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Wave–vortex dynamics in drift and β-plane turbulence

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For the theory of drift plasma and β-plane geophysical dynamics both large-scale vortex and small-scale wave components are important: linear excitation and dissipation occur mainly at small scales, while concentration of the energy spectrum takes place (through the inverse cascade) at large vortices. Based on the time and space separation of these scales averaged evolution equations are derived. The equation for the small scales describes the propagation of high-frequency quanta on the background of a flow produced by large-scale vortices; this equation provides the conservation of the spectral density of the potential enstrophy of small scales. The equation for the large-scale component is the Charney–Hasegawa–Mima equation with a source term having the form of the ponderomotive force and providing the inverse energy cascade from small to large scales. A new computational approach for the modeling of drift and β-plane turbulence is proposed on the basis of the equations obtained – the quantum in the cell method.

1. One of the most widely used models for the description of drift turbulence in an inhomogeneous magnetized plasma as well as Rossby wave turbulence on the β-plane in geophysical hydrodynamics is based on the Charney–Hasegawa–Mima equation

$$\partial_t(\triangle\Psi-\Psi)-\partial_x\Psi+(\partial_x\triangle\Psi)\partial_y\Psi -(\partial_y\triangle\Psi)\partial_x\Psi=0\,. \tag{1}$$

Here we use the dimensionless time t, coordinates x, y and the stream function $\Psi=\Psi(x,y,t)$; their meaning in a wide variety of particular situations in plasma physics and hydrodynamics is explained, e.g., in ref. [1].

At present it is well recognized that large-scale vortex motions play a very important role in physical processes of practical significance, e.g., in anomalous plasma transport across a magnetic field, in global atmospheric and oceanic circulation. Concentration of the turbulent spectrum at large scales is usually attributed to the existence of an inverse energy cascade in 2D turbulent media. On the other hand, the small-scale motions are also of great importance for the drift and Rossby wave turbulence theory: both the excitation and the dissipation of turbulent motion are most important at small scales, not at large scales. Therefore, small-scale waves provide (through the inverse energy cascade) the source for large-scale motions, and, thus, determine their level.

A selfconsistent theory of the nonlinear evolution of drift and Rossby wave systems has been developed in the weak turbulence limit, when the level of excitation is so small that large-scale motions appear to be a set of dispersive waves instead of vortices [1,2]. In particular, the following saturation mechanism of the level of turbulence at large scales has been found: when reaching some threshold level large-scale turbulence forces small-scale motions to disappear (through the enhancing of the coefficient of their diffusion in $\boldsymbol{k}$-space from the source to the domain of dissipation) and, hence, turns off the source of large-scale motions.

The generic situation in experiments and nature is that the width of the frequency spectra of drift and Rossby wave turbulence is as large as the eigenfrequency of the linear waves. Then, large-scale tur-

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bulence is related with vortex motions of the medium, not with dispersive waves, and weak turbulence theory is not applicable. To consider the problem of the formation of large-scale vortices from a small-scale background, we can still use the time and space separation between large and small scales.

The idea is that the small-scale turbulence may be considered to consist of a number of high-frequency quanta moving on the background of a mean flow formed by the large-scale motions and acting on the large scales by some pondermotive force.

In some sense, such an approach is analogous to the description of superfluid ^{4}He as a mixture of two liquids (superfluid and normal components), which is possible due to the presence in the spectrum of excitations of two distinct parts – large-scale phonon and small-scale roton components [3].

