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Optical turbulence: weak turbulence, condensates and collapsing filaments in the nonlinear Schrödinger equation*

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The nonlinear Schrödinger (NLS) equation $i\Psi_t + \nabla^2\Psi + \alpha|\Psi|^s\Psi = 0$ is a canonical and universal equation which is of major importance in continuum mechanics, plasma physics and optics. This paper argues that much of the observed solution behavior in the critical case $sd = 4$, where d is dimension and s is the order of nonlinearity, can be understood in terms of a combination of weak turbulence theory and condensate and collapse formation. The results are derived in the broad context of a class of Hamiltonian systems of which NLS is a member, so that the reader can gain a perspective on the ingredients important for the realization of the various equilibrium spectra, thermodynamic, pure Kolmogorov and combinations thereof. We also present time-dependent, self-similar solutions which describe the relaxation of the system towards these equilibrium states. We show that the number of particles lost in an individual collapse event is virtually independent of damping. Our numerical simulation of the full governing equations is the first to show the validity of the weak turbulence approximation. We also present a mechanism for intermittency which should have widespread application. It is caused by strongly nonlinear collapse events which are nucleated by a flow of particles towards the origin in wavenumber space. These highly organized events result in a cascade of particle number towards high wavenumbers and give rise to an intermittency and a behavior which violates many of the usual Kolmogorov assumptions about the loss of statistical information and the statistical independence of large and small scales. We discuss the relevance of these ideas to hydrodynamic turbulence in the conclusion.

1. Introduction

The nonlinear Schrödinger (NLS) equation [1]

$$\frac{\partial\Psi}{\partial t} + \sum_{j=1}^{d} \frac{\partial\omega}{\partial k_j}\frac{\partial\Psi}{\partial x_j} - \frac{i}{2}\sum_{j,l=1}^{d} \frac{\partial^2\omega}{\partial k_j \partial k_l}\frac{\partial^2\Psi}{\partial x_j \partial x_l} + i\left(\frac{\partial\omega}{\partial|\Psi|^2}\right)_0 |\Psi|^2\Psi = 0 \tag{1.1}$$

plays a profound role in mathematical physics. The reason for its importance and ubiquity is that it describes the evolution of the envelope $\Psi(x,t)$ of an almost monochromatic wave in a conservative system of weakly nonlinear dispersive waves. It is in fact nothing other than the nonlinear dispersion relation

$$\left[\omega - \omega(k, |\Psi|^2)\right]\Psi = 0, \tag{1.2}$$

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valid for wavetrains $\Psi \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\omega t)+(*)$ for which Ψ is constant and the first factor in (1.2) zero, modified to take account of the fact that for a wavepacket the envelop amplitude $\Psi(\boldsymbol{x},t)$ is slowly varying. In that case, (1.2) becomes a partial differential equation, obtained by replacing ω and $\boldsymbol{k}$ by $\omega+\mathrm{i}\,\partial/\partial t$ and $\boldsymbol{k}-\mathrm{i}\nabla$ respectively where ω and $\boldsymbol{k}$ are the frequency and wavevector of the underlying carrier wave and expanding (1.2) in a Taylor series to second order in the amplitude and gradient variables. In this paper we shall be interested in situations where the dispersion tensor $(\partial^2\omega/\partial k_j\,\partial k_l)$ is positive definite in which case (1.1) can be written

$$\mathrm{i}\Psi_t+\Delta\Psi+\alpha|\Psi|^2\Psi=0, \quad \alpha=\pm1. \tag{1.3}$$

In this form, (1.3) describes the propagation of optical pulses in nonlinear dielectrics [2] and trains of capillary waves on the fluid surface [3]. It also applies to the description of Langmuir waves in plasmas [4] and describes the behavior of a weakly nonlinear Bose-gas in the classical limit.

When the dimension d is 1, (1.3) is completely integrable and for the self-focusing case, $\alpha=1$, has a class of very special solutions called solitons. They are stable, scatter elastically and can combine to form clusters with a quasi-periodic time dependence. It also has an infinite number of motion integrals. For $d\geq 2$, eq. (1.3) is not integrable. It has only three integrals of motion. The integral

$$N=\int|\Psi|^2\,\mathrm{d}\boldsymbol{r} \tag{1.4}$$

has different names: "number of particles", "power", or "wave action". We mainly use the first name. The integral

$$H=\int\left(|\nabla\Psi|^2-\tfrac{1}{2}\alpha|\Psi|^4\right)\mathrm{d}\boldsymbol{r} \tag{1.5}$$

usually is called "energy". It is the Hamiltonian for eq. (1.3) which can be rewritten in a form

$$\mathrm{i}\Psi_t=\frac{\partial H}{\partial\Psi^*}. \tag{1.6}$$

The last integral is momentum

$$\boldsymbol{P}=\tfrac{1}{2}\mathrm{i}\int(\Psi\nabla\Psi^*-\Psi^*\nabla\Psi)\,\mathrm{d}\boldsymbol{r}. \tag{1.7}$$

In a field of statistically homogeneous turbulence, it can be taken to be zero. The properties of eq. (1.3) depend dramatically on the sign of α. If the nonlinearity is positive ($\alpha=1$), the eq. (1.3) when $d=2,3$ also has soliton solutions but they are unstable and do not play any significant role in the theory. The most important nonlinear phenomenon for $d\geq 2$ is wave collapse. The identity

$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2}=8\left(H-\tfrac{1}{2}d(\tfrac{1}{2}d-1)\int|\Psi|^4\,\mathrm{d}\boldsymbol{r}\right), \tag{1.8}$$

where $V = \int r^2 |\Psi|^2 \, \mathrm{d}\boldsymbol{r}$, holds by virtue of the equation. For $d \geq 2$,

$$V \leq 4Ht^2 + c_1 t + c_2 \tag{1.9}$$

obtains and leads to contradiction if $H < 0$. Namely, the positive quantity V becomes negative in a finite time!

The resolution of this contradiction is that it is not possible to continue the solution of eq. (1.3) to long times for certain classes of initial data. If $H < 0$, a solution of (1.3) leads to a singularity in a finite time t_0. The theory of this singularity, or *wave collapse*, depends critically on the dimension d. Although some questions still remain open, the behavior of the solution near the collapse point is well understood, both from the numerical and analytic viewpoints (see ref. [5] and references therein).

Whereas $H < 0$ is a sufficient condition for singularity development, it is not necessary. On the other hand, a sufficient condition for the solution to remain regular for all time when $d = 2$ is that $N < N_0$ where N_0 is that value of particle number corresponding to a soliton solution of (1.3),

$$\Psi = \mathrm{e}^{\mathrm{i}t} R\left(r = \left(x^2 + y^2\right)^{1/2}\right), \tag{1.10}$$

where $R(r)$ satisfies

$$R_{rr} + \frac{1}{r} R_r - R + \alpha R^3 = 0, \quad R'(0) = 0, \quad R(\infty) = 0, \tag{1.11}$$

and $\Psi(x, y, 0) = \phi(x, y)$ obeys the relatively weak condition $\int (|\phi|^2 + |\nabla \phi|^2) \mathrm{d}\boldsymbol{r} < \infty$. Multiplication of (1.11) by $r^2 R_r$ followed by integration in r over $(0, \infty)$ reveals that $\frac{1}{2}\alpha \int_0^\infty r R^4 \, \mathrm{d}r = \int_0^\infty r R^2 \, \mathrm{d}r$. Multiplication of (1.11) by rR followed by integration in r over $(0, \infty)$ yields $\int_0^\infty r R_r^2 \, \mathrm{d}r = \int_0^\infty \alpha r R^4 \, \mathrm{d}r - \int_0^\infty r R^2 \, \mathrm{d}r$. Therefore, for a soliton solution,

$$N = N_0 = 2\pi \int_0^\infty r R^2 \, \mathrm{d}r = 2\pi \int_0^\infty r R_r^2 \, \mathrm{d}r = \pi\alpha \int_0^\infty r R^4 \, \mathrm{d}r \tag{1.12}$$

and H is identically zero. These observations suggest, but do not prove, that as soon as the number of particles exceeds the critical value N_0, wave collapses will begin to occur. We return to this point in a few paragraphs when we consider the validity of the weak turbulence approximation.

The existence of the singularity forces us to seek a regularization of (1.3). From the physical point of view, the most natural way to do this is to include a damping term which switches on in the vicinity of the collapse point. Therefore, we will consider

$$\mathrm{i}\left(\frac{\partial \Psi}{\partial t} + \hat{\gamma}\Psi\right) + \Delta\Psi + \alpha|\Psi|^2\Psi = 0, \tag{1.13}$$

where $\hat{\gamma}$ is pseudodifferential operator acting on Ψ with symbol $\hat{\gamma}$ meaning that the Fourier transform of $\hat{\gamma}\Psi$ is $\gamma_k A(k)$ where $A(k)$ is the Fourier transform of Ψ. (Sometimes, it is also convenient to use nonlinear damping $\hat{\gamma} = \epsilon|\Psi|^{2m}$ for m large, which simulates multiphoton absorption.) Regularization takes place if γ_k grows fast enough at $k \to \infty$. With this regularization the NLS equation (1.13) has a global solution in time and space. A collapse event is now a flash of damping of the integral N ("power" or "particle number"), localized in time and space in a very small domain. It will be important to estimate the loss of this integral in a single collapse in the limit $\hat{\gamma} \to 0$. We give the solution of this problem in section 5 for the most interesting case $d = 2$ and this is one of the new results contained in this paper. However, the main goal of our work is to understand the *turbulence* described by eq. (1.10) in the case $d = 2$.

What do we mean by turbulence? The term is generally used to describe the chaotic behavior of solutions of a system of nonlinear partial differential equations. A minimum requirement is that the power spectrum of the output signal is broadband (temporal, weak or wimpy turbulence). We, however, are mainly interested in what is called fully developed or macho turbulence in which the number of active degrees of freedom is very large, there is energy and power at all length scales and spatial correlations decay rapidly. In hydrodynamics, this corresponds to behavior seen in the large Reynolds number limit. The turbulence associated with (1.13) is often called optical turbulence because of its relevance in describing the propagation of almost monochromatic light beams in media with a nonlinear refractive index. We point out that, in optical contexts, the dimension of (1.13) does not necessarily coincide with the dimension of the medium. For typical light beams, diffraction is much stronger than dispersion and so often the study of the case $d = 2$ is more relevant and it is that case which commands most of our attention here.

There are also additional reasons for studying optical turbulence. It is our contention that there are several universal features common to turbulence in general and it is therefore natural to study those models which both display these features and are analytically and numerically tractable. The two-dimensional nonlinear Schrödinger equation is ideal. First, we can do extremely accurate long time simulations. We use an implicit spectral method, described in section 6, defined on an enlarged grid in order to avoid all aliasing errors. Second, (1.3) is a simple example among a class of Hamiltonian systems given by

$$H = \int \omega_k A_k A_k^* \,\mathrm{d}\boldsymbol{k} + \tfrac{1}{2}\int T_{kk_1k_2k_3} A_k^* A_{k_1}^* A_{k_2} A_{k_3} \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3. \tag{1.14}$$

For (1.3), $\mathrm{i}\Psi_{kt} = \delta H/\delta\Psi_k^*$ with $\omega_k = k^2$ and $T_{kk_1k_2k_3} = -\alpha/(2\pi)^d$. This class of Hamiltonians is very broad and represents many physical systems, surface gravity waves on deep water [6], spin waves in ferromagnets and antiferromagnets [7] and, even, in the special case of a helicity free flow, hydrodynamics in an incompressible fluid [8]. In the last example, after introducing Clebsch variables we obtain (1.11) with $\omega_k \equiv 0$ and

$$T_{kk_1k_2k_3} = (\boldsymbol{\phi}_{kk_2} \cdot \boldsymbol{\phi}_{k_1k_3}) + (\boldsymbol{\phi}_{kk_3} \cdot \boldsymbol{\phi}_{k_1k_2}) \tag{1.15}$$

with

$$\boldsymbol{\phi}_{k_1k_2} = \frac{\mathrm{i}}{2(2\pi)^{3/2}}\left(-\boldsymbol{k}_1 - \boldsymbol{k}_2 + \frac{(k_1^2 - k_2^2)}{|\boldsymbol{k}_1 - \boldsymbol{k}_2|^2}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\right). \tag{1.16}$$

It contrasts sharply with (1.14) in that the quadratic term is absent and therefore represents a fully nonlinear flow.

