

ON THE BENNEY EQUATIONS

V.E. Zakharov

L.D. Landau Institute for Theoretical Physics,
USSR Academy of Sciences, Moscow, USSR

I. The system of Benney equations describing long shallow water waves has the form (without an assumption on the potential flow):

$$h_t + \operatorname{div} \int_0^h \vec{u} dz = 0 \quad (1)$$

$$\vec{u}_t + (\vec{u} \nabla) \vec{u} + w \frac{\partial \vec{u}}{\partial z} + \nabla h = 0 \quad (2)$$

$$\frac{\partial w}{\partial z} + \operatorname{div} \vec{u} = 0 \quad (3)$$

Here $h = h(\vec{r}, t)$ ($\vec{r} = (x, y)$ $0 < z < h$) is the fluid surface shape, $\vec{u} = \vec{u}(\vec{r}, z)$ - the horizontal velocity vector, $w = w(\vec{r}, z)$ - the vertical velocity component. Gravity is assumed to be equal to unity. Let us show that the system of (1)-(3) can be reduced to an infinite system of two-dimensional hydrodynamics equations.

Let us divide the fluid volume into layers in the z -direction and enumerate them by means of the index ξ ($0 \leq \xi < 1$), thus giving the location of each layer. Thus we obtain the function:

$$z = z(\vec{r}, \xi, t) \quad h(\vec{r}, t) = z(\vec{r}, 1, t)$$

It is clear that this function must satisfy the condition:

$$\frac{\partial z}{\partial t} + (\vec{u} \nabla) z = w \quad (4)$$

which means that the fluid does not penetrate through the boundary of the layer. Assuming that $\xi = 1$, we see that Eq.(1) follows from Eq.(4). The derivatives taken at the constants z and ξ are interrelated by the formulae:

$$\left(\frac{\partial}{\partial t} \right)_z = \left(\frac{\partial}{\partial t} \right)_\xi - \frac{z_t}{\eta} \frac{\partial}{\partial \xi} \quad (\nabla)_z = (\nabla)_\xi - \frac{\nabla z}{\eta} \frac{\partial}{\partial \xi} \quad (5)$$

Besides we have

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{\eta} \frac{\partial}{\partial \xi} \quad (6)$$

Here

$$\eta(\vec{r}, \xi, t) = \frac{\partial \bar{z}}{\partial \xi} \quad (7)$$

Differentiating the relation (4) with respect to ξ and using formulae (5)-(7), one easily obtains the following equation:

$$\frac{\partial \eta}{\partial t} + \operatorname{div} \vec{u} \eta = 0 \quad (8)$$

Here and elsewhere the derivatives are taken at the constant ξ .

Applying formulae (5)-(6) to Eq.(2) (involving derivatives at constant \bar{z}) and using relation (4) we obtain after simple manipulations:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \nabla) \vec{u} + \nabla h = 0 \quad (9)$$

Besides, it is clear that

$$h(\vec{r}, t) = \int_0^1 \eta(\vec{r}, \xi, t) d\xi$$

So, Eqs.(8)-(9) represent an infinite (parametrized by ξ) system of two-dimensional equations of hydrodynamics of perfect compressible fluid with self-consistent pressure. Such representation of Benney equations solves the problem of their Hamiltonian structure reducing it to the problem of the Hamiltonian structure of hydrodynamics equations.

In fact, if the fluid flow is potential in the (x, y) plane

$$\vec{u} = \nabla \varphi(\vec{r}, \xi, t) \quad (10)$$

the functions $\varphi(\vec{r}, \xi, t)$ and $\eta(\vec{r}, \xi, t)$ are canonical variables

$$\frac{\partial \varphi}{\partial t} = - \frac{\delta H}{\delta \eta} \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \varphi}$$

where

$$H = \frac{1}{2} \int d\vec{x} d\vec{v} h(\vec{v}) (\nabla \Phi(\vec{v}, \vec{x}, t))^2 + \frac{1}{2} \int h^2(\vec{v}, t) d\vec{v} \quad (11)$$

For a more general flow, introducing the Klebsch variables (see, e.g. [2]), one can obtain the Hamiltonian structure.