2. To obtain the equation for the evolution of the large-scale component, let us Fourier transform eq. (1) and average over the characteristic times of small scales:

$$-\partial_t(p^2+1)\langle\Psi_p\rangle - \mathrm{i}p_x\langle\Psi_p\rangle + \int [\boldsymbol{p}_2,\boldsymbol{p}_1]p_2^2\langle\Psi_{p_1}\rangle\langle\Psi_{p_2}\rangle\delta(\boldsymbol{p}-\boldsymbol{p}_1-\boldsymbol{p}_2)\,\mathrm{d}\boldsymbol{p}_1\,\mathrm{d}\boldsymbol{p}_2 + \int [\boldsymbol{k}_2,\boldsymbol{k}_1]k_2^2\langle\Psi_{k_1}\Psi_{k_2}\rangle\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{p})\,\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2 = 0\,, \tag{2}$$

where the Fourier images Ψ_{p_i}, Ψ_{k_i} are the functions of 2D wave vectors $\boldsymbol{p}_i=(p_{ix}, p_{iy})$, $\boldsymbol{k}_i=(k_{ix}, k_{iy})$ corresponding to the large and small scales; $|\boldsymbol{p}_i| \ll |\boldsymbol{k}_i|$, $i,j=(0,1,2)$. The bracket $\langle\ \rangle$ means averaging over the fast time of small scales, while the bracket [ ] means the z-component of vector multiplication $[\boldsymbol{a},\boldsymbol{b}]=a_xb_y-a_yb_x$.

Taking into account that the second integral in eq. (2) can be rewritten in the form

$$\int [\boldsymbol{k},\boldsymbol{p}](\boldsymbol{k}+\tfrac{1}{2}\boldsymbol{p})^2\langle\Psi_{p/2-k}\Psi_{p/2+k}\rangle\,\mathrm{d}\boldsymbol{k} \equiv \int [\boldsymbol{k},\boldsymbol{p}](\boldsymbol{k}\cdot\boldsymbol{p})^2\langle\Psi_{p/2-k}\Psi_{p/2+k}\rangle\,\mathrm{d}\boldsymbol{k}\,,$$

where $\boldsymbol{k}=\frac{1}{2}(\boldsymbol{k}_2-\boldsymbol{k}_1)$, and performing an inverse Fourier transformation on eq. (2) we get the following equation for the evolution of large scales,

$$\partial_t(\Delta\Psi_\mathrm{L}-\Psi_\mathrm{L})-\partial_x\Psi_\mathrm{L}+(\partial_x\Delta\Psi_\mathrm{L})\partial_y\Psi_\mathrm{L} - (\partial_y\Delta\Psi_\mathrm{L})\partial_x\Psi_\mathrm{L} = -\partial^2_{xx}A-\partial^2_{xy}B+\partial^2_{yy}A\,, \tag{3}$$

where

$$A\equiv A(\boldsymbol{r},t)=2\int\frac{k_xk_y}{k^2(1+k^2)}n_k\,\mathrm{d}\boldsymbol{k}\,,$$

$$B\equiv B(\boldsymbol{r},t)=2\int\frac{k_y^2-k_x^2}{k^2(1+k^2)}n_k\,\mathrm{d}\boldsymbol{k}\,,$$

$$n_k\equiv n(\boldsymbol{k},\boldsymbol{r},t)=\tfrac{1}{2}k^2(1+k^2) \times\int\langle\Psi_{p/2-k}\Psi_{p/2+k}\rangle\exp(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r})\,\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^2}\,, \tag{4}$$

and $\Psi_\mathrm{L}\equiv\Psi_\mathrm{L}(\boldsymbol{r},t)$ is the stream function of large-scale motions. Notice that eq. (3) involves only slow time and space variations, the action of the small-scale motions having the form of a pondermotive force proportional to the density of high-frequency quanta $n(\boldsymbol{k})$.

3. To derive the evolution equation for the density of high-frequency quanta, $n(\boldsymbol{k})$, let us add the evolution equation for the Fourier image Ψ_k multiplied by $\Psi_{k'}$ to the equation for $\Psi_{k'}$ multiplied by Ψ_k; as a result we get

$$\partial_t(\Psi_k\Psi_{k'})+\mathrm{i}(\omega_k^0+\omega_{k'}^0)\Psi_k\Psi_{k'}+R=0\,, \tag{5}$$

where

$$R=\int\left(\frac{[\boldsymbol{k},\boldsymbol{p}](\boldsymbol{k}-\boldsymbol{p})^2}{1+k^2}\Psi_{k-p}\Psi_{k'} + \frac{[\boldsymbol{k}',\boldsymbol{p}](\boldsymbol{k}'-\boldsymbol{p})^2}{1+k'^2}\Psi_{k'-p}\Psi_k\right)\Psi_p\,\mathrm{d}\boldsymbol{p}\,,$$

and

$$\omega_k^0\equiv k_x/(1+k^2) \tag{6}$$

is the eigenfrequency of the linear waves.

