeroes, in addition to the function G, one must prescribe the positions of the zeroes of the function $\Psi_1(\lambda_1,\lambda_2,\ldots,\lambda_n)$, the positions of the zeroes of the function $\Psi_2(\mu_1,\mu_2,\ldots,\mu_n)$, and also the two collections of subspaces

$$\hat{M}_i=\mathrm{Im}\,\Psi_1|_{\lambda=\lambda_i},\quad \hat{N}_i=\mathrm{Ker}\,\Psi_2|_{\lambda=\mu_i}.$$

This problem can be reduced to the regular Riemann problem by means of successive annihilation of the zeroes. We represent the solution in the form

$$\Psi_1=\Psi_1^{(1)}\tilde{\Psi}_1,\quad \Psi_2=\tilde{\Psi}_2\Psi_2^{(1)},$$

where $\Psi_1^{(1)},\Psi_2^{(1)}$ is a solution of the type (5.1) with the projector P_1 constructed according to the subspaces $\hat{M}_1,\hat{N}_1$, at the points $\lambda=\lambda_1$, $\lambda=\mu_1$. It is obvious that the functions Ψ_1,Ψ_2 no longer have zeroes at the points $\lambda=\lambda_1,\lambda=\mu_1$, respectively. At the points λ_i^{μ} $(i=2,\ldots,n)$

$$\hat{M}_i=\mathrm{Im}\,\tilde{\Psi}_1=\Psi_2^{(1)}|_{\lambda=\lambda_i}\hat{M}_i,\quad \hat{N}_i=\mathrm{Ker}\,\tilde{\Psi}_2=\Psi_1^{(1)}|_{\lambda=\mu_i}\hat{N}_i,$$

$$\tilde{G}=\Psi_2^{(1)}G\Psi_1^{(1)}.$$

For $G=1$ we obtain $\tilde{G}=1$, therefore the soliton problem with an arbitrary number of zeroes can be solved purely algebraically.

Going over to the case under consideration of a principal chiral field, we choose the matrix Ψ_0 in the form

$$\Psi_0(\lambda)=\exp\left\{\frac{i}{\lambda+1}\int^{\xi}A_0(\xi')d\xi'-\frac{i}{\lambda-1}\int^{\eta}B_0(\eta')d\eta'\right\}. \tag{5.8}$$

In terms of g the one-soliton solution has the form

$$g=\left(1+\frac{\lambda_1-\bar{\lambda}_1}{\bar{\lambda}_1}P_1\right)\Psi_0(0). \tag{5.9}$$

We limit ourselves to a consideration of the groups $SU(2)$ and $SU(3)$. In these cases we can confine ourselves to projectors with one-dimensional range. Such a projector is characterized by a single vector c_j:

$$c_j=\exp\left\{\frac{i}{\lambda_1+1}\int^{\xi}A_{0j}(\xi')d\xi'-\frac{i}{\lambda_1-1}\int^{\eta}B_{0j}(\eta')d\eta'\right\}c_j^{(1)} \tag{5.10}$$

($c_j^{(1)}$ is an arbitrary constant complex vector), and has the form

$$P_{\alpha\beta}=c_\alpha\bar{c}_\beta\Big/\sum_{\gamma}|c_\gamma|^2. \tag{5.11}$$

It is obvious that the vector c is defined only accurate to multiplication by a complex number.

In the case of the group $SU(2)$ the soliton solution depends on two complex parameters ($\lambda_1=\lambda_1'+i\lambda_1''$, $c_2^{(1)}/c_1^{(1)}$) and represents a soliton proper, i.e., a solitary wave which has constant velocity at constant $A_0(\xi)$ and $B_0(\eta)$

$$v=\frac{(b_1-b_2)|1+\lambda_1|^2-(a_1-a_2)|1-\lambda_1|^2}{(b_1-b_2)|1+\lambda_1|^2+(a_1-a_2)|1-\lambda_1|^2}$$

One can calculate the energy and momentum of the soliton from (2.24) and (2.25) and obtain an expression for its mass

$$M^2=4\lambda_0''^2(a_1-a_2)(b_1-b_2)/|1-\lambda_0^2|^2. \tag{5.12}$$

For the soliton the matrix P_{ij} has the form

$$P=\begin{bmatrix} e^{y}/2\,\mathrm{ch}\,y & e^{-is}/2\,\mathrm{ch}\,y \\ e^{is}/2\,\mathrm{ch}\,y & e^{-y}/2\,\mathrm{ch}\,y \end{bmatrix}$$

where

$$y=\lambda_1''\alpha_{12}{}^{0}(x-vt-x_0),$$

$$s=(\lambda_1'\alpha_{12}{}^{1}+\alpha_{12}{}^{0})t-(\lambda_1'\alpha_{12}{}^{0}+\alpha_{12}{}^{1})x+\arg(c_1^{(1)}/c_2^{(1)}),$$

$$x_0=-\ln|c_1^{(1)}/c_2^{(1)}|/\lambda_1''\alpha_{12}^{0},$$

$$\alpha_{12}{}^{0}=\frac{a_1-a_2}{|\lambda_1+1|^2}+\frac{b_1-b_2}{|\lambda_1-1|^2}\qquad \alpha_{12}{}^{1}=\frac{a_1-a_2}{|\lambda_1+1|^2}-\frac{b_1-h}{|\lambda_1-1|^2}$$