The third reason for studying (1.3) is that it admits a weak turbulence description in which the quadratic part of the Hamiltonian dominates the quartic. In these circumstances, the dispersive properties of the linear waves lead to a long time behavior of the statistical moments which is sufficiently close to Gaussian that a natural closure of the hierarchy of moment equations is achieved [9, 10]. In particular, this leads to the irreversible kinetic equation for the spectral particle number n_k,

$$\frac{\partial n_k}{\partial t} = \mathrm{st}(n,n,n) = 4\pi \int |T_{kk_1k_2k_3}|^2 \, \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_2)\, \delta(\omega+\omega_1-\omega_2-\omega_3) \times (n_{k_1}n_{k_2}n_{k_3} + n_k n_{k_2} n_{k_3} - n_k n_{k_1} n_{k_2} - n_k n_{k_1} n_{k_3})\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3, \tag{1.17}$$

where $\langle A_k A^*_{k'} \rangle = n_k \delta(\boldsymbol{k}-\boldsymbol{k}')$. Observe that because the kinetic equation involves only the square of the modulus of coupling coefficient $T_{kk_1k_2k_3}$, weak turbulence theory does not distinguish between the dynamical behavior of the focusing ($\alpha = +1$) and defocusing ($\alpha = -1$) cases. Our starting point will be an examination of the nature of solutions of (1.17) and their relevance to the regularized NLS equation (1.13). To put the discussion in context, we will first seek solutions for the class of equations given by the Hamiltonian (1.14) and will find their relevance depends in no small way on the properties of the dispersive relation and the nonlinear coupling coefficient $T_{kk_1k_2k_3}$ and in particular their ratio as k becomes small or large. The domination of the quadratic term over the quartic term in the Hamiltonian (1.17), and the applicability of the weak turbulence description over all k depends on these properties. For example, in NLS, the ratio $T_{kk_1k_2k_3}/\omega_k$ becomes infinite as $k \to 0$ and therefore one might anticipate, correctly, as it turns out, that near $k = 0$, a fully nonlinear description of the dynamics becomes necessary. Indeed we have already seen that, for localized fields obeying $\int (|\Psi|^2 + |\nabla\Psi|^2)\,\mathrm{d}\boldsymbol{r} < \infty$, it is possible for collapsing filaments to occur as soon as the number of particles N exceeds N_0. They will certainly appear as soon as N exceeds N_0 by an amount sufficient for H to be negative. For the statistical initial value problem, there is no analogous condition on the average number of particles $N = \int n_k \,\mathrm{d}\boldsymbol{k}$, or average energy H, so that we do not have an explicit condition for the validity of weak turbulence theory. However, experience with the behavior of solutions of the NLS equation (1.3), and in particular with the onset of the Benjamin–Feir or modulational instability which leads, when $d \geq 2$, to collapsing filaments, shows that the multigap (multi-periodic) states are much less unstable than the singly periodic or condensed states and in an infinite geometry, with $\boldsymbol{P} = \boldsymbol{0}$, the only condensed state likely to be realized is the one with zero wavenumber. In the one-dimensional case, the onset can be expressed in terms of functionals of the spectral data which is associated with the periodic NLS inverse scattering problem and presently one of the authors and Ercolani are attempting to express these criteria in terms of the conserved densities. Because of the inverse cascade property we will shortly discuss, it is to be expected that as soon as enough spectral number density has accumulated at low wavenumbers, modulation instabilities will be triggered and lead to collapsing filaments, which structures are fully nonlinear. Therefore, it is likely that the weak turbulence theory for focusing ($\alpha = +1$) NLS is never valid for all time, especially when we continuously excite the medium and add number (and energy) density at intermediate wavenumbers. However, if the pumping is very weak and applied at sufficiently high wavenumbers, so that the pumping rate divided by the frequency at which the input is maximum is small, there will be enough time for the weak turbulence equilibrium states to be realized before enough particle number density has accumulated at small wavenumbers to cause the onset of intermittent collapses. Furthermore, the frequency of collapses in both time and space will depend on the inverse cascade flux rate and if this is small enough, the collapses, although fully nonlinear events, will be sufficiently rare so as not to affect the weak turbulence equilibria in any major way. We verify this numerically. The spectra shown in figs. 5, 6, and 7 below are for the defocusing ($\alpha = -1$) case, and the cases of both strong and weak damping at large scales. Because of the similarity of the spectra, we conclude that, even in the focusing case, weak turbulence theory is still relevant. Nevertheless, this discussion clearly points to the need of carrying out the statistical initial value problem for spatially homogeneous random fields $\Psi(x,y,t)$ which contain both weakly nonlinear wavetrains and collapsing filaments. This investigation is underway.

The solutions of (1.17) which we examine correspond to

(a) thermodynamic equilibria

$$n_k = T/(\mu + \omega_k) \tag{1.18}$$

for which the fluxes of both particle number density n_k and energy density $E_k = \omega_k n_k$ are zero, and

(b) pure Kolmogorov spectra

$$n_k = a_1 Q^{1/3} \omega_k^{-1-\gamma/3} \tag{1.19}$$

and

$$n_k = a_2 P^{1/3} \omega_k^{-4/3-\gamma/3} \tag{1.20}$$

with γ depending on dimension d, the linear dispersion relation and the properties of $T_{kk_1k_2k_3}$. These solutions correspond respectively to a constant flux Q of particle number density and zero flux P of energy density to low wavenumbers and constant flux P of energy density and zero flux Q of particle number density to high wavenumbers. It will turn out that in two-dimensional optical turbulence, none of these solutions are relevant and another solution which is a combination of (a) and (b), given by

$$n_k = n_k(P,Q,T,\mu,k) = \frac{T}{\mu + \omega_k + \phi(\omega_k)} \tag{1.21}$$

is particularly important. It is best described as a finite flux Kolmogorov spectrum on a thermodynamic background.

The fourth reason for the study of optical turbulence is that the unforced, undamped model (1.3) possesses two nontrivial integrals N and H. This translates to the conservation of $\int n_k\,\mathrm{d}\boldsymbol{k}$ and $\int \omega_k n_k\,\mathrm{d}\boldsymbol{k}$ in (1.17) by virtue of the fact that the integrals $I_1 = \int \mathrm{st}(n,n,n)\mathrm{d}\boldsymbol{k}$ and $I_2 = \int \omega_k \mathrm{st}(n,n,n)\mathrm{d}\boldsymbol{k}$ are zero. But these integrals can vanish in one of either two ways. After averaging over wavevector angles, we will find that we can write the integrands of I_1 and I_2 as $\partial Q/\partial k$ and $-\partial P/\partial k$ respectively. Here Q and P are the fluxes of number density and energy density. If either I_1 or I_2 is zero because the corresponding flux Q or P is zero at the ends of the integration interval, then we will call $\int n_k\,\mathrm{d}\boldsymbol{k}$ or $\int \omega_k n_k\,\mathrm{d}\boldsymbol{k}$ is true integral of the motion. Since no particles or energy leave the interval, one might expect that, at least in the case of a finite interval, the nonlinear interactions would lead to an equal sharing of particle number or energy over all wavenumbers and that therefore the thermodynamic equilibria (1.18) are relevant. On the other hand, the presence of damping acting at large wavenumbers $k \ge k_\mathrm{d}$ means that there will be a flux of energy density towards $k = \infty$ and this flux will settle down to a constant value determined by a balance between the input of energy at $k = k_0 \ll k_\mathrm{d}$ and the dissipation of energy for $k > k_\mathrm{d}$. Across the window of transparency (k_0, k_d) in which neither forcing or damping is important, the flux of energy will be constant. However, this flux of energy density to high wavenumbers is necessarily accompanied by a flux of particle number to low wavenumbers simply because energy conservation means that not all particles introduced at $k = k_0$ can find their way to $k = k_\mathrm{d}$. The fact that $\omega_0 n_0 = \omega_\mathrm{d} n_\mathrm{d}$ means that

$n_d/n_0 = \omega_0/\omega_d \ll 1$. What happens is that, in the four-wave resonant interactions, some particles pick up energy and most others lose theirs. There is necessarily an inverse cascade of particles towards low wavenumbers k_2. Therefore in the windows of transparency $k_2 \ll k \ll k_0$ and $k_0 \ll k \ll k_d$, where neither amplification or damping are important, one might expect that the pure Kolmogorov finite flux solutions (1.19) and (1.20) or the finite flux modification of the thermodynamic spectrum (1.12) are more relevant.

Whether any of the pure states (1.18), (1.19) or (1.20) is exactly relevant, however, is less important than understanding the fate of the particles carried by the inverse cascade. If the low-wavenumber region has infinite capacity, such as is the case in gravity driven, deep ocean waves, in the sense that the small wavenumbers can absorb indefinitely a flux of particles without changing the weakly nonlinear dynamics, then no correction to the theory is necessary. However, in optical turbulence, the ratio of the quartic part of the Hamiltonian to the quadratic becomes increasingly large as more and more particles reach the neighborhood of $k = 0$, and therefore the constant flux of particles towards $k = 0$ will lead to the building of fully nonlinear structures, one of which is the well known Bose condensate or, in the wave context, the monochromatic beam solution

$$\Psi(x,t) = |\Psi_0| \exp\left(i\alpha|\Psi_0|^2 t + i\Phi\right), \tag{1.22}$$

and the other is the wave collapse solution which we discuss in section 5. The sign of the nonlinearity, which is not important in the weak turbulence theory determines which of these two nonlinear states is more important. In the defocusing case of $\alpha = -1$, the Bose condensate is stable and the inverse cascade simply causes it to grow. This is very clearly evident in our numerical experiments. The presence of the condensate does not destroy weak turbulence theory but radically changes it because we now have to study fluctuations not about the zero state but about the condensate state. This changes the dispersion relation to one which admits three-wave (decay type) resonant mixing processes. The weak turbulence for this case is given in section 4. The weak turbulence theory for the fluctuations about a condensate which includes defects will be given in a later publication. In the focusing case, $\alpha = +1$, the condensate is unstable to the modulational instability and therefore cannot form. Instead a series of collapsing filaments, which occur randomly in time and space, are formed and they carry number density very quickly and in a very organized fashion back to large wavenumbers where each collapse deposits a finite and approximately constant number of particles. This secondary cascade of number density to high wavenumbers, caused by nonlinear behavior near $k = 0$, gives rise to intermittent behavior which we clearly observe in the dissipation function. As we have shown, the energy, that is the value of H associated with each of these filaments, is zero. The structure of each of these collapses is identical and well understood and in phase space corresponds to a heteroclinic connection to infinity (HCI) spoken about in earlier articles [11]. We calculate, for the first time in the literature, the number of particles dissipated by each of these events in the limit as $\hat{\gamma}$, the damping, goes to zero. As already mentioned, the frequency of collapses will be determined by the rate of flow of particles to the origin in k space.

In summary, turbulence in the focusing case consists of a coexistence of weak turbulence, dominated by resonantly interacting quartets of wavepackets whose statistics is almost Gaussian, and a field of randomly occurring collapsing filaments whose statistics has the character of a Poisson distribution in both time and space and whose parameters are functions of the inverse cascade rate of particles. A high frequency rate of collapses in both time and space will alter the probability density function (pdf) for the field $\Psi(x,t)$ and in particular the tails of this distribution will show a significant deviation from Gaussianity, a property which can be used as a definition of intermittency. Moreover, the Kolmogorov spectrum (1.20), corresponding to constant energy flux to high wavenumbers will be altered by the presence of the intermittent collapses. We can directly control the amount of intermittency by (a) not exciting large-scale structures susceptible to fast instabilities and (b) applying damping in the low-wavenumber region so as to inhibit the feeding of the same unstable large-scale structures through the inverse cascade. In the limit of large damping at low wavenumbers, the intermittent behavior is entirely suppressed and a pure Kolmogorov spectrum is obtained.

