Structural stability of wave collapse in media with a local instability

V. E. Zakharov, A. G. Litvak, E. I. Rakova, A. M. Sergeev, and V. F. Shvets

Institute of Applied Physics, Academy of Sciences of the USSR

(Submitted 1 October 1987)
Zh. Eksp. Teor. Fiz. **94**, 107–111 (May 1988)

We study the nonlinear dynamics of waves in systems with two possible kinds of collapse: "weak" collapse with zero energy going into the singularity and "quasi-classical" collapse with a finite amount of trapped energy. We use the example of the one-dimensional Schrödinger equation with a high-power non-linearity ($s = 6$) to show that the evolution of an initial wave distribution makes it multiply subdivided and accompanied by weak field singularities.

1. A broad class of physical problems in which wave collapse occurs is described by the three-dimensional nonlinear Schrödinger equation

$$-i\partial\Psi/\partial t+\Delta\Psi+\Psi|\Psi|^2=0. \tag{1}$$

In the framework of this equation the evolution of an arbitrary field distribution with a negative Hamiltonian

$$H=\int(|\nabla\Psi|^2-{}^1/_2|\Psi|^4)d\mathbf{r}$$

comes to an end after a finite period of time through the formation of a singularity. The analysis of the asymptotic state of the singularity formation indicates two alternative regimes of self-similar collapse[1,2]—the so-called "weak collapse" ($\Psi = (t_0 - t)^{-1/2}\varphi(\mathbf{r}(t_0 - t)^{-1/2})$) with zero energy $I = \int|\Psi|^2 d\mathbf{r}$ going into the singularity, and "quasi-classical strong" collapse where the amount of trapped energy is finite. It was shown in Ref. 2 that this latter collapse regime is unstable to the excitation of small-scale perturbations; the problem of the stability of the weak self-similar collapse remains so far unsolved. The problem of the quantitative collapse scenario of an arbitrary wave packet is also unsolved.

These problems might be solved by means of a numerical study of Eq. (1). A serious difficulty consists here in the fact that to find a physically valid answer it is necessary to solve an essentially three-dimensional problem with an approach to the point of collapse adequate for reliable conclusions.

The difficulties connected with the high dimensionality of Eq. (1) can to a considerable extent be avoided if we use the fact (see, for instance, Ref. 3) that a number of the most important properties of the nonlinear Schrödinger equation

$$-i\partial\Psi/\partial t+\Delta\Psi+|\Psi|^s\Psi=0 \tag{2}$$

are determined in a space of dimensionality d solely by the product ds. The equation describes stable solitons when $ds < 4$, the "critical" collapse for $ds = 4$, and collapse "in the general sense" in the case $ds > 4$, to which the three-dimensional Eq. (1) also belongs. It is therefore expedient to consider the one-dimensional problem of the general form of collapse.

Concretely, we consider the case $s = 6$, when we have instead of (2) (see also Refs. 4, 5)

$$-i\Psi_t+\Psi_{xx}+|\Psi|^6\Psi=0. \tag{3}$$

Equation (3) is similar to the original problem (1) in the sense of the laws for the formation of a singularity.

2. To study the properties of the self-similar solutions of Eq. (3) we apply to it the modified lens transformation[6–8]

$$\Psi=\frac{1}{(t_0-t)^{1/6}}\varepsilon(\xi,\tau)e^{i\xi^2/8},\qquad \tau=\ln\left(\frac{1}{t_0-t}\right),\quad \xi=\frac{x}{(t_0-t)^{1/2}}, \tag{4}$$

corresponding to the transition to a frame of reference which is compressed into the point $x = 0$ according to the weak collapse rule. In the new (ξ, τ) frame in which the moment at which the singularity is formed becomes infinity, Eq. (3) can be written in the form

$$-i\frac{\partial\varepsilon}{\partial\tau}+\frac{\partial^2\varepsilon}{\partial\xi^2}+\varepsilon|\varepsilon|^6=-i\frac{\varepsilon}{12}-\frac{\xi^2}{16}\varepsilon. \tag{5}$$