For $(a_1-a_2)(b_1-b_2)>0$ the soliton is an ordinary particle, and for $(a_1-a_2)(b_1-b_2)<0$ it is a tachyon.

In the case of the group $SU(3)$ the soliton solution is characterized in addition to the number λ_1 by the integer vector ($c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$). If one of the components of this vector vanishes we obtain the simple soliton solution; in this case there are evidently three types of soliton. Two of these solitons (for $c_1^{(1)}=0$ and $c_2^{(1)}=0$) will be called simple solitons and the third ($c_2^{(1)}=0$) will be called composite. It follows from the mass formula (5.12) that for the case of normal solitons the mass of the composite soliton is larger than the sum of the masses of the simple ones:

$$(a_1-a_3)(b_1-b_3)>(a_1-a_2)(b_1-b_2)+(a_2-a_3)(b_2-b_3)$$

for $a_1>a_2>a_3$ and $b_1>b_2>b_3$.

Thus, decay of the composite soliton into simple solitons is possible. In the case of the general formulation (all three $c_i^{(1)}\neq 0$) the soliton solution describes indeed such a decay. From an analysis of the general solution (5.10), (5.11) one can easily deduce that as $t\to-\infty$

$$P_{12}\to 0,\quad P_{23}\to 0,$$
$$|P_{13}|\to 1/2\,\mathrm{ch}\,(\lambda_1''\alpha_{13}{}^0x-\lambda_1''\alpha_{13}{}^1t+\ln|c_1/c_3|).$$

As $t\to+\infty$

$$P_{13}\to 0,\quad |P_{12}|\to 1/2\,\mathrm{ch}\,(\lambda_1''\alpha_{12}{}^0x-\lambda_1''\alpha_{12}{}^1t+\ln|c_1/c_2|),$$
$$|P_{23}|\to 1/2\,\mathrm{ch}\,(\lambda_1''\alpha_{23}{}^0x-\lambda_1''\alpha_{23}{}^1t+\ln|c_2/c_3|).$$

Thus, the numbers $c_2/c_1, c_3/c_1$ characterize the final coordinates of the decay products. The equations (2.2) are time-reversible, therefore there exists the inverse process of fusion of simple solitons into a composite one. The processes are contained in the soliton solution in which the image (range) of the complementary projector is a one-dimensional subspace.

The solutions of the Riemann problem with $G=1$ with two zeroes of the functions Ψ_1, Ψ_2 describe collisions of soliton solutions, in particular, collisions of simple solitons as well as processes of "induced" decay of composite solitons in collisions with simple ones. The corresponding formulas can be easily derived but we do not list them here since they are rather clumsy.

An analysis of soliton solutions for the groups $SU(N)$ is not trivial. Already for $N=4$ three exist composite solitons consisting of several (two, three) simple ones, and the processes of nontrivial interaction of solitons are quite varied. The corresponding analysis will be published elsewhere.

§ 6. INTEGRATION OF CHIRAL FIELDS ON GRASSMANN MANIFOLDS

We now consider the procedure of integration of fields which appear as a result of reduction. We restrict ourselves to the reduction $g^2=1$, considered in Sec. 3 for the principal chiral field on the group $SU(N)$, and leading to chiral fields on complex Grassmann manifolds. In order to carry out the integration it is necessary to clarify how the reduction reflects the data of the Riemann problem, i.e., what restrictions does it impose on the matrices $A_0, B_0, G(\lambda)$, on the position of the zeroes of the functions Ψ_1 and Ψ_2, and on the structure of the subspaces $\hat{N}_i$ and $\hat{M}_i$.

We first note that Eqs. (2.3) and the condition $g^2=1$ imply the relations

$$Ag+gA=0,\quad Bg+gB=0, \tag{6.1}$$

and also, taking into account $g=1-2P$, the relations

$$A=-2i[PP_\xi],\quad B=-2i[PP_\eta]. \tag{6.2}$$

From the relations (1.1) and the fundamental formulas (1.10) it follows that one can select a fundamental matrix of solutions Ψ, satisfying the conditions

$$\Psi(\lambda^{-1})=g\Psi(\lambda),\quad g=\Psi(0). \tag{6.3}$$

In other words, the zeroes of the function Ψ_1 are disposed symmetrically with respect to inversion in the unit circle and can be divided into simple zeroes for $|\lambda|=1$ and double zeroes. In the theory of the SG equation this fact has been known for a long time.[3] Accordingly there appear simple and double soliton solutions.