Let us suppose that the concentration of the turbulence spectrum at large scales is high enough to neglect the interactions of small scales among themselves in comparison with their interaction with large scales. Then we can expand the expression under the integral R in small $|\boldsymbol{p}|$, $|\boldsymbol{p}|\ll|\boldsymbol{k}|$.

Notice that in this case the small-scale motions can be considered to be linear waves propagating on the background of a weakly inhomogeneous flow produced by the large-scale motions. Therefore, the correlator of the small-scale field $\langle \Psi_k \Psi_{k'} \rangle$ is of an appreciable value only for the the small values of $\boldsymbol{k}+\boldsymbol{k}'$ which are comparable to or less than the characteristic wave vector of large-scale motions. Taking into account these arguments, let us expand the integrand of R in $|\boldsymbol{p}|$, average eq. (5) over the fast time and then expand it in $\boldsymbol{k}+\boldsymbol{k}'$. After straightforward transformations of the resulting equation we finally obtain

$$\left(\partial_t + \frac{\partial \omega}{\partial \boldsymbol{k}} \partial_{\boldsymbol{r}} - \frac{\partial \omega}{\partial \boldsymbol{r}} \partial_{\boldsymbol{k}}\right) n_k = 0\,, \tag{7}$$

where

$$\omega = \omega_k^{\ell} + \omega_k^{\mathrm{n}} = \frac{k_x + (\boldsymbol{v}_{\mathrm{L}} \cdot \boldsymbol{k}) k^2}{1+k^2} \tag{8}$$

is the frequency of the quanta; the term

$$\omega_k^{\mathrm{n}} \equiv \omega^{\mathrm{n}}(\boldsymbol{k}, \boldsymbol{r}, t) = \frac{(\boldsymbol{v}_{\mathrm{L}} \cdot \boldsymbol{k}) k^2}{1+k^2} \tag{9}$$

is the nonlinear correction owing to the motion of the large-scale background with velocity

$$\boldsymbol{v}_{\mathrm{L}} = (\partial_y \Psi_{\mathrm{L}}, -\partial_x \Psi_{\mathrm{L}})\,. \tag{10}$$

It is worth to point out that eq. (5) describes the conservation of the quantum density along the phase-space trajectories

$$\dot{\boldsymbol{r}} = \partial_{\boldsymbol{k}} \omega\,, \quad \dot{\boldsymbol{k}} = -\partial_{\boldsymbol{r}} \omega\,. \tag{11}$$

It is easy to show that the total number of quanta N, $N = \int n_k \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\boldsymbol{r}$, coincides with the potential enstrophy of small scales. This fact is in accordance with the notion that potential enstrophy cannot be transferred from small scales to large scales in 2D turbulence.

Instead of the enstrophy, the total energy of high-frequency quanta, $E_{\mathrm{S}} = \int (n_{\boldsymbol{k}}/k^2) \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\boldsymbol{r}$, is not conserved: there exists an inverse energy cascade from the small scales to the large scales. Meanwhile, the set of equations (3), (7) provide conservation of the total energy:

$$E = E_{\mathrm{L}} + E_{\mathrm{S}} = \mathrm{const}\,, \tag{12}$$

where

$$E = \tfrac{1}{2} \int [\Psi_{\mathrm{L}}^2 + (\nabla \Psi_{\mathrm{L}})^2] \, \mathrm{d}\boldsymbol{r}$$

is the energy of large-scale motions.