We suggest that this scenario may also be relevant for explaining the deviation from a pure $k^{-5/3}$ Kolmogorov spectrum in three-dimensional hydrodynamic turbulence and more importantly the deviation from the Kolmogorov behavior of the higher order moments of velocity gradients [12]. It is consistent with the pictures of Kraichnan [13] and She [14]. Kraichnan shows how the deviation from Gaussian behavior in the probability density function (pdf) for velocity gradients can be explained by following the dynamical and nonlinear evolution of the pdf due to the combined influences of straining and viscous relaxation. She follows Kraichnan but is more specific in attributing the non-Gaussian behavior to local structures with high-amplitude fluctuations in the velocity gradient field. By contrast we suggest a physical mechanism for intermittency by identifying a source (the inverse cascade) for building the large-scale structures whose instabilities lead to intermittent events, a source which is present even when these structures are not directly forced or even when the external forcing has been switched off. It requires the presence of a second integral of the motion which causes a drift of some conserved density J_k towards low wavenumbers where large structures attempt to form. When these large structures are unstable and when the growth of the instability is not saturated at finite amplitudes but rather is explosive in nature, then this leads to highly organized random and almost singular events which greatly alter the premises of Kolmogorov theory. In a parallel Letter [15] and in the conclusion of this article, we explore, for the hydrodynamic case, two candidates for the second finite flux motion invariant. They correspond to the averages of the squares of linear and angular momentum (the Loitsyanskii invariant) respectively. The existence of the latter depends on a zero value of the former [16]. Neither is the latter an exact invariant because long-range pressure correlations lead to a leakage of squared angular momentum spectral density through $k=0$ just as the energy is not an exact invariant because of the leakage of its spectral density through $k=\infty$. We argue that the inverse cascade in three-dimensional hydrodynamics has similar consequences to the inverse cascade in NLS. Large vortical structures attempt to form but are unstable to small-scale instabilities. By contrast, in two-dimensional hydrodynamics, energy drifts to large scales and creates highly stable large vortices so that the Kolmogorov spectrum corresponding to a constant flux of entropy to high wavenumbers ($E_k \sim k^{-3}$) is undisturbed.

We freely admit that the analogy is far from complete and that the arguments in the hydrodynamic case are much less compelling than they are for optical turbulence. In particular, the identification of the spectral density whose inverse cascade builds large-scale structures susceptible to collapse-like instabilities and the nature of the instabilities themselves can best be considered as suggestions. Nevertheless, the importance of an inverse cascade in so many contexts, in optical turbulence, in feeding large vortices of the two-dimensional Euler equations, in ocean waves where it is the only mechanism that explains the presence of "old" waves travelling faster than the wind, suggests that the ramifications of its presence also be explored in other situations.

The paper is organized as follows. In section 2, we reproduce the weak turbulence description of NLS in the case where the mean $\langle \Psi \rangle$ can be taken zero for all time and derive the kinetic equation. In section 3, we study for the general case the solutions corresponding to thermodynamic equilibria, pure Kolmogorov and modified spectra and combinations thereof, as well as the self-similar solutions which describe the relaxation of the turbulence either the zero or Kolmogorov states. We shall see that, in the case of two-dimensional optical turbulence, neither of the first two stationary solutions turn out to be relevant. They are relevant for three dimensions although the high-frequency equilibrium state (1.20), corresponding to a finite energy density flux, requires logarithmic corrections. In section 4, we discuss weak turbulence in the presence of a condensate produced, in the defocusing case, by a constant flux of particles towards $\omega = 0$, and obtain a kinetic equation with entirely different properties. Again, we look at the relaxation of the turbulence to its stationary states. In section 5, we return to the focusing case and discuss the nature of a collapsing filament and in particular show that the dissipated power depends very weakly on damping (i.e. like $\ln\ln(\hat{\gamma}^{-1})$). In section 6, we describe the numerical algorithm and, in section 7, discuss the numerical results. In the conclusion, we speculate about the ramifications of the ideas to other situations and in particular uses the optical turbulence as a paradigm for discussing fully developed hydrodynamic turbulence. In appendix A, we derive some useful formulae.

2. Weak turbulence description of the NLS equation

Let $\Psi(\boldsymbol{r},t)$ be a spatially homogeneous random field satisfying the equation

$$\mathrm{i}(\dot{\Psi} + \hat{\gamma}\Psi) + \Delta\Psi + \alpha|\Psi|^2\Psi = 0. \tag{2.1}$$

Here $\hat{\gamma}\Psi$ is a linear term, describing an interaction with the "external world" - wave damping and instability. We want to develop a statistical description of the field Ψ. It can be done self-consistently if the instability and the damping are small ($\hat{\gamma}\Psi \ll \Delta\Psi$). In this case the nonlinearity basically can be considered also small, and perturbation series for the moments of Ψ can be effectively exploited. Here we discuss the results as they apply to (2.1). It is convenient to introduce a generalized Fourier transform,

$$A(\boldsymbol{k},t) = \frac{1}{(2\pi)^{d/2}}\int \Psi(\boldsymbol{r},t)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{r}, \quad \Psi(\boldsymbol{r},t) = \frac{1}{(2\pi)^{d/2}}\int A(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\boldsymbol{k}. \tag{2.2}$$

The function $A(\boldsymbol{k},t)$ satisfies

$$\left(\frac{\partial}{\partial t} + \gamma(k) + \mathrm{i}\omega(k)\right)A(k) = \frac{\mathrm{i}\alpha}{(2\pi)^2}\int A^*_{k_1}A_{k_2}A_{k_3}\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3. \tag{2.3}$$

Here $\omega_k = k^2$, $\gamma(k)$ is a decrement of damping if $\gamma_k > 0$ or a growth rate of an instability if $\gamma_k < 0$. In all cases considered, the damping component of $\gamma(k)$ has support only near $k = 0$ and $k = \infty$ while the excitation component of $\gamma(k)$ will have support in a band surrounding some intermediate wavenumber k_0.

We are particularly interested in the evolution of the two-point correlation function,

$$\langle A_k(t)\,A^*_{k'}(t)\rangle = n_k\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{2.4}$$

We will call n_k the "wave action" or "particle number" distribution. The latter name reminds us that the field Ψ can be considered as a classical limit of a quantum field describing a weakly interacting Bose-gas.

We also introduce the fourth-order correlation function

$$\begin{aligned}&\langle A_k(t)\, A_{k_1}(t)\, A^*_{k_2}(t)\, A^*_{k_3}(t)\rangle\\&\quad= n_k n_{k_1}[\delta(\boldsymbol{k}-\boldsymbol{k}_2)\,\delta(\boldsymbol{k}_1-\boldsymbol{k}_3)+\delta(\boldsymbol{k}-\boldsymbol{k}_3)\,\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)]+I_{kk_1,k_2k_3}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\\&\quad=\hat{I}_{kk_1,k_2k_3}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3),\end{aligned} \tag{2.5}$$

which we write as the sum of products of second-order moments and the fourth-order cumulant I_{kk_1,k_2k_3}. I is zero if the process is exactly Gaussian. $\hat{I}$ is the fourth-order moment. We find, for $d=2$,

$$\frac{\partial n_k}{\partial t}+2\gamma(k)\,n_k=\frac{\alpha}{2\pi^2}\int \operatorname{Im}\hat{I}_{kk_1,k_2k_3}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3. \tag{2.6}$$

To close eq. (2.6) one must estimate $\hat{I}_{kk_1,k_2k_3}$ through n_k. This is done in refs. [9, 10]. One obtains

$$\operatorname{Im}\hat{I}_{kk_1,k_2k_3}=\frac{1}{\pi}\,\frac{\left(\tilde{\Gamma}_k+\tilde{\Gamma}_{k_1}+\tilde{\Gamma}_{k_2}+\tilde{\Gamma}_{k_3}\right)F_{kk_1,k_2k_3}}{\left(\tilde{\omega}_k+\tilde{\omega}_{k_1}-\tilde{\omega}_{k_2}-\tilde{\omega}_{k_3}\right)^2+\left(\tilde{\Gamma}_k+\tilde{\Gamma}_{k_1}+\tilde{\Gamma}_{k_2}+\tilde{\Gamma}_{k_3}\right)^2}, \tag{2.7}$$

where

$$F_{kk_1,k_2k_3}=\frac{4\pi\alpha}{(2\pi)^2}\left(n_{k_1}n_{k_2}n_{k_3}+n_k n_{k_2}n_{k_3}-n_k n_{k_1}n_{k_2}-n_k n_{k_1}n_{k_3}\right), \tag{2.8}$$

$$\tilde{\omega}=k^2-\frac{2\alpha}{(2\pi)^2}\int n_k\,\mathrm{d}k, \tag{2.9}$$

$$\tilde{\Gamma}_k=\gamma_k+\Gamma_k, \tag{2.10}$$

$$\Gamma_k=\frac{2\pi\alpha^2}{(2\pi)^2}\int\left(n_{k_1}n_{k_2}+n_{k_1}n_{k_3}-n_{k_2}n_{k_3}\right)\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3. \tag{2.11}$$

The last formula has a simple physical explanation. As a result of mutual interaction, the waves change their dispersion law ($\omega_k \to \tilde{\omega}_k$) and acquire an additional damping ($\gamma_k \to \tilde{\Gamma}_k$). However, these changes are small,

$$\frac{\tilde{\Gamma}_k}{\omega_k}\ll 1,\qquad \frac{\tilde{\omega}_k-\omega_k}{\omega_k}\ll 1,$$

and so, approximately,

$$\operatorname{Im}\hat{I}_{kk_1,k_2k_3}=F_{kk_1,k_2k_3}\,\delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}). \tag{2.12}$$

Substituting (2.12) into (2.6) we obtain the fundamental equation of a weak turbulent theory – the kinetic equation for waves,

$$\frac{\partial n_k}{\partial t}+2\gamma_k n_k=\operatorname{st}(n,n,n). \tag{2.13}$$

Here

$$\mathrm{st}(n,n,n) = \frac{4\pi\alpha^2}{(2\pi)^4}\int (n_{k_1}n_{k_2}n_{k_3} + n_k n_{k_1} n_{k_2} - n_k n_{k_1} n_{k_3} - n_k n_{k_1} n_{k_2}) \times \sigma(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 \tag{2.14}$$

is a collision term that can be rewritten as

$$\mathrm{st}(n,n,n) = f_k - 2\Gamma_k n_k, \tag{2.15}$$

where

$$f_k = \frac{4\pi\alpha^2}{(2\pi)^2}\int n_{k_1}n_{k_2}n_{k_3}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 > 0. \tag{2.16}$$

The particle number at k is increased by the contributions from all its four wave resonant partners and is decreased by its own interaction with the same resonant partners. So we have

$$\frac{\partial n_k}{\partial t} + 2\tilde{\Gamma}_k n_k = f_k. \tag{2.17}$$

The stationary solution of (2.17) is

$$n_k = \frac{f_k}{2\tilde{\Gamma}_k} > 0.$$

Hence $\tilde{\Gamma}_k > 0$. It means that in a stationary state all waves have some positive damping decrement. We note, however, that the irreversible nature of (2.13) does not require the presence of any real damping $\gamma(k)$. As is well known, irreversible behavior can result from reversible systems when one considers certain limits [10]. In obtaining the kinetic equation, we allow the time $T_0 = \omega_k t$, measured in units of inverse wave frequency ω_k^{-1}, to tend to infinity while keeping the resonant interaction time $T_1 = \Gamma_k t$ fixed. The resulting neglect of all nonresonant interactions as seen through replacing $\sin[(\omega+\omega_1-\omega_2-\omega_3)t]/(\omega+\omega_1-\omega_2-\omega_3)$ by $\pi\,\mathrm{sgn}\,t\,\delta(\omega+\omega_1-\omega_2-\omega_3)$ introduces the arrow of time.

Eq. (2.3) can be rewritten in the form

$$\left(\frac{\partial}{\partial t} + \gamma_k\right)A_k + \mathrm{i}\frac{\partial H}{\partial A_k^*} = 0, \tag{2.18}$$

where

$$H = \int \omega_k A_k A_k^*\,\mathrm{d}\boldsymbol{k} - \frac{\alpha}{2(2\pi)^2}\int A_k^* A_{k_1}^* A_{k_2} A_{k_3}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3. \tag{2.19}$$

Thus, (2.18) is a Hamiltonian system if $\gamma(k) = 0$. Weak turbulent theory also obtains for systems (2.18) with a more general Hamiltonian,

$$H = \int \omega_k A_k A_k^*\,\mathrm{d}\boldsymbol{k} + \tfrac{1}{2}\int T_{kk_1,k_2k_3} A_k^* A_{k_1}^* A_{k_2} A_{k_3}\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3, \tag{2.20}$$

where, in order for the system to be Hamiltonian, we must have

$$T_{kk_1,k_2k_3} = T^*_{k_2k_3,kk_1} = T_{k_2k,k_2k_3} = T_{kk_1,k_3k_2}. \tag{2.21}$$

The kinetic equation for a system with Hamiltonian (2.20) is

$$\frac{\partial n_k}{\partial t} + 2\gamma_k n_k = \int T_{kk_1,k_2k_3} \operatorname{Im} \hat{I}_{kk_1,k_2k_3} \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3 = \mathrm{st}(n,n,n), \tag{2.22}$$

where

$$\mathrm{st}(n,n,n) = 4\pi \int |T_{kk_1,k_2k_3}|^2 (n_{k_1}n_{k_2}n_{k_3} + n_k n_{k_2} n_{k_3} - n_k n_{k_1} n_{k_2} - n_k n_{k_1} n_{k_3}) \\ \times \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3. \tag{2.23}$$

The formulae corresponding to (2.9)–(2.11) are

$$\tilde{\omega}_k = \omega_k + \int T_{kk_1,k_2k_3} n_{k_1}\, \mathrm{d}k_1, \tag{2.24}$$

$$\tilde{\Gamma}_k = 2\pi \int |T_{kk_1,k_2k_3}|^2 (n_{k_1}n_{k_2} + n_{k_1}n_{k_3} - n_{k_2}n_{k_3}) \\ \times \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3})\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3. \tag{2.25}$$

These formula are valid for a medium of any dimension $d \geq 2$.