If the flow parameters depend on the x coordinate only, the Hamiltonian structure has a simple form in variables η, u

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \frac{\delta H}{\delta u} = 0 \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \frac{\delta H}{\delta \eta} = 0 \quad (12)$$

where H is the total fluid energy given by (11).

Another approach to the Hamiltonian structure of Benney equations for one-dimensional flow is described in [3].

2. Let us consider the function

$$f(\vec{v}, \vec{v}, t) = \int_0^1 \eta(\vec{v}, \xi, t) \delta(\vec{v} - \vec{u}(\vec{v}, \xi, t)) d\xi \quad (13)$$

It is clear that

$$h = \int f(\vec{v}, \vec{v}, t) d\vec{v} \quad (14)$$

Straightforward calculations make it clear that the function satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{v}} - \nabla h \frac{\partial f}{\partial \vec{v}} = 0 \quad (15)$$

with the self-consistent field (14).

In the one-dimensional case $\frac{\partial}{\partial y} = 0$ the system of (8), (9) generates all the solutions of the Vlasov equation. For example, to obtain the solution satisfying the condition $f(x, u, t) = f_0(x, u)$ at $t = 0$ the solution of the system (8), (9) with the initial condition

$$\eta|_{t=0} = f_0(x, \xi) \quad u|_{t=0} = \xi$$

is substituted into Eq.(13). This correspondence is a one-to-one correspondence as far as the function $u = u(x, \xi, t)$ is a

single-valued reversible function of ξ . If f is an analytical function of v , the one-dimensional equation (15) is equivalent to the following equation:

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial x} + n \frac{\partial A_0}{\partial x} A_{n-1} = 0 \quad (16)$$

where

$$A_n = \int v^n f(v) dv \quad A_0 = h = \int f(v) dv$$

This equation is known as a moment Benney equation [3].

Thus, the moment Benney equation is equivalent to the original Benney system only if v is a single-valued reversible function of ξ .

Note that in this case the function f has a simple physical meaning - $f(v) dv$ is the thickness of a fluid layer with a horizontal velocity in the interval $(v, v+dv)$.

In the two-dimensional case the Vlasov equation is equivalent to an infinite system of hydrodynamics equations which is parametrized by the vector $\vec{\xi} = (\xi_1, \xi_2)$ under the condition that the mapping $\vec{\xi} \leftrightarrow \vec{u}$ is a one-to-one mapping (the correspondence as above is given by (13) where $\vec{\xi}$ is a vector). In this case only very specific solutions of the Vlasov equation correspond to the solutions of the Benney equation.

3. In what follows only one-dimensional Benney system will be considered.

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \eta u &= 0 \\ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} &= 0 \\ h &= \int_0^1 \eta d\xi \end{aligned} \quad (17)$$

Let us show that the system (17) is naturally related to an infinite system of Schrodinger equations for the function $\Psi = \Psi(x, \xi, t)$

$$i\Psi_t = \frac{1}{2} \Psi_{xx} - h\Psi \quad (18)$$

Here $h = \int_0^1 |\Psi|^2 d\xi$

, Actually, the substitution

$$\Psi = \sqrt{\eta} e^{-i \int_{-\infty}^x u dx} \quad (19)$$

reduces Eq.(18) to the form

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \eta u = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} = \frac{\partial}{\partial x} \frac{1}{\sqrt{\eta}} \frac{\partial^2}{\partial x^2} \sqrt{\eta}$$

In the limiting case $h \gg \frac{\partial^2}{\partial x^2}$ this system is reduced to the system (17). So we see that the Benney equation is a quasiclassical limit of the system of nonlinear Schrodinger equations (18) (for the two-dimensional case it is valid only for the potential flow in the (x, y) -plane $u = \nabla \Phi$).