For sufficiently small levels of large-scale turbulence, $\Psi_{\mathrm{L}} p k^2 \ll 1$, the set of equations (3), (7) can be reduced to the nonlocal evolution equation for the spectrum of weak turbulence derived in refs. [1,2]. In this case the quanta move in $\boldsymbol{k}$-space along one-dimensional curves $\omega k^2 = \mathrm{const}$, so that their energy has a form usual for the weak turbulence theory, $E_{\mathrm{S}} = \int \omega_{\boldsymbol{k}} n_{\boldsymbol{k}}$.

4. Eq. (7) for the evolution of the quantum density in $\boldsymbol{k}$–$\boldsymbol{r}$ space, $n(\boldsymbol{k}, \boldsymbol{r}, t)$, bears resemblance to the 2D Vlasov equation for the plasma electron distribution function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ in velocity–coordinate phase-space. This fact allows us to propose a numerical method for the simulation of the drift wave–vortex turbulence – *the quantum in the cell method* (QIC), analogous to the particle in the cell method (PIC) in plasma physics [4]. The state of the 2D medium is characterized by the large-scale (vortex) stream function $\Psi_{\mathrm{L}}(\boldsymbol{r}, t_{i-1/2})$ defined in the nodes of a 2D $\boldsymbol{r}$-grid at time $t_{i-1/2}$, and a large number $\mathcal{N}$ of quanta (waves) each having a definite value of the wave vector $\boldsymbol{k}_j(t_i)$ and position $\boldsymbol{r}_j(t_i)$, $j = 1, ..., \mathcal{N}$ at time t_i. To obtain the function Ψ_{L} at the next time step $t_{i+1/2}$ one should calculate the grid functions $A(\boldsymbol{r}, t_i)$, $B(\boldsymbol{r}, t_i)$ according to eq. (4) (compare, e.g., with the calculations of the electron density, $\int f(\boldsymbol{v}, \boldsymbol{r}, t) \, \mathrm{d}\boldsymbol{v}$, or the mean velocity, $\int f(\boldsymbol{v}, \boldsymbol{r}, t) \, \mathrm{d}\boldsymbol{v}$, in the PIC method [4]). Then, the function $\Psi_{\mathrm{L}}(\boldsymbol{r}, t_{i+1/2})$ can be obtained as the solution of eq. (3), e.g., by means of the spectral code.

Afterwards, one should find the new values of the wave vectors and the positions of the quanta using eqs. (11) for their trajectories with the velocity function $\boldsymbol{v}_{\mathrm{L}}(\boldsymbol{r}, t_{i+1/2})$ (see (10)). Let us divide a square grid cell in coordinate space by the diagonal from the top left icon of the cell to the bottom right one, and approximate the large-scale field $\Psi_{\mathrm{L}}(\boldsymbol{r}, t_{i+1/2})$ in each of the triangular cells by a linear function (one can always plot the plane through the three points corresponding to the nodes of the grid). Then, the large-scale velocity $\boldsymbol{v}_{\mathrm{L}}(\boldsymbol{r}, t_{i+1/2})$ inside these triangular cells will have some constant value (see eq. (10)). As long as the function Ψ_{L} is continuous at all the boundaries of the triangular cells, the component of the velocity field $\boldsymbol{v}_{\mathrm{L}}$ normal to the boundary does not change when crossing this boundary, while the parallel component of $\boldsymbol{v}_{\mathrm{L}}$ changes (in the general case) by a finite value.

The important fact is that eqs. (11) for trajectories of quanta can be integrated analytically under under such an approximation of large-scale vortex motion: according to eqs. (8), (6), (9)–(11) the velocities of quanta $\dot{\boldsymbol{r}}_j$ and wave vectors $\boldsymbol{k}_j$ are constant inside the triangular cells ($\dot{\boldsymbol{r}}_j=\text{const}$, $\dot{\boldsymbol{k}}_j=0$), while the wave vectors of quanta undergo some finite changes when crossing the boundaries of these cells.

