# CANONICAL VARIABLES FOR ROSSBY WAVES AND PLASMA DRIFT WAVES

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The Poisson bracket defining a hamiltonian formulation of the Rossby wave equation is transformed to the Gardner bracket via a special functional change. The diagonal form of the bracket enables us to introduce the normal canonical variables in the considered hamiltonian system. The first terms of the hamiltonian expansion in powers of the canonical variables are calculated. The proposed method of the Poisson bracket diagonalization is relevant for other physically significant problems: barotropic waves above an uneven bottom, waves in the presence of a scalar nonlinearity and quasigeostrophic flow of a vertically stratified fluid, including the baroclinic effects of topography as dynamical boundary conditions.

The equation for the stream function $\Psi(x, y)$ of barotropic Rossby waves,

$$\frac{\partial(\triangle\Psi - k_0^2\Psi)}{\partial t} + \beta\frac{\partial\Psi}{\partial x} = J(\triangle\Psi, \Psi), \tag{1}$$

where $\triangle$ is the Laplace operator, $J$ is the jacobian, $\beta$, $k_0$ are constants, can be written in the hamiltonian form [1,2]

$$\frac{\partial\Omega}{\partial t} = \{\Omega, H\}, \tag{2}$$

where $\Omega = \triangle\Psi - k_0^2\Psi$ is the vortex density, the hamiltonian

$$H = \tfrac{1}{2}\int (|\nabla\Psi|^2 + k_0^2\Psi^2)\,dx\,dy \tag{3}$$

represents a kinetic energy and the Poisson bracket for the functionals $F = F(\Omega)$, $G = G(\Omega)$ is defined by

$$\{F, G\} = \int \Omega_{\text{tot}}J\left(\frac{\delta F}{\delta\Omega}, \frac{\delta G}{\delta\Omega}\right)dx\,dy, \tag{4}$$

where $\Omega_{\text{tot}} = \Omega + \beta y$ is the total vortex of the Rossby waves.

Eq. (1) is also used to describe drift waves in a plasma [3]. In order to apply the methods of perturbation theory [1] to study the hamiltonian system it is expedient to introduce into the system normal canonical variables, having the sense of wave amplitudes ($k$ is a wave vector) where eq. (2) is equivalent to

$$\frac{\partial a_k}{\partial t} = -i\frac{\delta H}{\delta a_k^*}. \tag{5}$$

From the mathematical point of view the introduction of the canonical variables is reduced to the problem of diagonalization of the Poisson bracket. For the finite-dimensional case this problem has a relatively complete solution [4,5]. At the same time there do not exist any general recipes of diagonalization of the Poisson bracket

of the continuum hamiltonian systems; in each special case of solution of this problem some artificial tricks [6,7] are used.

We introduce the Gardner bracket known in the theory of integrable systems,

$$\{F, G\}_1 + \beta \int \frac{\delta F}{\delta \Omega} \frac{\partial}{\partial x} \frac{\delta G}{\delta \Omega} \, dx \, dy \,.$$

Let the function $\xi = \xi(x, y)$ be determined by one of the two equivalent relationships,

$$\Omega(x, y) = \xi(x, y + \beta^{-1} \Omega(x, y)) \,,$$

$$\xi(x, y) = \Omega(x, y - \beta^{-1} \xi(x, y)) \,. \tag{6}$$

Each of them can be fulfilled for all $y$ (at fixed $x$) only in the case of open lines of the vortex. The solution of the diagonalization problem of the Poisson bracket (4) giving the continuum hamiltonian system (2) is defined by the following proposition:

*Theorem 1.* If (4) is fulfilled then the following identity holds,

$$\{F, G\}_\Omega = \{F, G\}_{1,\xi} \quad \text{for all } F, G \,. \tag{7}$$

We emphasize that relationships (6) imply that the functions

$$w(y) = \beta^{-1} \Omega(x, y) + y \tag{8}$$

and $y(w) = \beta^{-1} \xi(x, w) - w$ are mutually invertible. Relationship (7) is immediately obtained if in the integral (4) one makes the substitution $w = \beta^{-1} \Omega_{\text{tot}}$ and takes into account the relation

$$\xi(x, w) = \Omega(x, y) \,. \tag{9}$$

In terms of the new variable $\xi$ eq. (2) is equivalent to the equation $\partial \xi / \partial t = (\partial / \partial x) \delta H / \delta \xi$, and henceforth, canonical variables can be introduced by supposing $a_k = \xi_k \sqrt{-2\beta p}$, where $k = (p, q)$ and $\xi_k$ is the Fourier transformation of the function $\xi$. Using an expansion sequence in degrees of nonlinearity,