The main remarkable feature of (18) is that the method of the inverse scattering problem can be applied to it [4]. This is the system that is a compatibility condition for the following system of linear differential equations for the functions $\chi(x, \xi, t)$, $\chi_0(x, t)$

$$\chi_x = i\lambda \chi + \Psi \chi_0$$

$$\chi_{0x} = -i\lambda \chi_0 + R \quad (20)$$

$$\chi_t = i\lambda^2 \chi + \lambda \Psi \chi_0 - \frac{i}{2} \Psi_x \chi_0 + \frac{i}{2} \Psi R$$

$$\chi_{0t} = -i\lambda^2 \chi_0 + \lambda R + \frac{i}{2} S - \frac{i}{2} u \chi_0 \quad (21)$$

$$R = \int_0^1 \bar{\Psi}(x, \xi, t) \chi(x, \xi, t) d\xi; \quad S = \int_0^1 \bar{\Psi}_x(x, \xi, t) \chi(x, \xi, t) d\xi$$

Here λ is a spectral parameter. This enables us to advance in the theory of Benney equation.

Below it would be more convenient to deal with a particular case of solutions of the Benney system when the function Ψ is a step function of ξ , i.e. the flow is divided into layers, in each of them the horizontal velocity is not changed in the vertical direction. In this case

$$\eta = \sum_{k=1}^N \eta_k(x, t) \delta(\xi - \xi_k) \quad h = \sum_{k=1}^N \eta_k \quad (22)$$

and the system (17) is reduced to the following finite system

$$\begin{aligned}\frac{\partial \eta_k}{\partial t} + \frac{\partial}{\partial x} \eta_k u_k &= 0 \\ \frac{\partial u_k}{\partial t} + u_k \frac{\partial u_k}{\partial x} + \frac{\partial h}{\partial x} &= 0 \\ h &= \sum_{k=1}^N h_k\end{aligned}\quad (23)$$

The linear system (20), (21) also becomes finite

$$\begin{aligned}y_{kx} &= i\lambda y_k + \psi_k f_k \\ y_{0x} &= -i\lambda y_0 + R\end{aligned}\quad (24)$$

$$\begin{aligned}y_{kt} &= i\lambda^2 y_k + \lambda \psi_k y_0 - \frac{i}{2} \psi_{kx} y_0 + \frac{i}{2} \psi_k R \\ y_{0t} &= -i\lambda^2 y_0 + \lambda R + \frac{i}{2} S - \frac{i}{2} h y_0\end{aligned}\quad (25)$$

$$R = \sum_{k=1}^N \bar{\psi}_k y_k \quad S = \sum_{k=1}^N \bar{\psi}_{kx} y_k$$

Now the system of the Schrodinger equations is of the form:

$$i \psi_{kt} = \frac{1}{2} \psi_{kxx} - \psi_k \sum_{l=1}^N |\psi_l|^2 \quad (26)$$

The system (26) has an infinite number of motion integrals which must also be the integrals of the system (23). Yet, it is easier to calculate the integrals of the system (23) directly but not from the integrals of the system (26). Then it is necessary to consider quasiclassical limit, to make a substitution (18)

$$\begin{aligned}\psi_n &= \sqrt{\eta_n} e^{-i \int_{-\infty}^x u_n dx} \text{ and to assume} \\ y_0 &= e^{i \int_{-\infty}^x x dx} \quad y_n = \xi_n e^{-i \int_{-\infty}^x (y - u_n) dx}\end{aligned}\quad (27)$$