Since the equation

$$\Psi_0(\lambda^{-1})=g_0\Psi_0(\lambda),\quad g_0=\Psi(0), \tag{6.4}$$

is also valid for the original prescribed solution of the system (1.1), the functions Ψ_1 and Ψ_2 satisfy the involution

$$\Psi_1(\lambda^{-1})=g\Psi_1(\lambda)g_0,\quad \Psi_2(\lambda^{-1})=g_0\Psi_2(\lambda)g, \tag{6.5}$$

which implies

$$G(\lambda^{-1})=g_0G(\lambda)g_0,\quad G_0(\lambda^{-1})=G_0(\lambda), \tag{6.6}$$

it follows that Eq. (6.6) represents a condition on the "dressing" matrix $G_0(\lambda)$.

From Eqs. (6.2) follow the relations

$$\begin{aligned}&PA+AP=A,\quad P_0A_0+A_0P_0=A_0,\\&PB+BP=B,\quad P_0B_0+B_0P_0=B_0.\end{aligned} \tag{6.7}$$

Here P_0 is the "bare" projector, $g_0=1-2P_0$, and from (6.5) follow the formulas for "dressing" the bare projector:

$$P=f_1P_0f_1^{-1},\quad f_1=\Psi_1(1). \tag{6.8}$$

Therefore for the complete solution it is necessary to establish the form of P_0 and A_0.

Remembering that A_0 and B_0 are diagonal, it follows from (6.7) that

$$P_{0ik}(a_i+a_k)=\delta_{ik}a_k,\quad P_{0ik}(b_i+b_k)=\delta_{ik}b_k. \tag{6.9}$$

We order the squares of the numbers a_i in decreasing order. The diagonal of the matrix A^2 splits into segments with equal squares of the eigenvalues. In each of these segments we let the negative eigenvalues follow the positive ones. It follows from (6.9) that the matrix P_0 has block-diagonal form, and in each block the matrix P_0 reduces to a matrix P_0^k of the form

$$P_0{}^k=\frac{1}{2}\begin{bmatrix}1 & R_k\\ R_k{}^+ & 1\end{bmatrix},\quad R_kR_k{}^+=R_k{}^+R_k=1, \tag{6.10}$$

where R_k is a unitary matrix.

From the equations (6.7) it is easy to establish that the matrix B_0, which commutes with A_0 must also decompose, for the above ordering of the numbers a_0, into diagonal matrices in accordance with the structure of the matrix A_0. Finally, the k-th block of the matrices A_0 and B_0 is characterized by the functions $a_k(\xi)$ and $b_k(\eta)$. Substituting (6.10) into (6.2) we obtain the expressions for the matrices R_k:

$$R_k=R_k^0 \exp\left(i\int^{\xi} a_k d\xi'+i\int^{\eta} b_k d\eta'\right). \tag{6.11}$$

Here R_k^0 is an arbitrary constant unitary matrix. The general dimension of the image of the projector P_0 is determined by the total rank of the matrices P_0^k. The equations (6.10), (6.11) solve the problem of the structure of the asymptotic states of chiral fields on Grassmann manifolds.

The remainder of the construction of the solutions for these fields does not differ from that described in Sec. 4 and 5. We also remark that in the simplest case of a chiral field on a complex projective space, the matrices A_0 and B_0 can contain at most one nontrivial block, so that the asymptotic states are here characterized by one pair of functions, a and b. From the results of Sec. 5 it follows that in this case there is no nontrivial interaction of solitons. The whole theory exposed here can be transposed also to chiral fields on real Grassmann manifolds, and to **n**-fields on real spheres. The corresponding theory will be published elsewhere. We only note that from the preceding it obviously follows that there is no nontrivial interaction of solitons in **n**-fields on spheres for any dimension.

CONCLUSION

In quantizing **n**-fields on spheres an important role is played by the problem of integrals of the motion. In this paper we have not touched this problem at all, but we would like to underline the fact that infinite series of integrals of the motion of our system can be calculated in a regular way. The pair of linear equations (1.11) for the U-V system differs only in the second equation from the pair of equations which generate the integrable model of N-wave interaction, studied earlier.[12] In this model there are also nontrivial interactions of the solitons analogous to those considered in Sec. 5.

In the present paper we have limited our attention to the case when the functions U, V have one simple pole. Important and simple integrable systems are contained in the more complicated systems of type (1.13) which have a larger number of poles. Thus, the massive Thirring model corresponds to the case when the matrices U and V each contain a pole of second order.[4]

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