We can greatly simplify the kinetic equation (2.22) if the original system, such as the NLS equation (1.10), is isotropic. It means that the frequency ω depends only on the modulus of the wavevector $\boldsymbol{k}$, and T_{kk_1,k_2k_3} is invariant with respect to an arbitrary rotation of all vectors $\boldsymbol{k}_i$ through the same angle. In this case one can find an isotropic solution of the kinetic equation $n = n(|k|)$. It is then convenient to introduce a variable $\omega = \omega(k)$, $k = |k|$. In the new variables, the kinetic equation becomes

$$\frac{\partial N}{\partial t} + 2\gamma(\omega)\, N(\omega) = \iiint (n_{\omega_1}n_{\omega_2}n_{\omega_3} + n_\omega n_{\omega_2} n_{\omega_3} - n_\omega n_{\omega_1} n_{\omega_2} - n_\omega n_{\omega_1} n_{\omega_3}) S_{\omega\omega_1,\omega_2\omega_3} \\ \times \delta(\omega + \omega_1 - \omega_2 - \omega_3)\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\omega_3 \\ = T[n(\omega)]. \tag{2.26}$$

Here N_ω is a frequency distribution, corresponding to the number of particles in the frequency band $(\omega, \omega + \mathrm{d}\omega)$, defined by the following equation:

$$\int N_\omega\, \mathrm{d}\omega = \int n_k\, \mathrm{d}\boldsymbol{k}, \tag{2.27}$$

or $N_\omega = \Omega_0 k(\mathrm{d}k/\mathrm{d}\omega) n_\omega$, where Ω_0 is the solid angle in d dimensions and $n_\omega = n_k$ expressed as function of ω through $\omega = k^\alpha$.

The coefficient S is given by

$$S_{\omega\omega_1,\omega_2\omega_3} = 4\pi\Omega_0(kk_1k_2k_3)^{d-1} \frac{dk}{d\omega}\frac{dk_1}{d\omega_1}\frac{dk_2}{d\omega_2}\frac{dk_3}{d\omega_3}\left\langle |T_{kk_1,k_2k_3}|^2 \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\right\rangle, \tag{2.28}$$

where by the brackets $\langle \quad \rangle$ we mean that we have integrated over unit spheres in $\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3$ space, namely we have integrated over all the angular contributions. For the simplest case of NLS ($T_{kk_1,k_2k_3} = -\alpha/(2\pi)^2$) in two dimensions this averaging is carried out in appendix A. In that case, on the resonant manifold $\omega + \omega_1 = \omega_2 + \omega_3$,

$$S_{\omega\omega_1,\omega_2\omega_3} = \frac{1}{2\pi}\frac{1}{(\omega\omega_1)^{1/2} + (\omega_2\omega_3)^{1/2}} F\left(\frac{2(\omega\omega_1\omega_2\omega_3)^{1/4}}{(\omega\omega_1)^{1/2} + (\omega_2\omega_3)^{1/2}}\right), \tag{2.29}$$

where

$$F(q) = \int_0^{\pi/2} \frac{d\phi}{(1-q^2\sin^2\phi)^{1/2}}$$

has integrable logarithmic singularities on $\omega_2 = \omega$ (whence $\omega_3 = \omega_1$) and $\omega_3 = \omega$ (whence $\omega_2 = \omega_1$). The function $S_{\omega\omega_1,\omega_2\omega_3}$ has natural symmetries inherited from T_{kk_1,k_2k_3},

$$S_{\omega\omega_1,\omega_2\omega_3} = S_{\omega_2\omega_3,\omega\omega_1} = S_{\omega_1\omega,\omega_2\omega_3} = S_{\omega\omega_1,\omega_3\omega_2}. \tag{2.30}$$

In the absence of damping and instability ($\gamma_k \equiv 0$), the equation for n_k has the following "formal" integrals of motion:

$$N = \int n_k \, d\boldsymbol{k}, \tag{2.31}$$

$$\boldsymbol{P} = \int \boldsymbol{k} n_k \, d\boldsymbol{k}, \tag{2.32}$$

$$E = \int \omega_k n_k \, d\boldsymbol{k}. \tag{2.33}$$

In a statistically homogeneous medium, we can take $\boldsymbol{P} = 0$, and, if the turbulence is isotropic,

$$N = \int_0^\infty N_\omega \, d\omega, \tag{2.34}$$

$$E = \int_0^\infty \omega N_\omega \, d\omega. \tag{2.35}$$

We can write (2.26) as

$$\frac{\partial N_\omega}{\partial t} + 2\gamma_\omega N_\omega = T[n] = \frac{\partial^2 R}{\partial \omega^2} = \frac{\partial Q}{\partial \omega} \tag{2.36}$$

and then the equation for $E_\omega = \omega N_\omega$ is

$$\frac{\partial E_\omega}{\partial t} + 2\gamma_\omega E_\omega = \omega T[n] = -\frac{\partial P}{\partial \omega}, \tag{2.37}$$

where

$$R = \int_0^\omega (\omega - \omega')T[n]\,d\omega', \quad Q = \int_0^\omega T[n]\,d\omega', \quad P = -\int_0^\omega \omega' T[n]\,d\omega'. \tag{2.38}$$

At this point, it is very important to distinguish between two types of integral of motion. We will imagine that the support for forcing and damping occurs over certain intervals of frequency space, damping near $\omega = 0$ to absorb the inverse cascade of number density, excitation in a small interval about $\omega = \omega_0$, and finally high-frequency damping for $\omega > \omega_d$. Between these intervals there are windows of transparency in which only the transfer term $T[n]$ in (2.26) is important. Let (a, b) be an interval in one of these windows of transparency. Then $(\partial/\partial t)\int_a^b N_\omega\,d\omega$ and $(\partial/\partial t)\int_a^b \omega N_\omega\,d\omega$ can be written as $Q(b) - Q(a)$ and $P(a) - P(b)$ respectively. We call the integrals $\int_a^b N_\omega\,d\omega$ and $\int_a^b \omega N_\omega\,d\omega$ true integrals of motion if there is no flux into or out the interval through either boundary, i.e. $Q(b) = Q(a) = P(a) = P(b) = 0$. These integrals are associated with stationary solutions $T[n] = 0$ of (2.26) which are thermodynamic equilibria (an equal sharing of number and/or energy density by all frequencies). The presence of damping at high frequencies, however, renders these solutions of little interest because damping causes a finite flux of energy.

Therefore the motion constants which are of most interest are those for which $\int_a^b N_\omega\,d\omega$ and $\int_a^b \omega N_\omega\,d\omega$ are constant because the fluxes Q and P are equal at the ends of the interval. We call these finite flux motion constants. They are particularly important because they are compatible with a class of finite and constant flux stationary solutions of (2.26), namely the Kolmogorov solutions for which $T[n] = 0$ and either Q or P is a nonzero constant. We now turn to a discussion of the different possible solutions of (2.26) and in particular their relevance in the context of optical turbulence.

3. Solutions of the kinetic equation

Let us now study the principal properties of the kinetic equation for waves and its most important special solutions. It is convenient to use a quantum-mechanical language and interpret n_k as a distribution function for particles of a strongly generated Bose-gas. Then N_ω is a "density of particles" in frequency space and E_ω is a "density of energy". Q and P are the fluxes of these quantities, respectively. A positive $Q(P)$ corresponds to a flux of particle numbers (energy) towards low (high) frequencies. We imagine that the system is driven by instabilities at intermediate values of the frequency and damped at large and small frequencies. Suppose that $\gamma(\omega) < 0$ in some vicinity $\omega_0 - \Delta\omega < \omega < \omega_0 + \Delta\omega$ of a frequency ω_0. It means that there is an instability in this interval of frequencies. Due to nonlinear effects, this instability saturates on some stationary level and the particles are produced at the rate

$$\frac{\partial N}{\partial t} = -2\int_{\omega_0 - \Delta\omega}^{\omega_0 + \Delta\omega} \gamma(\omega)\, N_\omega\,d\omega = Q_p, \tag{3.1}$$

and energy at the rate

$$\frac{\partial E}{\partial t} = -2\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} \gamma(\omega)\omega N_\omega \, d\omega = E_p. \tag{3.2}$$

Suppose also that the saturation of the instability leads to the establishment of a stationary state on the whole axis $0 < \omega < \infty$ without a leakage of particles and energy at $\omega = 0, \infty$. Integrating the kinetic equation over ω, we get the obvious balances

$$\int_0^\infty \gamma(\omega) N_\omega \, d\omega = 0, \tag{3.3}$$

$$\int_0^\infty \omega\gamma(\omega) N_\omega \, d\omega = 0. \tag{3.4}$$

Since $N_\omega \geq 0$, conditions (3.3), (3.4) mean that there must exist regions of damping where $\gamma(\omega) > 0$. The rates of damping of particle numbers Q_d and energy E_d must equal those of production, namely

$$Q_d = Q_p, \quad E_d = E_p. \tag{3.5}$$

Introducing averaged frequencies of pumping and damping,

$$\omega_p = E_p/Q_p, \quad \omega_d = E_d/Q_d, \tag{3.6}$$

we see that

$$\omega_p = \omega_d. \tag{3.7}$$

This very simple relation leads to the nontrivial conclusion: there must be at least two regions of damping placed at both sides of the instability region in order to achieve a stationary state. Suppose we have only one damping region, for instance at $\omega > \omega_0 + \Delta\omega$. It is then obvious that $\omega_d > \omega_0 + \Delta\omega$. On the other hand, since ω_p lies inside the interval of instability $\omega_0 - \Delta\omega < \omega_p < \omega_0 + \Delta\omega$, the equality (3.7) cannot be satisfied. Therefore because there is a flux in both directions, a stationary state requires damping at both high and low frequencies.

We can think of this also in the following way. In real physical situations, various different mechanisms cause damping at high frequencies $\omega \sim \omega_d \gg \omega_0$. If this is the only damping, then it is impossible to reach equilibrium. Why? A particle born in the instability region carries with it an energy ω_0 but a particle dying in the high-frequency region carries an energy $\omega \sim \omega_d \gg \omega_0$. Energy balance, therefore, requires that less particles die than are born by at least a factor ω_0/ω_d. Some particles increase their energy as a result of nonlinear interactions. In this process, the "lucky" particle picks up energy from many unlucky ones and escapes to infinity. The "unlucky" particles now carry lower energies $\omega < \omega_0$ so there is a natural drift of a number of particles to low frequencies. It is clear that in order to establish an equilibrium state, a low-frequency damping of particle number must exist. The only alternative would be that the origin $\omega = 0$ has an infinite capacity and can forever absorb a finite number of particles flowing towards it. In the most interesting case this damping is concentrated at very low frequencies $\omega \sim \omega_2 \ll \omega_0$. Then almost all the energy produced by the instability is absorbed at high frequencies $\omega \sim \omega_d$, and almost all particles are absorbed in the low-frequency region $\omega \sim \omega_2$. In other words the nonlinear interaction causes energy transport to the high frequencies and particle transport to the low frequencies.