From (24) we have

$$\begin{aligned}\xi_{nx} + i(y - u_n - \lambda) \xi_n &= \sqrt{\eta_n} \\ R &= y_0 \sum_{n=1}^N \sqrt{\eta_n} \xi_n\end{aligned}\quad (28)$$

$$i(y + \lambda) = \sum_{n=1}^N \sqrt{\eta_n} \xi_n \quad (29)$$

In the quasiclassical limit we must neglect the differential term

ξ_{nx} in (28). Then we have:

$$\xi_n = \frac{i\sqrt{\eta_n}}{\lambda + u_n - x} \quad R = iy_0 \sum_{n=1}^N \frac{\eta_n}{\lambda + u_n - x} \quad (30)$$

$$y + \lambda = \sum_{n=1}^N \frac{\eta_n}{\lambda + u_n - x}$$

With the quasiclassical accuracy we get

$$S = - \sum_{n=1}^N \frac{\eta_n u_n}{\lambda + u_n - x}$$

Therefore it follows from (25) that

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x} \left(-\lambda^2 - \frac{h}{2} + \sum_{n=1}^N \frac{\eta_n (\lambda - \frac{1}{2} u_n)}{\lambda + u_n - x} \right) \quad (31)$$

With the help of (30) we reduce (31) to the form:

$$\frac{\partial y}{\partial t} = - \frac{\partial}{\partial x} \left(\frac{1}{2} (y - \lambda)^2 + h \right) \quad (32)$$

Eqs.(30),(32) represent an overdetermined system of equations one of which is a differential equation and the other one is algebraic. Let us check their compatibility. For this purpose let us find the derivatives y_t, y_x from Eq.(30) and substitute them into (32). The rational function of $y - \lambda$ has to vanish identically and the condition of this coincides with Eqs.(23).

4. Let us calculate the integrals of the system (23). Obviously, the integral dependent on the parameter λ is an integral

$$I(\lambda) = \int_{-\infty}^{\infty} y(\lambda, x, t) dx \quad (33)$$

Eq.(30) gives y as a $(N+1)$ sheet algebraic function of λ . When $\lambda \rightarrow \infty$ this function can be expanded in asymptotic series over $\frac{1}{\lambda^n}$. At every sheet. All the coefficients of these expansions are motion integrals for the system (23).

At one of the sheets we have

$$y = -\lambda + \sum_{n=1}^{\infty} \frac{y_n^{(0)}}{(2\lambda)^n} \quad (34)$$

Here $\lambda \rightarrow \infty$ at $\lambda - y \rightarrow 0$, therefore (using the expansion in powers of $\frac{1}{y - \lambda}$) we can write (30) in the form:

$$\lambda + \lambda + \sum_{n=0}^{\infty} \frac{A_n}{(\lambda - \lambda)^n} = 0 \quad (35)$$

Here $A_n = \sum_{k=1}^N \beta_k u_k^n$ are moments of the Benney system. Hence, it is clear that all integrals of this type are expressed by A_n in a polynomial way. These integrals have been found in [5] by Miura; The convenient form to write them down has been found by Lebedev and Manin in [6.] The first three of these integrals

$$\lambda_1 = A_0 \quad \lambda_2 = A_1 \quad \lambda_3 = A_3 + A_0^2$$

have a physical meaning of conservation laws for mass, momentum and energy. The first non-trivial integral is the following

$$\lambda_4 = A_3 - 2A_0 A_2 + A_1^2$$

In terms of [6] we can rewrite (35) in the form

$$\xi + \varphi(\xi) + 2\lambda = 0$$

where

$$\varphi(\xi) = \sum_{n=0}^{\infty} \frac{A_n}{\xi^{n+1}}$$

Hence the result of [6] follows immediately:

$$I_n = \text{res}(\xi + \varphi(\xi))^n$$

Here I_n is any of the above-mentioned motion integrals.