To find the values of such changes of $\boldsymbol{k}_j$ let us first suppose the boundary between the near triangular cells to be of finite width d with a continuous parallel velocity $v_{\mathrm{L}\parallel}$ profile inside (the normal velocity $v_{\mathrm{L}\perp}$ is constant when crossing the boundary, see above), and then pass to the limit $d\to 0$ in the resulting expressions for quantum trajectories in $\boldsymbol{k}$–$\boldsymbol{r}$ space.

Suppose that a quantum is moving toward the boundary parallel to the x-axis, so that $v_{\mathrm{L}x}=v_{\mathrm{L}x}(y)$, $v_{\mathrm{L}y}=\text{const}$ within the boundary layer (all the other situations can be considered in the same way). Such a form of the velocity profile implies conservation of "energy", ω, and "x-momentum" of the quantum, k_x (ω does not depend on time and the x-coordinate, see eqs. (6), (8), (9)). Therefore, the Hamiltonian equations (11) for quantum motion are integrable – there exist two integrals of motion in the involution in 4D phase-space.

It is useful to consider the result of the integration in the form of an expression for the parallel velocity $v_{\mathrm{L}x}$ as a function of k_y (on the quantum trajectory) resulting from eqs. (6), (8), (9) and the conditions ω, k_x, $v_{\mathrm{L}y}=\text{const}$, namely,

$$v_{\mathrm{L}x}=\frac{\omega/k_x-1}{k_x^2+k_y^2}+\omega/k_x-\frac{v_{\mathrm{L}y}}{k_x}k_y\,, \tag{13}$$

see fig. 1a. The value of k_y that the quantum has at a point $y=y^*$ within the boundary layer can be obtained by solving eq. (13) with $v_{\mathrm{L}x}=v_{\mathrm{L}x}(y^*)$ with respect to k_y, it is easy to see that this equation can be rewritten as a cubic equation:

$$(v_{\mathrm{L}y}/k_x)k_y^3+(v_{\mathrm{L}x}+\omega/k_x)k_y^2+(v_{\mathrm{L}y}k_x)k_y$$
$$+(k_x^2v_{\mathrm{L}x}+1-\omega/k_x-\omega k_x)=0\,. \tag{14}$$

It is clear that this cubic equation has only one real root if

$$|\omega-k_x|<(\tfrac{4}{3})^{3/2}|v_{\mathrm{L}y}k_x^3|\,, \tag{15}$$

while in the opposite case it has either one real root

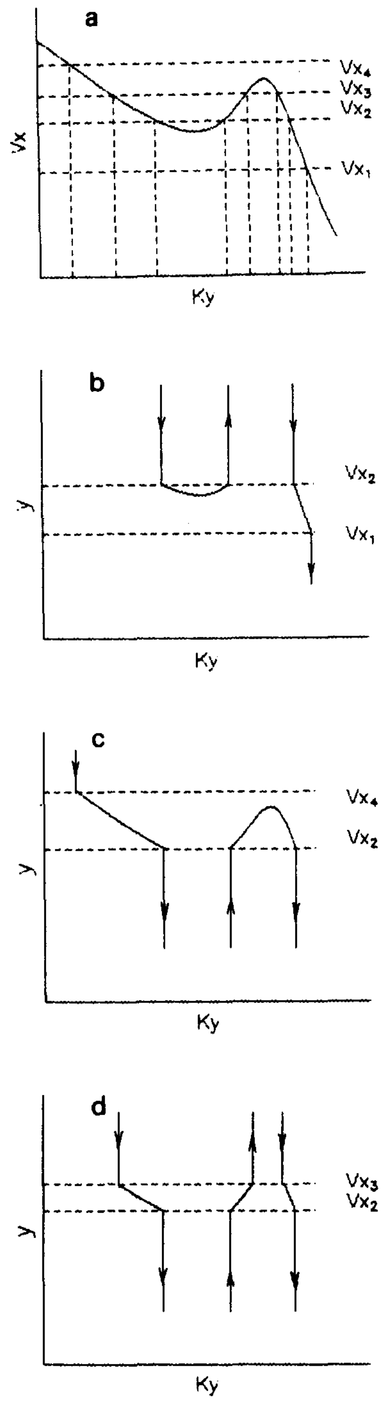

Fig. 1. (a) Relation between the tangent component of the velocity and the normal component of the wave vector in the boundary layer perpendicular to the y-axis (with $v_{\mathrm{L}y}=2$, $k_x=0.5$, $\omega=1.5$); (b)–(d) Phase trajectories of a quantum passing the boundary layer (between the dashed lines) for different values of the tangent velocity in the adjoining cells.

or three real roots depending on the value of v_{Lx}.