The existence of the inverse cascade has been argued on the basis of weakly nonlinear theory for which the spectral density of energy E_ω is approximated by ωN_ω. On the other hand, for the two-dimensional Euler equations, one can show that both the energy and mean squared vorticity densities are conserved by each triad interaction between wavevectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ with $\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = 0$. Because each density is a positive definite (each is also quadratic) functional of the Fourier components of the velocity field, one can argue the direct cascade of enstrophy is accompanied by an inverse cascade of energy. For NLS, for each four-wave interaction, not necessarily resonant, between the Fourier amplitudes A_j, $j = 0, 1, 2, 3$ of wavevectors $\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_0 + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3$, one can find that the number density $N = |A_0|^2 + |A_1|^2 + |A_2|^2 + |A_3|^2$ and energy density $H = \omega_0|A_0|^2 + \omega_1|A_1|^2 + \omega_2|A_2|^2 + \omega_3|A_3|^2 - 2\alpha(A_0^* A_1^* A_2 A_3 + A_0 A_1 A_2^* A_3^*) - \alpha N^2 + \frac{1}{2}\alpha(|A_0|^4 + |A_1|^4 + |A_2|^4 + |A_3|^4)$ are conserved. Because H is not positive definite in the focusing case $\alpha = +1$, one cannot argue that the inverse cascade will persist when the fields are fully nonlinear. In the defocusing case, $\alpha = -1$, one can. We anticipate that, even in strongly nonlinear fields which include many collapsing filaments, the nonlinear wavetrain component of the solution will still produce an inverse cascade. In particular, we conjecture that the radiation produced after an incomplete burnout of a collapsing filament will drift to low wavenumbers and participate once again in the nucleation of collapsing filaments.

In many physical situations, the transport of particle number density and energy density is "diffusion-like". It means that only particles having frequencies of the same order interact strongly. This property of turbulence is called "locality". The picture of "local" weak turbulence could be compared with the Kolmogorov's picture of a well-developed turbulence in an incompressible fluid. The advantage of the present situation is that, because it allows closure, the theory of weak turbulence is in principle much simpler than its counterpart in incompressible fluids. For example, the property of the local nature of energy transfer in hydrodynamic turbulence is nothing but a very plausible hypothesis. In the theory of weak turbulence this property can be checked concretely. What does locality mean? It means that there exist certain windows in frequency and wavenumber space in which the rate of change of number or energy density is given purely by the undamped, unexcited kinetic equation

$$\frac{\partial n_k}{\partial t} = \mathrm{st}(n, n, n) \tag{3.8}$$

and consequently we require the r.h.s. of (3.8) to converge for values of k in these windows. Convergence of the integral means that the strength of the resonant interaction, as measured by $S(\omega, \omega_1, \omega_2, \omega_3)$, decays sufficiently fast as $|\boldsymbol{k}_j - \boldsymbol{k}|$, $|\omega_j - \omega|$, $j = 1, 2, 3$ becomes large. This is not a very strong condition. It does not imply that only those frequencies close to ω contribute to the change of N_ω. It does mean, however, that these interactions are more important than the nonlocal ones. We will mention later, and give more details in a later paper, that if one *assumes* the interaction to be *strongly* local, then $\mathrm{st}(n, n, n)$ can be replaced by a differential term proportional to

$$\frac{\partial^2}{\partial \omega^2} \omega^s n^4 \frac{\partial^2}{\partial \omega^2} \frac{1}{n}, \tag{3.9}$$

where $s = \frac{13}{2}$ if $d = 3$ and $s = 5$ if $d = 2$.

Let us now assume that locality is guaranteed. It means that in the windows of transparency $\omega_2 \ll \omega \ll \omega_0$ and $\omega_0 \ll \omega \ll \omega_d$, we can neglect damping and instability. A stationary state is described by

$$T[n] = \frac{\partial^2 R}{\partial \omega^2} = 0 \tag{3.10}$$

with

$$\begin{aligned} R(\omega) &= \omega Q + P \\ &= \int_0^\infty (\omega - \omega')\,\mathrm{d}\omega' \int_{\omega_i > 0}^\infty n_\omega n_{\omega_1} n_{\omega_2} n_{\omega_3} \left(\frac{1}{n_\omega} + \frac{1}{n_{\omega_1}} - \frac{1}{n_{\omega_2}} - \frac{1}{n_{\omega_3}} \right) \\ &\quad \times S_{\omega' \omega_1 \omega_2 \omega_3}\, \delta(\omega' + \omega_1 - \omega_2 - \omega_3)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3. \end{aligned} \tag{3.11}$$

We now discuss several classes of solutions.

3.1. Thermodynamic equilibria

It is easy to see that if

$$n(\omega, T, \mu) = \frac{T}{\mu + \omega_k}, \tag{3.12}$$

(3.10) is satisfied because the integrand of $\mathrm{st}(n,n,n)$ is identically zero, and in particular the flux of number density Q and flux of energy P are also zero. This is the Rayleigh–Jeans distribution and by analogy with standard thermodynamics the parameter T is called temperature and μ is called the chemical potential. The presence of damping at large frequencies means that this solution cannot be relevant there because $E_\omega = \omega N_\omega \to T$ as $\omega \to \infty$. Because E_ω must tend to zero as $\omega \to \infty$, the effective temperature of the thermodynamic solution would have to be zero!

These solutions also lead to an exactly Gaussian final state [10]. In the perturbation analysis which leads to the kinetic equation, the only surviving part of the fourth-order cumulant is the integrand of $\mathrm{st}(n,n,n)$, which is zero on the thermodynamic equilibria. This is not the case for the finite flux solutions.

We will return to a modification of the solution (3.12) corresponding to a finite flux, Kolmogorov-type spectrum on a thermodynamic background which has particular relevance to the two dimensional NLS equation after we discuss the pure Kolmogorov spectra.

3.2. The Kolmogorov solutions

We next turn to the class of solutions for which the fluxes of number density Q and energy P are finite. Such solutions depend on two parameters and frequency

$$n = n(\omega, P, Q) \tag{3.13}$$

and, for $P, Q > 0$, we assume the existence of an energy source at $\omega = 0$ and a number of particles source at $\omega = \infty$. It is difficult to find general formulae, but solutions with a power law behavior

$$n = c\omega^{-x} \tag{3.14}$$

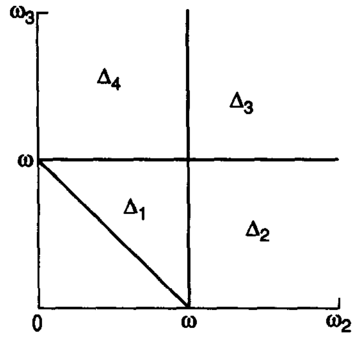

Fig. 1. The integration domain $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$.

are possible whenever either P or Q is zero. Substitute (3.14) into the expression of $T(n)$,

$$T(n) = \int_{\omega_i > 0}^{\infty} S(\omega, \omega_1 \omega_2, \omega_3)\, n_\omega n_{\omega_1} n_{\omega_2} n_{\omega_3} \left(\frac{1}{n_\omega} + \frac{1}{n_{\omega_1}} - \frac{1}{n_{\omega_2}} - \frac{1}{n_{\omega_3}} \right)$$
$$\times \delta(\omega + \omega_1 - \omega_2 - \omega_3)\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\omega_3. \tag{3.15}$$

We can perform the integration over ω_1 and then the integration region Δ in the ω_2, ω_3 plane is $\omega_2 > 0$, $\omega_3 > 0$, $\omega_1 = \omega_2 + \omega_3 - \omega > 0$, namely all ω_2, ω_3 is the positive quadrant above the straight line $\omega_2 + \omega_3 = \omega_2$ as shown in fig. 1.

The following conformal transformations:

$$\omega_2 = \frac{\omega^2}{\omega_2'}, \quad \omega_3 = \frac{\omega(\omega_2' + \omega_3' - \omega)}{\omega_2'}, \qquad \text{implies } \omega_1 = \frac{\omega\omega_3'}{\omega_2'},$$

$$\omega_2 = \frac{\omega\omega_2'}{\omega_2' + \omega_3' - \omega}, \quad \omega_3 = \frac{\omega\omega_3'}{\omega_2' + \omega_3' - \omega}, \quad \text{implies } \omega_1 = \frac{\omega^2}{\omega_1'},$$

$$\omega_2 = \frac{\omega(\omega_2' + \omega_3' - \omega)}{\omega_3'}, \quad \omega_3 = \frac{\omega^2}{\omega_3'}, \qquad \text{implies } \omega_1 = \frac{\omega\omega_2'}{\omega_3'}, \tag{3.16}$$

with Jacobians $(\omega/\omega_2')^4$, $(\omega/\omega_1')^4$ and $(\omega/\omega_3')^4$ $(\omega_1' = \omega_2' + \omega_3' - \omega)$ respectively, transform the regions Δ_2, Δ_3 and Δ_4 onto Δ_1. Now recall that $S(\omega, \omega_1, \omega_2, \omega_3)$ is the angle average over $|T_{kk_1,k_2k_3}|^2$ and if T be homogeneous of degree β, then

$$S(\epsilon\omega, \epsilon\omega_1, \epsilon\omega_2 \epsilon\omega_3) = \epsilon^\gamma S(\omega, \omega_1, \omega_2, \omega_3), \tag{3.17}$$

where

$$\gamma = (2\beta + 3d)/\alpha - 4, \tag{3.18}$$

and $\omega = k^\alpha$ and d is dimension. In the case NLS, $\beta = 0$, $\alpha = 2$, $\gamma = \frac{3}{2}d - 4$. Recalling from (2.30) that S

is symmetric under the interchange of index pairs, we obtain

$$T(n) = c^3 \int_{\Delta_1} S(\omega, \omega_1, \omega_2, \omega_3) (\omega\omega_1\omega_2\omega_3)^{-x} (\omega^x + \omega_1^x - \omega_2^x - \omega_3^x) \delta(\omega + \omega_1 - \omega_2 - \omega_3)$$
$$\times \left[1 - \left(\frac{\omega_2}{\omega}\right)^y + \left(\frac{\omega_1}{\omega}\right)^y - \left(\frac{\omega_3}{\omega}\right)^y\right] d\omega_1\, d\omega_2\, d\omega_3,$$

which, by writing $\omega_j = \omega\xi_j$, $j = 1,2,3$, can be written as

$$T(n) = c^3 \omega^{-y-1} I(x, y), \tag{3.19}$$

where

$$I(x, y) = \int_\Omega S(1, \xi_1, \xi_2, \xi_3) (\xi_1\xi_2\xi_3)^{-x} \delta(1 + \xi_1 - \xi_2 - \xi_3)$$
$$\times (1 + \xi_1^x - \xi_2^x - \xi_3^x)(1 + \xi_1^y - \xi_2^y - \xi_3^y)\, d\xi_1\, d\xi_2\, d\xi_3, \tag{3.20}$$

and

$$y(x, \gamma(\beta, d, \alpha)) = 3x + 1 - (2\beta + 3d)/\alpha = 3x - \gamma - 3. \tag{3.21}$$

The region of integration in the ξ_2, ξ_3 plane is the triangle $0 < \xi_2 < 1$, $0 < \xi_3 < 1$, $\xi_2 + \xi_3 > 1$. In what follows, it will be convenient to think of the integral I as a function of the two independent variables x and y rather than x and γ. The existence of $T(n)$ or what we have called locality requires that $I(x, y)$ converges for values of x and y in the neighborhood of those values for which $I(x, y)$ vanishes. In $\Omega(0 < \xi_2 < 1, 0 < \xi_3 < 1, \xi_2 + \xi_3 > 1)$, the neighborhood of $\xi_2 \sim \xi_3 \sim \xi_1 = \xi_2 + \xi_3 - 1 \sim 1$ corresponds to interactions between neighboring wavevectors and frequencies. The lines $\xi_2 = 1$ (whence $\xi_3 \simeq \xi_1$) and $\xi_3 = 1$ (whence $\xi_2 = \xi_1$) correspond to modal interactions and the integrand is zero there. The line $\xi_2 + \xi_3 = 1$ or $\xi_1 = 0$ corresponds to interactions involving frequencies $\omega_1 = \omega\xi_1 \simeq 0$, $\omega_2, \omega_3 \neq 0$ which are not close together. If the Kolmogorov exponent x is positive, then the product $\xi_1^{-x} S(1, \xi_1, \xi_2, \xi_3)$ will need to tend to zero as ξ_1^r with $r > -1$ in order for the integral $I(x, y)$ to converge. From (2.28), we see that for NLS, $S \to \xi_1^{d/2-1}$ and the condition for locality is that $x < \frac{1}{2}d$. We will see below that the relevant values of x are $x = x_1 = \frac{1}{3}\gamma + 1 = \frac{1}{2}d - \frac{1}{3}$ and $x = x_2 = \frac{1}{3}\gamma + \frac{4}{3} = \frac{1}{2}d$ so that the locality condition holds in the first case but not in the second. Nonetheless, the second case is marginal in that the divergence is very weak, proportional to $\ln(1/\xi_1)$ as $\xi_1 \to 0$, and can be overcome by introducing a correction to (3.14) involving a logarithmic factor.