The weak collapse mode is a stationary solution of (5) of the form $\varepsilon(\xi, \tau) = g(\xi)e^{-i\alpha\tau}$ and is determined by the parameters $g_0 = g(\xi = 0)$ and α [if we assume symmetry, $g'(\xi = 0) = 0$]. One can show that the corresponding asymptotic form of the amplitude has the form

$$|g(\xi)|\sim\xi^{-1/3}. \tag{6}$$

If there is no right-hand side, the stationary localized field distributions have the form

$$\varepsilon(\xi,\tau)=2^{1/3}u_0\,\mathrm{ch}^{-1/3}(3u_0^3\xi)\exp(-iu_0^6\tau) \tag{7}$$

and are, clearly, on the whole unstable. When we take the extra terms into account the structure of the mode is deformed, to begin with at the periphery [in accordance with (6)]. The first term on the right-hand side of (5) corresponds to an increase in the number of quanta in the $\varepsilon(\xi, \tau)$ distribution, and it follows from the properties of the solution (7) that it acts to broaden it. However, the effect of the second term, which describes emission of waves from the periphery of the bunch and a decrease in the number of quanta in the central part of the distribution, increases. One may assume that under conditions when these factors are in equilibrium Eq. (5) has a stable stationary solution in the form of a solitary dissipative structure. It is just this fact which guarantees an important role for the weak collapse process in the dynamics of nonlinear systems.

3. Another possible regime for the formation of a singularity is the strong quasi-classical collapse.[1,2] We apply to Eq. (3) the appropriate compression transformation:

$$\Psi=\frac{1}{a^{1/2}}\,\varepsilon(\xi,\tau)\exp\left(-i\xi^2\frac{da}{d\tau}\frac{1}{4a^2}\right),$$

$$\xi=\frac{x}{a},\qquad \tau=\int\frac{dt}{a^3},\qquad a=(t_0-t)^{2/5}=\frac{25}{\tau^2}. \tag{8}$$

The equation for the self-similar mode can then be written in the following form:

$$-i\frac{\partial\varepsilon}{\partial\tau}+a(\tau)\,\frac{\partial^2\varepsilon}{\partial\xi^2}+\varepsilon|\varepsilon|^6+\frac{3}{50}\,\xi^2\varepsilon=0. \tag{9}$$

There are no exact stationary localized solutions of Eq. (9), so that the quasi-classical collapse mode $\varepsilon(\xi, t) = A(\xi)\exp(-iA_0^6\tau)$ is determined in the limit as $a(\tau)\to 0$ and has the form (see also Ref. 9)

$$A(\xi)=A_0(1-\xi^2/\xi_0^2)^{1/6}\ \text{when}\ \xi^2<\xi_0^2=50A_0^6/3,$$
$$A(\xi)=0\ \text{when}\ \xi^2>\xi_0^2. \tag{10}$$

The term with the spatial derivative, which was dropped to obtain (10) turns out to be small for this distribution everywhere except in a narrow region $\Delta\xi\approx 1/\tau A_0^3$ near the points $\xi=\pm\xi_0$. This region serves as an unusual generator of small-scale perturbations which develop against the background of the stationary structure (10). An analysis of the linear stage of the field instability, given in the form

$$\varepsilon=[A(\xi)+a(\xi,\tau)]\exp(-iA_0^6\tau),\qquad |a|\ll A,$$

shows that amplitude perturbations with a wavelength $\lambda=2\pi/\varkappa\sim\Delta\xi\ll\xi_0$ grow in the central region, starting from a level $1\gg|a|/A_0\gg 1/6\tau^2A_0^{12}$, according to the formula

$$|a|\propto\exp\left\{5\cdot 6^{1/2}\varkappa A_0^3\left[\operatorname{Arch}\frac{6^{1/2}\tau A_0^3}{5\varkappa}-\left(1-\frac{25\varkappa^2}{6A_0^6\tau^2}\right)^{1/2}\right]\right\}$$

and lead to the destruction of the quasi-classical solution.