Evidently, if condition (15) is satisfied then the new value of the quantum wave vector can be found as the only root of eq. (14) with the value of the x-velocity in the adjoining cell (while k_x is not changed).

A more complex situation arises if condition (15) is not satisfied: then for some values of v_{Lx} in the cells eq. (14) may have three roots. So, to define the final value of k_y we must find out how the trajectories behave inside the boundary layer. Consider, for example, the case $|\omega-k_x|>(\frac{4}{3})^{3/2}|v_{Ly}k_x^3|$, $\omega>k_x$, $v_{Ly}k_x>0$, represented in figs. 1a–1d.

(1) Suppose first that the velocity v_{Lx} varies in the boundary layer (between the dashed lines in figs. 1b–1d) from the value v_{x1} in the lower cell to the value v_{x2} in the upper cell as shown in fig. 1a. In this case eq. (14) has three roots k_{1y}, k_{2y}, k_{3y}, $k_{1y}>k_{2y}>k_{3y}$, in the upper cell and only one root in the lower cell, k'_{1y}, see fig. 1b. Hence, a quantum approaching the boundary may have one of two values of k_y: k_{1y} or k_{3y}. If the initial value of the wave vector corresponds to the largest root (k_{3y}) then according to figs. 1a, 1b the quantum will cross the boundary with a final value of k_y corresponding to the only root of eq. (14) in the lower cell (k'_{1y}). If the initial value of k_y corresponds to the smallest root (k_{1y}) then the boundary will reflect this quantum and it will move back with a wave vector corresponding to the middle root of eq. (14) in the upper cell (k_{2y}).

(2) Let the x-velocity v_{Lx} vary in the boundary layer from some value v_{x2} to v_{x4} (see figs. 1a, 1c). Then eq. (14) has one root in the upper cell and three roots in the lower cell. If the quantum approaches from the upper cell (initial k_y is equal to k_{1y}) then it will cross the boundary along the smallest root (final k_y is equal to k'_{1y}), while if the quantum moves from the lower cell (initial k_y is equal to k'_{2y}) it will be reflected along the largest root in the lower cell (final k_y is equal to k'_{3y}).

(3) If the parallel velocity profile varies from value v_{x2} to v_{x3} then there are three roots of eq. (14) in both the upper and the lower cells (see figs. 1a, 1d). There is no reflection in this case: the smallest, middle and largest roots of eq. (14) in the upper cell, k_{1y}, k_{2y}, k_{3y}, are joined by trajectories with respectively the smallest, middle and largest roots in the lower cell, k'_{1y}, k'_{2y}, k'_{3y} ($k_{1y}\to k'_{1y}$, $k_{2y}\to k'_{2y}$, $k_{3y}\to k'_{3y}$).

5. Notice that the quanta can cross several cells during one time step, so the QIC method described above seems to be very fast. Another advantage of the QIC method is that there is no limitation on the smallest scale of the turbulence, so we can simultaneously take account of dynamics of sufficiently different scales.

In our next paper the QIC method will be applied to the problem of excitation of large-scale drift and Rossby vortices by a small-scale turbulent background and the modeling of nonlocal strong drift and β-plane turbulence (cf. the nonlocal weak turbulence theory developed in refs. [1,2]).

The area of applicability of the QIC method is not restricted by the Charney–Hasegawa–Mima equation; it can be generalized to more complex models of drift turbulence. A similar method can be developed also for any nonlinear medium where mainly essentially separated scales interact; e.g. for nonlinear optics based on the nonlinear Schrödinger equation [5].

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