Observe that

$$I(0, y) = I(1, y) = I(x, 0) = I(x, 1) = 0. \tag{3.22}$$

The thermodynamic equilibrium solutions correspond to the choices $x = 0$ or 1. the Kolmogorov solutions correspond to the choices

$$y = 0 \quad \text{or} \quad y = 1,$$

for which

$$x = x_1 = 2\beta/3\alpha + d/\alpha - \tfrac{1}{3} = \tfrac{1}{3}\gamma + 1, \tag{3.23}$$

or

$$x = x_2 = 2\beta/3\alpha + d/\alpha = \tfrac{1}{3}\gamma + \tfrac{4}{3}. \tag{3.24}$$

From (3.19), let us now compute the fluxes for the two cases $y = 0, 1$. For Q, we obtain

$$Q(x,y) = \int_0^{\omega} T(n)\,\mathrm{d}\omega' = c_1^3 I(x,y)\frac{\omega^{-y}}{-y},$$

which in the limit $y \to 0$, $x \to x_1$, is

$$Q = -c_1^3 \frac{\partial I(x_1,y)}{\partial y}\bigg|_{y=0}. \tag{3.25}$$

Note $I(x_2, 1) = 0$ so that $Q(x_2, 1) \equiv 0$.
For P we obtain

$$P(x,y) = -\int_0^{\omega} \omega' T(n)\,\mathrm{d}\omega' = -c_2^3 I(x,y)\frac{\omega^{-y+1}}{-y+1},$$

which in the limit $y \to 1$, $x \to x_2$ is

$$P = c_2^3 \frac{\partial I(x_2,y)}{\partial y}\bigg|_{y=1}. \tag{3.26}$$

Note $I(x_1, 0) = 0$, so that $P(x_1, 0) \equiv 0$. From (3.25) and (3.26), we find c_1 and c_2. From the positivity of n, P and Q, we require c_1 and c_2 to be positive which will establish a range of γ for which these solutions exist. A negative c_1 would correspond to a negative Q and a flux of particles towards high frequencies which is incompatible with energy conservation and the fact that the presence of viscosity induces an energy density flux towards infinity.

A little calculation shows that

$$-\frac{\partial I(x_1,0)}{\partial y} = \int_{\Omega} S(1,\xi_1,\xi_2,\xi_3)(\xi_1\xi_2\xi_3)^{-x_1}\delta(1+\xi_1-\xi_2-\xi_3) \times (1+\xi_1^{x_1}-\xi_2^{x_1}-\xi_3^{x_1})\ln\left(\frac{\xi_2\xi_3}{\xi_1}\right)\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \tag{3.27}$$

and

$$\frac{\partial I(x_2,1)}{\partial y} = \int_{\Omega} S(1,\xi_1,\xi_2,\xi_3)(\xi_1\xi_2\xi_3)^{-x_2}\delta(1+\xi_1-\xi_2-\xi_3)(1+\xi_1^{x_2}-\xi_2^{x_2}-\xi_3^{x_2}) \times[\xi_2\ln(1/\xi_2)+\xi_3\ln(1/\xi_3)-\xi_1\ln(1/\xi_1)]\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3, \tag{3.28}$$

which are each positive providing

$$F(x) = 1 - \xi_1^x - \xi_2^x - \xi_3^x. \tag{3.29}$$

$F(x)$ is positive when $x < 0$ or $x > 1$. The fact that $\ln(\xi_2\xi_3/\xi_1) > 0$ follows from the positivity of the product $(\xi_2 - 1)(\xi_3 - 1)$. The positivity of $\xi_2 \ln(1/\xi_2) + \xi_3 \ln(1/\xi_3) - \xi_1 \ln(1/\xi_1)$ in the triangle Ω is a little more difficult to prove. Convergence of (3.27) and (3.28) requires further considerations. We have learned that $S(1, \xi_1, \xi_2, \xi_3)$ has logarithmic singularities on $(\xi_3 = 1, \xi_3 = \xi_1)$ and $(\xi_3 = 1, \xi_2 = \xi_1)$. So does $\ln(\xi_2\xi_3/\xi_1)$. But these are canceled by the zero of $F(x_1)$ and $F(x_2)$. The obstacle to the convergence of $\partial I(x_1, 0)/\partial Y$ and $\partial I(x_2, 1)/\partial y$ is the behavior of the integrand near the line $\xi_1 = \xi_2 + \xi_3 - 1 = 0$, just as it was for $I(x, y)$ itself. If $S \sim \xi_1^\sigma$ as $\xi_1 \to 0$, the change of coordinates to $\xi_1 = \xi_2 + \xi_3 - 1$ and either ξ_2 or ξ_3 near $\xi_1 = 0$ gives

$$\int_0 \xi_1^{-x+\sigma} (\ln \xi_1)^j \, d\xi_1, \tag{3.30}$$

where $j = 1$, $x = x_1$ in (3.27) and $j = 0$, $x = x_2$ in (3.28). The behaviors are

$$\xi_1^{\sigma+1-x}\left(\frac{\ln \xi_1}{x_1 - 1 - \sigma} + \frac{1}{(x_1 + \sigma - 1)^2}\right) \quad \text{and} \quad \frac{\xi_1^{\sigma+1-x_2}}{x_2 - 1 - \sigma_2}.$$

Convergence of (3.27) requires $x_1 < 1 + \sigma$. For NLS, $\sigma = \frac{1}{2}d - 1$ and $x_1 = \frac{1}{2}d - 1$ so that this condition is satisfied for all d. Convergence of (3.28) requires $x_2 < 1 + \sigma$ which for NLS where $x_2 = \frac{1}{2}d$ and $\sigma = \frac{1}{2}d - 1$ is marginal. Thus $\partial I_2(x_2, 1)/\partial y$ diverges weakly as $\ln(1/\xi_1)$ as $\xi \to 0$. We remedy this weakly divergent behavior by introducing a cutoff ω_c so that the region of integration Ω is $\omega_c < \omega_2$, $\omega_3 < \infty$, $\omega_2 + \omega_3 > \omega_c$. This means that c_2 is weakly frequency dependent and we can incorporate this into (3.14) by introducing the multiplicative factor $[\ln(\omega_c/\omega)]^{-1/3}$.

When all integrals do exist and when the fluxes P and Q are positive, the pure Kolmogorov solutions are

$$n = \left(-\frac{\partial I(x_1, 0)}{\partial y}\right)^{-1/3} Q^{1/3} \omega^{-\gamma/3 - 1} \tag{3.31}$$

with finite flux Q and zero flux P and

$$n = \left(\frac{\partial I(x_2, 1)}{\partial y}\right)^{-1/3} P^{1/3} \omega^{-4/3 - \gamma/3} \tag{3.32}$$

with zero flux Q and finite flux P. For NLS when $d = 2$, the sign of Q is negative so that (3.31) does not hold and we are forced to seek an alternative solution for the stationary state in the left transparency window $\omega_2 < \omega < \omega_0$.

3.3. Finite temperature Kolmogorov solutions

We conjecture that in general there should be a four-parameter family of solutions

$$n = n(\omega, T, \mu, P, Q) \tag{3.33}$$

which have approximately the Rayleigh–Jeans thermodynamic character of equal energy density or particle number densities but yet a finite flux of either one or both quantities. We do not know how to build such solutions in general but will give a perturbation construction which leads to a solution which seems to be realized in the left transparency window of the two dimensional NLS equation. The reason that the nonzero temperature thermodynamic solution may have relevance in the left transparency window is that the source of particles and energy acts as a buffer at $\omega = \omega_0$ and isolates this portion of the spectrum from the dissipation sink at high frequencies.

Let us suppose that a solution of (3.10) is close to the thermodynamic one (3.12). Then it is reasonable to search for a solution in the form

$$n = \frac{T}{\mu + \omega + \phi(\omega)}, \quad \phi \ll \mu + \omega. \tag{3.34}$$

The linearized equation (3.10) can be written as

$$T(n) = T^3 \int_{\omega_i > 0} \frac{\phi(\omega) + \phi(\omega_1) - \phi(\omega_2) - \phi(\omega_3)}{(\mu + \omega)(\mu + \omega_1)(\mu + \omega_2)(\mu + \omega_3)} S(\omega, \omega_1, \omega_2, \omega_3) \times \delta(\omega + \omega_1 - \omega_2 - \omega_3)\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\omega_3 \equiv 0. \tag{3.35}$$

In the region $\omega \gg \mu$, one can find a solution in the form of a power function,

$$\phi = c\omega^x. \tag{3.36}$$

We obtain, after again using the conformal transformations (3.16),

$$T(n) = cT^3 \omega^{-y-1} I(x, y), \tag{3.37}$$

where

$$y = 1 - x - \gamma, \tag{3.38}$$

γ is given by (3.18) and

$$I(x, y) = \int_{\Omega} S(1, \xi_1, \xi_2, \xi_3)\, (\xi_1 \xi_2 \xi_3)^{-1} \delta(1 + \xi_1 - \xi_2 - \xi_3) \times (1 + \xi_1^x - \xi_2^x - \xi_3^x)(1 + \xi_1^y - \xi_2^y - \xi_3^y)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3, \tag{3.39}$$

where Ω is $0 < \xi_1, \xi_2, \xi_3 < 1$ with $\xi_1 = \xi_2 + \xi_3 - 1$. The zeros of $T(n)$ are given by $x = 0, 1,$ $y = 0, 1$ or

$$x = 0, 1, 1 - \gamma, -\gamma. \tag{3.40}$$

The root $y = 0$ or $x = 1 - \gamma$ gives rise to constant number density flux,

$$Q(x, y) = cT^3 \omega^{-y} \left(-\frac{1}{y} I(x, y) \right) \xrightarrow{y \to 0} cT^3 \left(-\frac{\partial I(1 - \gamma, 0)}{\partial y} \right). \tag{3.41}$$

The constant c is given by

$$c=\left(\frac{Q}{T^3}\right)\left[\int_\Omega S(1,\xi_1,\xi_2,\xi_3)\,(\xi_1\xi_2\xi_3)^{-1}\,\delta(1+\xi_1-\xi_2-\xi_3)\right.$$
$$\left.\times\left(1+\xi_1^{1-\gamma}-\xi_2^{1-\gamma}-\xi_3^{1-\gamma}\right)\ln\left(\frac{\xi_2\xi_3}{\xi_1}\right)\mathrm{d}\xi_1\xi_2\xi_3\right]. \tag{3.42}$$

For positivity we require γ to lie outside the interval (0, 1). For two-dimensional NLS, $\gamma=-1$. However, again we have a problem with convergence because near $\xi_1=0$, the integral in (3.42) behaves as $\int(1/\xi_1)\ln(1/\xi_1)\mathrm{d}\xi_1$ which diverges at $\xi_1=0$ as $[\ln(1/\xi_1)]^2$. But this is a weak divergence which can be remedied by introducing a cutoff wavenumber ω_c so that the region of integration Ω becomes $\omega_c/\omega<\xi_2<1$, $\omega_c/\omega<\xi_3<1$, $\xi_2+\xi_3>\omega_c/\omega$. This means that c, instead of being constant, is (weakly) ω dependent like $\ln^2(\omega_c/\omega)$ but we could have absorbed this behavior in $n(\omega)$ by introducing this as a further correction in (3.26) by taking $\phi(\omega)=c\omega^2\ln^2(\omega_c/\omega)$. We find that the solution

$$n=\frac{T}{\mu+\omega+aQT^{-3}\omega^2\ln^2(\omega_c/\omega)} \tag{3.43}$$

is indeed relevant in the left transparency window.