Let us now show that we are able to calculate new integrals of the Benney system. We shall study the asymptotic behaviour of the function λ at other sheets of the Riemann surface. Let

$\lambda \rightarrow \lambda + u_n$ when $\lambda \rightarrow \infty$. Let us denote

$$\lambda = \lambda + u_n + \tilde{\lambda}$$

For $\tilde{\lambda}$ we obtain

$$\tilde{\lambda} = \frac{1}{\tilde{\lambda} + 2\lambda + u_n} \left(u_n + \tilde{\lambda} \sum_{m \neq n} \frac{u_m}{u_m - u_n - \lambda} \right) \quad (36)$$

Let $\tilde{\lambda}$ be expanded in asymptotical series

$$\tilde{\lambda} \approx \sum_{n=1}^{\infty} \frac{\tilde{\lambda}_n}{(2\lambda)^n}$$

(In the course of this expansion, we must consider $\tilde{\lambda} \ll u_m - u_n$, which accounts for the use of the discrete approximation of the Benney system).

Comparing the expansions we get

$$\begin{aligned}\tilde{\chi}_1 &= \eta_n \\ \tilde{\chi}_2 &= \eta_n \left(\sum_{m \neq n} \frac{\eta_m}{u_m - u_n} - u_n \right) \\ \tilde{\chi}_3 &= \dots\end{aligned}$$

In the continuous limit these quantities correspond to:

$$\begin{aligned}\tilde{\chi}_1 &= \eta(\xi, x, t) \\ \tilde{\chi}_2 &= -\eta(\xi, x, t) \left[u(\xi) - \oint_0^1 \frac{\eta(\xi', x, t)}{u(\xi', x, t) - u(\xi, x, t)} d\xi' \right] \quad (37)\end{aligned}$$

Here an integral is taken in its principle value.

5. The Hamiltonian structure (12) of the one-dimensional Benney equations enables us to determine the Poisson bracket for any functionals α and β of η_n, u_n

$$\{\alpha, \beta\} = \sum_{n=1}^N \int dx \left(\frac{\delta \beta}{\delta u_n} \frac{\partial}{\partial x} \frac{\delta \alpha}{\delta \eta_n} + \frac{\delta \alpha}{\delta u_n} \frac{\partial}{\partial x} \frac{\delta \beta}{\delta \eta_n} \right) \quad (38)$$

Let us calculate the Poisson bracket for $I(\lambda), I(\lambda')$

where

$$I(\lambda) = \int \lambda(x, t, \lambda) dx$$

Simple but long calculations result in the following relation

$$\{I(\lambda), I(\lambda')\} = 0 \quad (39)$$

which means that all the motion integrals of the Benney equations obtained by the asymptotical expansion of $I(\lambda)$ commute. This leads to a hypothesis of total integrability of the Benney system though it cannot be considered as a proof of total integrability since it is necessary to prove that the set of these integrals is complete. The "highest" Benney systems

$$\begin{aligned}\frac{\partial \eta_n}{\partial t} &= \frac{\partial}{\partial x} \frac{\delta H}{\delta u_n} \\ \frac{\partial u_n}{\partial t} &= \frac{\partial}{\partial x} \frac{\delta H}{\delta \eta_n}\end{aligned} \quad (40)$$

have the same motion integrals. Here H is an arbitrary functional of $I(\lambda)$. The simplest of the systems of type (40) when H is one of the integrals of the First (polynomial) series has been studied in [3] by Manin and Couperushmidt.

References

1. Benney D.J. Studies in Appl. Math., L11, 1, 1973, 45-50
2. Zakharov V.E. Izvestia VUZov - Rasiofizika, v. XVII, N°4, 431, 1974
3. Couperschmidt B.A., Manin Yu.I. Functional Analysis and Its Applications. II. 3 (1977), c31-42
12.1 (1978), 25-37
4. Manakov S.V. ZHETF, v. 65, 1392, 1973.
5. Miura R.M. Studies in Appl. Math., v. 53, N°1, 45, 1974.
6. Lebedev D.R., Manin Yu.I. Preprint ITTF, N°60, 1979