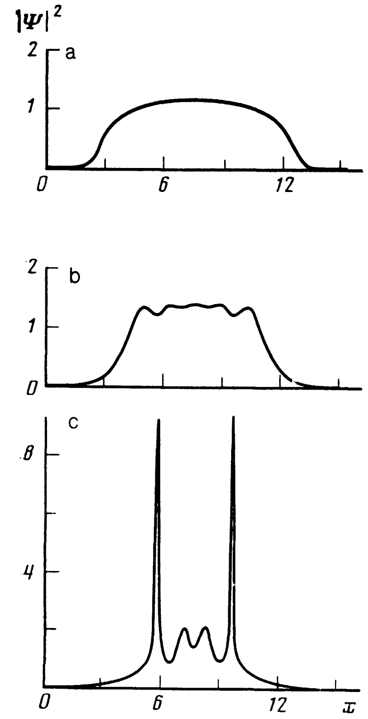

FIG. 1. Fragments of the evolution of an initial quasi-classical distribution with $A_0=1.15$, $a=1$: a) $t=0$, b) $t=0.36$, c) $t=0.62$.

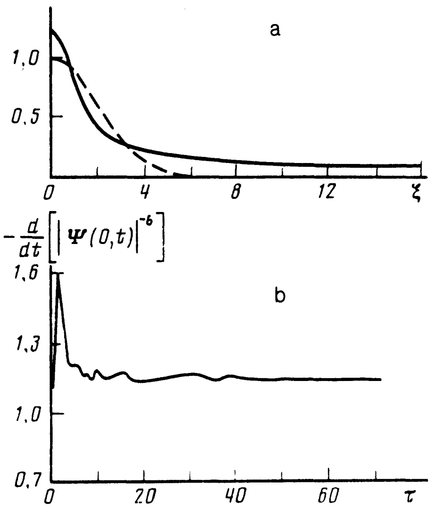

FIG. 2. Results of the simulation in the ξ, τ variables: a) the function $|\Psi|^2$ for $\tau=0$ (dashed curve) and $2\times10^{-7}|\Psi|^2$ for $\tau=40$ (full drawn curve); b) the τ-dependence of $-d[|\Psi(0,\tau)|^{-6}]/dt$.

4. The conclusions concerning the laws governing the behavior of the system studied were confirmed by a numerical experiment in which we studied the effects of the structural instability of broad quasi-classical distributions, the weak collapse process, and the effect of short-wave damping on the nature of the occurrence of singularities. We show in Fig. 1 fragments of the evolution of an initial field distribution in the form (10) with $A_0=1.15$ and $a=1$. Our calculations show that the wave clusters formed by breakdown of the quasi-classical structure of the wave packet change to a weak collapse regime which leads to the formation of singularities with zero trapped energy.

With the aim of a direct proof of the instability and a determination of the structure of the weak collapse mode, we undertook a study of the dynamics of a separate collapsing section. We succeeded in studying the detailed picture of the

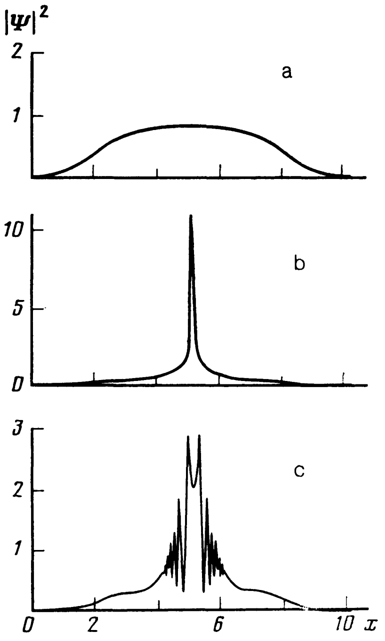

FIG. 3. Evolution of a spatial distribution in a system with small-scale damping: a) $t=0$, b) $t=0.87$, c) $t=0.88$.