The second correction of Kolmogorov type $x=-\gamma$ will appear to have more relevance for the flux of energy density. Writing

$$P=-\int_0^\omega \omega' T(n)\,\mathrm{d}\omega'=cT^3\omega^{-y+1}\left(\frac{1}{y-1}I(x,y)\right)\to cT^3\frac{\partial I(-\gamma,1)}{\partial y}\quad\text{as } x\to-\gamma \quad\text{and}\quad y\to 1.$$

But $I(x,y)$ has a double zero and

$$\left.\frac{\partial I(x,y)}{\partial y}\right|_{y=1}=\int_\Omega S(1,\xi_1,\xi_2,\xi_3)\,(\xi_1\xi_2\xi_3)^{-x}\,\delta(1+\xi_1-\xi_2-\xi_3)$$
$$\times\left\{(1+\xi_1^x-\xi_2^x-\xi_3^x)\left[\xi_2\ln(1/\xi_2)+\xi_3\ln(1/\xi_3)-\xi_1\ln(1/\xi_1)\right]\right\}\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \tag{3.44}$$

a single zero as $x=-\gamma=1$ and $y=1$, and therefore for two-dimensional NLS, it would appear $\partial I(1,1)/\partial y$ is zero. But without the factor $(1+\xi_1^x-\xi_2^x-\xi_3^x)$, the integral (3.44) diverges like $\ln(1/\xi_1)$ as $\xi_1\to 0$. Carefully taking the limit by setting $\gamma=-1+\delta$ and remedying and remaining logarithmic divergence by introducing a cutoff, we find that a solution $\phi(\omega)$ can be found in the form

$$\phi(\omega)=cPT^{-3}[\omega/\delta+\omega\ln(\omega_c/\omega)]. \tag{3.45}$$

The first terms can be absorbed into the thermodynamic part. Therefore a correction

$$\phi(\omega)=cPT^{-3}\omega\ln(\omega_c/\omega) \tag{3.46}$$

is possible. However, unfortunately it is too large to be a correction to the thermodynamic spectrum in precisely the region where we need it to cause $E_\omega=\omega N_\omega$ to decay, namely at $\omega=\infty$. Therefore, it is not likely to be relevant in the large ω region.

Let us now interpret these results. Distributions such as (3.34) can be understood as Kolmogorov spectra on a thermodynamic background. The distribution (3.43) is a Kolmogorov spectrum corresponding to a constant flux of particle number Q to the region of small ω. The distribution (3.46), if relevant, could be interpreted as a Kolmogorov spectrum corresponding to a constant flux of energy to the region of large ω. It is not easy to find a necessary and sufficient condition for realization of these spectra in real situations. The following two statements look plausible and are consistent with intuition about Kolmogorov-type spectra in turbulence. (1) The function $\phi(\omega)$ should be positive. Otherwise there is no guarantee that $n(\omega)$ will not acquire a singularity. (2) The functions $\phi(\omega)$ should satisfy the asymptotic conditions

$$\phi(\omega)/\omega \to 0 \quad \text{as } \omega \to 0 \tag{3.47}$$

for the spectrum (3.34). Statement 2 means that the Kolmogorov part of spectrum decreases inside the left window of transparency and that at low frequencies, a thermodynamic equilibrium obtains.

3.4. Logarithmic modifications of Kolmogorov spectra

The presence of a double root at $x = y = 1$ in $I(x, y)$ suggests that solutions such as

$$n = c\omega^{-1}(\ln \omega)^{z} \tag{3.48}$$

may be relevant. Indeed substitution of (3.48) into the expansion (3.19) for $T(n)$ gives

$$\begin{aligned} T(n) &= c^3\omega^{-y-1}(\ln \omega)^{3z} I_1(x, y) + c^3\omega^{-y-1} z(\ln \omega)^{3z-1} I_2(x, y) \\ &\quad + c^3\omega^{-y-1}(\ln \omega)^{3z-2}\left[\tfrac{1}{2}z(z-1)I_3(x, y) + z^2 I_4(x, y)\right] + \mathcal{O}\left(\omega^{-y-1}(\ln \omega)^{3z-3}\right), \end{aligned} \tag{3.49}$$

where $I_1(x, y) = I(x, y)$ and $I_2(x, y)$, $I_3(x, y)$ and $I_4(x, y)$ are expressions like $I(x, y)$ containing products of $\ln \xi_j$, $j = 1, 2, 3$. Assuming convergence of all integrals, the following properties hold:

$$\begin{aligned} &I_1(x, 1) = I_1(1, y) = I_2(1, 1) = 0, \\ &-3\frac{\partial I_1(x, 1)}{\partial y} = I_2(x, 1), \quad \frac{\partial I_1(1, 1)}{\partial y} = 0, \quad I_3(1, 1) = I_4(1, 1) \neq 0. \end{aligned} \tag{3.50}$$

Therefore, after little calculation, a lot of cancellation occurs, and

$$P = -\int^{\omega} \omega T(n)\, d\omega = -\tfrac{1}{2}c^3 z I_3(1, 1)(\ln \omega)^{3z-1} + \mathcal{O}(\ln \omega)^{3z-2}$$

and the choice of $z = \frac{1}{3}$ gives

$$n = \left(\frac{6P}{I_3(1, 1)}\right)^{1/3} \omega^{-1} \ln^{1/3}(1/\omega). \tag{3.51}$$

Unfortunately, $I_3(1, 1)$ is negative and $\omega n \to \infty$ as $\omega \to \infty$, so this solution also does not appear to be relevant. In fact, we have not yet been able to deduce the large-frequency behavior of the stationary spectra for two-dimensional optical turbulence. We find that in order that the turbulence remain weak in $\omega_0 < \omega < \omega_d$, the production rate must be very small. We do reach a quasi-equilibrium state close to ω^{-2} but it is not stationary and appears to be still relaxing.

3.5. The differential approximation for strongly "local" transfer

If the coupling coefficient $S(\omega, \omega_1, \omega_2, \omega_3)$ is strongly local in the sense that its value in the neighborhood of $\omega = \omega_1 = \omega_2 = \omega_3$ are significantly larger than its values anywhere else in the region of integration $0 < \omega_2 < \infty$, $0 < \omega_3 < \infty$, $\omega_1 = \omega_2 + \omega_3 - \omega > 0$, then one can approximate $T(n)$ as follows. Multiply (2.26) by an arbitrary function $f(\omega)$ and integrate over $(0, \infty)$ in ω to obtain

$$\int S(\omega, \omega_1, \omega_2, \omega_3)\, n n_1 n_2 n_3\, \delta(\omega + \omega_1 - \omega_2 - \omega_3) \left(\frac{1}{n} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right) f \,\mathrm{d}\omega\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\omega_3. \tag{3.52}$$

Now write (3.52) as the sum of four quarters, and in the second, third and fourth make the interchanges of variables $\omega \leftrightarrow \omega_1$, $\omega_2 \leftrightarrow \omega_3$; $\omega \leftrightarrow \omega_2$, $\omega_1 \leftrightarrow \omega_3$; $\omega \leftrightarrow \omega_3$, $\omega_1 \leftrightarrow \omega_2$ respectively so that (3.52) becomes

$$\begin{aligned} &\tfrac{1}{4}\int S(\omega, \omega_1, \omega_2, \omega_3)\, n n_1 n_2 n_3\, \delta(\omega + \omega_1 - \omega_2 - \omega_3) \left(\frac{1}{n} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right) \\ &\quad \times (f + f_1 - f_2 - f_3)\, \mathrm{d}\omega\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\omega_3. \end{aligned} \tag{3.53}$$

Write $\omega_j = \omega(1 + p_j)$, $j = 1,2,3$, and expand the two brackets $(1/n) + (1/n_1) - (1/n_2) - (1/n_3)$ and $f + f_1 - f_2 - f_3$ to second order in a Taylor series, approximate $n n_1 n_2 n_3$ by n^4 and use the homogeneity of

$$S(\omega, \omega(1 + p_1), \omega(1 + p_2), \omega(1 + p_3)) = \omega^{\gamma} S(1, 1 + p_1, 1 + p_2, 1 + p_3)$$

to find

$$\tfrac{1}{16}\int \mathrm{d}\omega\, \omega^{6+\gamma} n^4 \frac{\partial^2}{\partial\omega^2} \cdot \frac{1}{n} \frac{\partial^2 f}{\partial\omega^2} \int S(1, 1 + p_1, 1 + p_2, 1 + p_3)(p_1^2 - p_2^2 - p_3^2)\, \delta(p_1 - p_2 - p_3)\, \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}p_3,$$

and integrating twice by parts, obtain

$$S_0 \int \left(\frac{\partial^2}{\partial\omega^2} \omega^{6+\gamma} n^4 \frac{\partial^2}{\partial\omega^2} \cdot \frac{1}{n}\right) f(\omega)\, \mathrm{d}\omega,$$

where

$$S_0 = \tfrac{1}{16}\int S(1, 1 + p_1, 1 + p_2, 1 + p_3)\left(p_1^2 - p_2^2 - p_2^2\right)^2 \delta(p_1 - p_2 - p_3)\, \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}p_3. \tag{3.54}$$

The region of integration in (3.54) is all p_1, p_2, p_3 and convergence relies on the fact that $S(1, 1 + p_1, 1 + p_2, 1 - p_3)$ decays sufficiently fast as $p_1, p_2, p_3 \to \pm\infty$. Because $f(\omega)$ is arbitrary, we can equate integrands and obtain

$$\frac{\partial N_\omega}{\partial t} + 2\gamma(\omega)\, N(\omega) = T(n) = S_0 \frac{\partial^2}{\partial\omega^2} \cdot \omega^{6+\gamma} n^4 \frac{\partial^2}{\partial\omega^2} \cdot \frac{1}{n}, \tag{3.55}$$

where $N_\omega = \Omega_0(\mathrm{d}k/\mathrm{d}\omega)\cdot k^{d-1}n_k$, where n_k is written as function of ω through the dispersion relation $\omega = k^\alpha$ and Ω_0 is the solid angle of integration (i.e. $\Omega_0 = 2\pi$ if $d = 2$, $\Omega_0 = 4\pi$ if $d = 3$).

While the differential approximation to the kinetic equation is quantitatively useful when $S(1, 1+p_1, 1+p_2, 1+p_3)$ decays very rapidly to zero, it is also useful for qualitative understanding in general. We can identify the function $R(\omega)$ and the fluxes $Q(\omega)$ and $P(\omega)$ explicitly,

$$R(\omega) = S_0\omega^{6+\gamma}n^4\frac{\partial^2}{\partial\omega^2}\frac{1}{n},\quad Q(\omega) = \frac{\partial R}{\partial\omega},\quad P(\omega) = R - \omega\frac{\partial R}{\partial\omega}. \tag{3.56}$$

The thermodynamic solutions are given by

$$R(\omega) = S_0\omega^{6+\gamma}n^4\frac{\partial^2}{\partial\omega^2}\frac{1}{n} = 0 \tag{3.57}$$

whence $1/n = (1/T)(\mu + \omega)$ and $Q(\omega) = P(\omega) = 0$. For the Kolmogorov spectra $n = c\omega^{-x}$,

$$\begin{aligned}
R(\omega, x, y) &= c^3S_0\omega^{-y+1}x(x-1),\\
T(n) &= c^3S_0y(y-1)x(x-1)\omega^{-y-1},\\
Q(\omega, x, y) &= c^3S_0(1-y)x(x-1)\omega^{-y},\\
P(\omega, x, y) &= c^3S_0yx(x-1)\omega^{-y+1},
\end{aligned} \tag{3.58}$$

where $y = 3x - \gamma - 3$ and the integral $I(x, y) = S_0y(y-1)x(x-1)$. From (3.58), we see $T(n) = 0$ at $x = 0, 1$ (thermodynamic equilibria), $y = 0, 1$ (pure Kolmogorov spectra) and

$$\begin{aligned}
&Q(\omega, x_1, 0) = c^3S_0x_1(x_1 - 1),\quad P(\omega, x_1, 0) = 0,\\
&Q(\omega, x_2, 1) = 0,\quad P(\omega, x_2, 1) = c^3S_0x_2(x_2 - 1)
\end{aligned} \tag{3.59}$$

are constants whose positivity requires x_1 and x_2 lie outside the interval [0, 1]. One can also see that

$$n = c\omega^{-1}(\ln\omega)^z \tag{3.60}$$

with $z = \frac{1}{3}$ is an approximate solution and that its flux P is negative. In another paper, we shall exploit the differential approximation in considerable detail and discuss how its general stationary solution is related to the work of Bellman [17].