$$\Omega(x, y) = \sum_{n=0}^{\infty} \frac{1}{(n+1)! \beta^n} \frac{\partial^n}{\partial y^n} \xi^{n+1}(x, y) \,, \tag{10}$$

following from (6) and truncating after the first three terms of this sequence gives the expansion for the hamiltonian in terms of the canonical variables

$$H = 2 \int_{p<0} \sigma_k |a_k|^2 \, dk + \int_{G_{12}} (V_{kk_1k_2} a_k^* a_{k_1} a_{k_2} + \text{c.c.}) \delta_{k-k_1-k_2} \, dk \, dk_1 \, dk_2$$

$$+ \int_{G_{123}} [W^{(1)}_{kk_1k_2k_3} (a_k^* a_{k_1} a_{k_2} a_{k_3} + \text{c.c.}) \delta_{k-k_1-k_2-k_3}$$

$$+ W^{(2)}_{kk_1k_2k_3} (a_k^* a_{k_1}^* a_{k_2} a_{k_3} + \text{c.c.}) \delta_{k+k_1-k_2-k_3}] \, dk \, dk_1 \, dk_2 \, dk_3 \,.$$

$$G_{12} = \{k, k_1, k_2 : p < 0, p_1 < 0, p_2 < 0\}, \qquad G_{123} = \{k, k_1, k_2, k_3 : p < 0, p_1 < 0, p_2 < 0, p_3 < 0\}$$

are the integration demains, $\sigma_k = -\beta p (k^2 + k_0^2)^{-1}$ is the Rossby wave frequency and

$$V_{kk_1k_2} = \frac{i}{2\pi} \left( \frac{q}{k^2 + k_0^2} - \frac{q_1}{k_1^2 + k_0^2} - \frac{q_2}{k_2^2 + k_0^2} \right) \sqrt{-2\beta p p_1 p_2} \,,$$

$$W^{(1)}_{kk_1k_2k_3} = -\frac{\sqrt{pp_1p_2p_3}}{6\pi^2}\left(\frac{q^2}{k^2+k_0^2} + \frac{q_1^2}{k_1^2+k_0^2} + \frac{q_2^2}{k_2^2+k_0^2} + \frac{q_3^2}{k_3^2+k_0^2} - \frac{(q_1+q_2)^2}{(k_1+k_2)^2+k_0^2}\right.$$

$$\left. - \frac{(q_2+q_3)^2}{(k_2+k_3)^2+k_0^2} - \frac{(q_1+q_3)^2}{(k_1+k_3)^2+k_0^2}\right),$$

$$W^{(2)}_{kk_1k_2k_3} = -\frac{\sqrt{pp_1p_2p_3}}{8\pi^2}\left(\frac{q^2}{k^2+k_0^2} + \frac{q_1^2}{k_1^2+k_0^2} + \frac{q_2^2}{k_2^2+k_0^2} + \frac{q_3^2}{k_3^2+k_0^2} - \frac{1}{2}\frac{(q_1-q_2)^2}{(k_1+k_2)^2+k_0^2}\right.$$

$$\left. - \frac{1}{2}\frac{(q-q_3)^2}{(k+k_3)^2+k_0^2} - \frac{1}{2}\frac{(q_1-q_3)^2}{(k_1-k_3)^2+k_0^2} - \frac{1}{2}\frac{(q_1+q)^2}{(k_1-k)^2+k_0^2} - \frac{1}{2}\frac{(q_2+q_3)^2}{(k_2-k_3)^2+k_0^2} - \frac{1}{2}\frac{(q_2-q)^2}{(k_2-k)^2+k_0^2}\right)$$

are the interaction coefficients. We point out that if $\zeta(x, y)$ is analytical in the band $|\mathrm{Im}\,y| < 1/\beta$, then the sequence (10) can be summed and as a result we obtain a direct formula for the substitution of the variable (6)

$$\Omega(x, y) = -\frac{1}{2\pi i}\oint_{|z|=1} \ln\left(1 - \frac{\zeta(y+\beta^{-1}z)}{z}\right)\mathrm{d}z.$$

The above-mentioned method of diagonalization of the Poisson bracket can be also applied for other hamiltonian systems of two-dimensional hydrodynamics.