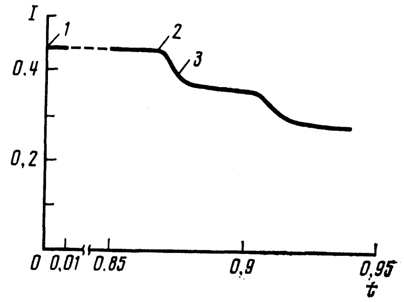

FIG. 4. Gradual diminution in the energy of a wavepacket in a system with damping.

formation of the singularity up to significant—by 6 to 7 or more orders of magnitude—excess of the field amplitude over the initial level, thanks to a specially developed experimental procedure[10,11] based upon changing to a (ξ, τ) reference frame which is compresssed according to the formula

$$\xi = x|\Psi(0, t)|^3, \quad d\tau/dt = |\Psi(0, t)|^6.$$

We show in Fig. 2a the self-similar structure established during the evolution of the initial field distribution (dashed curve) with a Gaussian form: $\Psi(x, 0) = \exp(-x^2/16)$. The self-similarity law in which we are interested was verified, in particular, according to the formula

$$w[t(\tau)] = -\frac{d}{dt}[|\Psi(0, t)|^{-6}],$$

which is shown in Fig. 2b. One sees easily that starting from $\tau \approx 10$ and $|\Psi(0, t)| \sim 10$ the distributions are trapped in the stable weak collapse regime. In the self-similar regime, the "tail" of the distribution agrees with excellent accuracy with the asymptotic form (6). The structure of the weak self-similar collapse mode is described in detail in Ref. 10.

The most important feature of this regime of self-similar collapse is the decrease of the energy I_c contained in the collapsing distribution $\Psi(x, t)$ as the time of the singularity approaches: $I_c \sim (t_0 - t)^{1/6}$. In actual systems with small-scale damping of the waves (for instance, in a high-temperature plasma) this leads to many bursts of field bunches and the corresponding "batched" energy dissipation. We observed such a process when we modified the numerical experiment by introducing into the right-hand side of Eq. (3) a dissipative term

$$\frac{i\alpha}{2\pi}\int\int \Psi(x', t)\exp\left[ik(x-x') - \frac{\beta^2}{k^2}\right]dk\,dx'$$

with $\alpha = 10^2$, $\beta = 7.5$ [guaranteeing damping (of the Landau type) of the high spatial harmonics of the $\Psi(x, t)$ field]. This enabled us to simulate the system for arbitrary evolution times (Fig. 3). The function $I(t)$ shown in Fig. 4 which characterizes the gradual decrease in the energy stored in the initial field distribution confirms the conclusion that the important role is played by the weak collapse in the dynamics of wave systems with a local nonlinearity. The problem of the behavior of the wave packet remains then on the whole unsolved. One may assume that the packet will also be compressed according to a so far unknown law that must be determined by a further theoretical analysis and numerical experiments.

[1] V. E. Zakharov, E. A. Kuznetsov, and S. L. Musher, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 125 (1985) [JETP Lett. **41**, 154 (1985)].
[2] V. E. Zakharov and E. A. Kuznetsov, Zh. Eksp. Teor. Fiz. **91**, 1310 (1986) [Sov. Phys. JETP **64**, 773 (1986)].
[3] J. J. Rasmussen and K. Rypdal, Phys. Scripta **33**, 481 (1986).
[4] G. Pelletier, Phys. Rev. Lett. **49**, 782 (1982).
[5] O. Russell, D. F. DuBois, and H. A. Rose, Phys. Rev. Lett. **56**, 838 (1986).
[6] V. I. Talanov, Pis'ma Zh. Eksp. Teor. Fiz. **11**, 303 (1986) [JETP Lett. **11**, 199 (1986)].
[7] G. M. Fraĭman, Zh. Eksp. Teor. Fiz. **88**, 390 (1985) [Sov. Phys. JETP **61**, 228 (1985)].
[8] N. A. Zharova, A. G. Litvak, T. A. Petrova *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 12 (1986) [JETP Lett. **44**, 13 (1986)].
[9] K. Rypdal and J. J. Rasmussen, Phys. Scripta **33**, 498 (1986).
[10] V. E. Zakharov and V. F. Shvets, Pis'ma Zh. Eksp. Teor. Fiz. **47**, 227 (1988) [JETP Lett. **47**, 275 (1988)].
[11] N. E. Kosmatov, I. V. Petrov, V. F. Shvets, and V. E. Zakharov, No. 1365, Preprint Space Res. Inst., Acad. Sc. USSR, 1988.

Translated by D. ter Haar