The equation for the barotropic Rossby waves above an uneven bottom can be written in the form (2) with the same hamiltonian of the form (4) where one should put $\Omega_{\mathrm{tot}} = \Omega + f_0 h'/\bar{h}$, where $\bar{h}$ and $h'(x, y)$ are the mean depth of the ocean and its variation respectively, and $f_0$ is the Coriolis parameter. In this case the function $\zeta$ should be defined by eq. (9) with

$$w = \beta^{-1}\Omega_{\mathrm{tot}}(x, y).$$

When a scalar nonlinearity (8) is present [8], i.e. while adding a term of the form $\alpha\Psi\,\partial\Psi/\partial x$ into the left-hand side of eq. (1), where $\alpha$ is a constant parameter, the hamiltonian form of the equation and the hamiltonian appearance are conserved and the Poisson bracket should be determined by eq. (4), with

$$\Omega_{\mathrm{tot}} = \Omega + \beta y + \alpha y\Psi. \tag{12}$$

In this case the substitution of the functional variable by eq. (9) where $w$ is determined by eq. (11) with $\Omega_{\mathrm{tot}}$ given by (12) also results in the relationship (7).

Finally, consider the nondimensional baroclinic quasigeostrophic flow equations, including the baroclinic effects of topography as dynamical boundary conditions [9]. The hamiltonian formulation of these equations is, following ref. [10],

$$\frac{\partial\Omega}{\partial t} = \{H, \Omega\}, \qquad \frac{\partial\phi_0}{\partial t} = \{H, \phi_0\}, \qquad \frac{\partial\phi_1}{\partial t} = \{H, \phi_1\}, \tag{13}$$

where the hamiltonian $H$ is given by

$$H = \frac{1}{2}\int \rho[\,|\nabla\Psi|^2 + S^{-1}(\partial\Psi/\partial z)^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and the Poisson bracket { } appearing in (13) is defined as

$$\{F, G\} = - \int \rho^{-1} \Omega_{\text{tot}} J\left(\frac{\delta F}{\delta \Omega}, \frac{\delta G}{\delta \Omega}\right) dx\, dy\, dz - \int \phi_0 \left[ S\rho^{-1} J\left(\frac{\delta F}{\delta \phi_0}, \frac{\delta G}{\delta \phi_0}\right) \right]_{z=0} dx\, dy$$

$$+ \int \phi_1 \left[ S\rho^{-1} J\left(\frac{\delta F}{\delta \phi_1}, \frac{\delta G}{\delta \phi_1}\right) \right]_{z=1} dx\, dy \tag{14}$$

for functionals $F = F(\Omega, \phi_0, \phi_1)$, $G = G(\Omega, \phi_0, \phi_1)$. Here $\rho = \rho(z)$ is the vertically stratified mass density, $S(z) = N^2(z)/f_0^2$, $N^2$ is the Brunt–Väsälaa frequency, $\Omega = \nabla^2 \Psi + (\partial/\partial z)(S^{-1} \partial \Psi/\partial z)$, $\Omega_{\text{tot}} = \Omega + \beta y$. The streamfunction $\Psi(x, y, z)$ depends on the vertical coordinate $z$ as parameter, $\phi_0 = [\partial \Psi/\partial z + S\eta]_{z=0}$, $\phi_1 = \partial \Psi/\partial z|_{z=1}$ are the boundary values, and the quantity $\eta(x, y)$ represents the effect of bottom topography (we preserve the main notation of ref. [10]).

Further, it is easy to show that the change of variables

$$w = \beta^{-1} \Omega_{\text{tot}}, \qquad w_0 = \phi_0 \,,$$

$$\xi(x, w, z) = \Omega(x, y, z), \qquad \xi_0(x, w_0) = \frac{\partial \Psi}{\partial z}\bigg|_{z=0} (x, y) \,,$$

which is analogous to (6), transforms the Poisson bracket (14) into the sum of the Gardner brackets for functionals depending on the variables $\Omega$ and $\phi_0$ only.

*Conclusion.* The results of this paper set two-dimensional hydrodynamics into the fruitful context explored earlier by the first author [1]. In particular, this approach enables one to derive the energy transform equation (ETE) for weakly interacting waves. As a result, Kolmogorov and Rayleigh–Jeans spectra may be found as solutions of stationary ETE. Note that derivation of ETE in the baroclinic case by other methods is unlikely to be possible for the present. The mentioned results have been published in Russian [11,12].

Dr. D.D. Holm has attracted our attention to the applicability of the transformation (6) to the baroclinic wave equation with dynamical boundary conditions.

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