3.6. *The relaxation of solutions to their equilibria*

The relaxation of solutions of (2.26)

$$\frac{\partial N_\omega}{\partial t} + 2\gamma(\omega)N_\omega = T(n) \tag{3.61}$$

towards their equilibria can be captured by self-similar solutions of the equivalent forms (Ω_0 the solid angle in d dimensions)

$$n_k = \frac{1}{t^a} n_0\left(\frac{k}{t^b}\right), \quad N_\omega = \frac{\Omega_0}{\alpha}\frac{\omega^{d/\alpha-1}}{t^a} n_0\left(\frac{\omega}{t^{\alpha b}}\right). \tag{3.62}$$

We obtain our first relation between a and b by direct substitution of (3.62) into

$$T(n) = \int S(\omega,\omega_1,\omega_2,\omega_3)\, n n_1 n_2 n_3\left(\frac{1}{n}+\frac{1}{n_1}-\frac{1}{n_2}-\frac{1}{n_3}\right)\delta(\omega+\omega_1-\omega_2-\omega_3)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\omega_3$$

as given by (3.17). We obtain, using (3.18),

$$2a = 1 + (2\beta + 2a - \alpha)b. \tag{3.63}$$

The next relation is obtained by considering the energy

$$E = \int \omega N_\omega \,\mathrm{d}\omega. \tag{3.64}$$

The first case we consider is that of a constant production of energy so that E is proportional to t. Balancing powers of t in (3.64) gives

$$-a + db + \alpha b = 1, \tag{3.65}$$

from which we find

$$a = \frac{3d + 2\beta}{3\alpha - 2\beta}, \quad b = \frac{3}{3\alpha - 2\beta}. \tag{3.66}$$

How does this solution relate to the Kolmogorov spectrum? Recall that the Kolmogorov solution

$$n = c\omega^{-x} = ck^{-\alpha x}, \tag{3.67}$$

with $x = x_2 = \frac{1}{3}\gamma + \frac{4}{3} = (2\beta + 3d)/3\alpha$, and corresponds to a constant energy flux in the right window of transparency. Observe that the solution

$$n = ck^{-(2\beta+3d)/3} \tag{3.68}$$

exactly matches

$$n = \frac{1}{t^a} n_0\left(\frac{k}{t^b}\right) \tag{3.69}$$

if

$$n_0 \sim c\left(\frac{k}{t^b}\right)^{-2\beta+3d/3}$$

because $a = \frac{1}{3}(2\beta + 3d)b$ and the dependence on t disappears. Thus the solution (3.69) can be interpreted as a front in wavenumber space which travels at speed

$$k \sim t^{3/(3\alpha - 2\beta)} \tag{3.70}$$

and joins the Kolmogorov spectrum (3.68) which obtains behind the front to an almost zero state for $k > t^b$. It is relevant in the situation when the energy calculated on the Kolmogorov spectrum

$$E = \int \omega n_k \,\mathrm{d}k \sim \int k^{(3\alpha - 2\beta - 1)/3} \,\mathrm{d}k$$

is strongly divergent as $k \to \infty$. Therefore a front is necessary in order to keep the energy finite for finite time. The total number of particles, however,

$$N = \int N_\omega \,\mathrm{d}\omega = \frac{1}{t^{a-db}} \int n_0(\Omega)\,\mathrm{d}\Omega,$$

is constant if $\beta = 0$ and decays if $a - db = 2\beta/(3\alpha - 2\beta) > 0$. This shows in the three-dimensional NLS that whereas the energy in the right transparency window increases with time, the number of particles does not and is consistent with our picture of a particle drift to low wavenumbers.

If instead of pumping energy in at a constant rate, we simply put some in initially in the vicinity of $\omega = \omega_0$ and then allow it decay, we find from the conservation of energy (3.64) in the window of transparency that

$$-a + db + \alpha b = 0 \tag{3.71}$$

which together with (3.63) gives

$$a = \frac{\alpha + d}{3\alpha - 2\beta}, \quad b = \frac{1}{3\alpha - 2\beta}. \tag{3.72}$$

Now there is no Kolmogorov spectrum. There is only a decaying lump of energy density which travels to $k = \infty$ with a rate given by

$$k \sim t^b. \tag{3.73}$$

For three-dimensional optical turbulence, $a = \frac{5}{6}$ and $b = \frac{1}{6}$.

Just as we can consider the solution which corresponds to a constant production of energy and the constant flux of energy density towards $k = \infty$, we can also consider the self-similar solutions which correspond to a front moving towards $k = 0$, driven by a constant production of particle number $N = \int N_\omega \mathrm{d}\omega$. If $N \sim t$, then the second relation between a and b is

$$-a + db = 1 \tag{3.74}$$

whence

$$a = \frac{2\beta - \alpha + 3d}{\alpha - 2\beta}, \quad b = \frac{3}{\alpha - 2\beta}. \tag{3.75}$$

We can see a front

$$n = \frac{1}{t^a} n_0\left(\frac{k}{t^b}\right) \tag{3.76}$$

will join the Kolmogorov spectrum

$$n = ck^{-\alpha x_1} = ck^{-(2\beta+3d-\alpha)/3}$$

for $k > t^{3/(\alpha-2\beta)}$ to a zero state for $k < t^{3/(\alpha-2\beta)}$ because $a - \alpha x_1 b = 0$.

4. Fluctuations about the condensate

Let us consider the following question: what happens if energy and power is injected into the system through an instability at $\omega = \omega_0$ and there is no damping at a small ω? It is obvious that the permanent particle flux, produced by the instability, will cause an accumulation of particles in the low-frequency region. What is the "fate" of these particles? The answer depends on the structure of the Hamiltonian. Suppose $\beta > \frac{1}{2}\alpha$ and hence $\gamma > 0$. Suppose also that the condition of locality is satisfied. In this case a Kolmogorov spectrum (3.21) is established in the region $\omega < \omega_0$. Written as function of k, it has the form

$$n_k = \frac{\alpha_1 Q^{1/3}}{k^{(d+2\beta-\alpha)/3}}. \tag{4.1}$$

The total number of particles $N = \int n_k \, dk$ diverges at $k \to 0$. It means that the region of small frequencies has an "infinite capacity" and can absorb an arbitrarily large number of particles. One can check, by comparing linear and nonlinear terms in the Hamiltonian (2.19), that the region $k \to 0$ is "asymptotically linear" under these conditions. On the other hand, if $\beta < \frac{1}{2}\alpha$, the Kolmogorov spectrum (4.1) can absorb only a finite number of particles. In this case, the self-consistency of the weak turbulent theory is violated in the region of small wavenumber, and some strongly nonlinear effects are inevitable. The simplest consequence is that a condensate – a coherent state with zero wavenumber – is generated. For the NLS equation, the condensate is the trivial solution

$$\Psi = \Psi_0 \, e^{i\alpha|\Psi_0|^2 t} \tag{4.2}$$

and exists if the damping $\gamma(k)$ at $k = 0$ is zero. In the general case with Hamiltonian (1.11), the equation for A_k is

$$\left(\frac{\partial}{\partial t} + \gamma(k)\right)A_k + i\omega_k a_k + i\int T_{kk_1,k_2k_3} A^*_{k_1} A_{k_2} A_{k_3} \, \delta(k + k_1 - k_2 - k_3) \, dk_1 \, dk_2 \, dk_3 = 0. \tag{4.3}$$

It, too, has a condensate solution

$$A_k(t) = A\delta(k) \, e^{-iT_0|A|^2 t}. \tag{4.4}$$

The presence of a condensate changes the dispersive properties of the medium. Consider

$$A_k = [A\delta(k) + a_k] \, e^{-iT_0|A|^2 t}. \tag{4.5}$$

Linearization of eq. (4.3) gives

$$\left(\frac{\partial}{\partial t} + \gamma k\right) + \mathrm{i}U_k a_k + \mathrm{i}V_k a^*_{-k} = 0. \tag{4.6}$$

In (4.6),

$$U_k = 2T_k - T_0|A|^2 + \omega_k, \quad T_k = T(0,k,0,k), \quad V_k = T(k,-k,0,0)A^2.$$

Assuming a_k, $a^*_{-k} \sim \mathrm{e}^{\lambda t}$ one finds

$$(\lambda + \gamma k + \mathrm{i}U_k)(\lambda - \gamma_{-k} - \mathrm{i}U^*_{-k}) - |V_k|^2 = 0. \tag{4.7}$$

Its solution gives a renormalization of a dispersion law in the presence of a condensate. The condensate is stable if $\operatorname{Re}\lambda_k > 0$.

We will now focus on the NLS equation (1.3). Write $\Psi = n^{1/2}\,\mathrm{e}^{\mathrm{i}\phi}$ and find, on ignoring $\hat{\gamma}$,

$$n_t + 2\nabla\cdot n\nabla\phi = 0, \tag{4.8}$$

$$\phi_t + (\nabla\phi)^2 - \alpha n + n^{-1/2}\Delta n^{1/2} = 0. \tag{4.9}$$

Eqs. (4.8), (4.9) are canonical,

$$\frac{\partial n}{\partial t} = \frac{\delta H}{\delta\phi}, \quad \frac{\partial\phi}{\partial t} = -\frac{\delta H}{\delta n}, \tag{4.10}$$

where

$$H = \int\left[n(\nabla\phi)^2 + (\nabla n^{1/2})^2 - \tfrac{1}{2}\alpha n^2\right]\mathrm{d}\boldsymbol{r}. \tag{4.11}$$

Assuming $n = n_0 + \delta n$, $\delta n \ll n_0$, $\phi = \alpha n_0 + \Phi$, and setting δn, Φ proportional to $\mathrm{e}^{\mathrm{i}\Omega_k t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$, we find

$$\Omega_k^2 = k_0^2 k^2 + k^4, \quad k_0^2 = -2\alpha n_0. \tag{4.12}$$

We see that the presence of condensate changes the dispersion law substantially. In the limit $k \ll k_0$ and $k \gg k_0$, we find

$$\Omega_k \simeq k k_0 + \frac{1}{2}\frac{k^3}{k_0} + \dots, \quad k \ll k_0, \tag{4.13}$$

$$\Omega_k \simeq k^2 + \tfrac{1}{2}k_0^2 + \dots, \qquad k \gg k_0. \tag{4.14}$$

Another effect of the condensate is that it changes the nature of the wave interaction. Expanding the Hamiltonian in powers of δn, we get

$$H = H_0 + H_1 + H_2 + \dots, \tag{4.15}$$

where

$$H_0 = \int \left(n_0 (\nabla\phi)^2 + \frac{1}{4n_0} (\nabla\,\delta n)^2 + \tfrac{1}{2} (\delta n)^2 \right) \mathrm{d}\boldsymbol{r}, \tag{4.16}$$

$$H_1 = \int \left(\delta n (\nabla\phi)^2 - \frac{\delta n}{4n_0^2} (\nabla\,\delta n)^2 \right) \mathrm{d}\boldsymbol{r}, \tag{4.17}$$

$$H_2 = \frac{1}{4n_0^3} \int \delta n^2 (\nabla\,\delta n)^2 \,\mathrm{d}\boldsymbol{r}. \tag{4.18}$$

The Hamiltonian is now an infinite series, and the first non-quadratic term H_1 is cubic in the dependent variables. Introducing

$$\delta n_k = \left(n_0 k^2 / \Omega_k \right)^{1/2} (a_k + a^*_{-k}),$$
$$\phi_k = \tfrac{1}{2}\mathrm{i} \left(\Omega_k / n_0 k^2 \right)^{1/2} (a_k - a^*_{-k}), \tag{4.19}$$

where δn_k and ϕ_k are the Fourier transforms of δn and ϕ respectively, we diagonalize the quadratic part of the Hamiltonian H_0 and obtain

$$H_0 = \int \Omega_k a_k a_k^* \,\mathrm{d}k. \tag{4.20}$$

For H_1 we get

$$H_1 = \int V_{kk_1k_2} (a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^*)\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2$$
$$+ \tfrac{1}{3} \int U_{kk_1k_2} (a_k^* a_{k_1}^* a_{k_2}^* + a_k a_{k_1} a_{k_2})\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2, \tag{4.21}$$

where

$$V_{kk_1k_2} = \frac{1}{4n_0^{1/2}(2\pi)^{d/2}} \left[\left(\frac{k^2 \Omega_{k_1} \Omega_{k_2}}{\Omega_k k_1^2 k_2^2} \right)^{1/2} (k_1 k_2) + \left(\frac{k_1^2 \Omega_k \Omega_{k_2}}{\Omega_{k_1} k^2 k_2^2} \right)^{1/2} (k k_2) + \left(\frac{k_2^2 \Omega_k \Omega_{k_1}}{\Omega_{k_2} k^2 k_1^2} \right)^{1/2} (k k_1) \right.$$
$$\left. + \left(\frac{k^2 k_1^2 k_2^2}{\Omega_k \Omega_{k_1} \Omega_{k_2}} \right)^{1/2} \left[(k_1 k_2) - k^2 \right] \right]. \tag{4.22}$$

We can compute $U_{kk_1k_2}$ and the higher terms in the Hamiltonian but since we do not use them, we will not write explicit expressions for them.

Again, we introduce the two-point correlation function

$$\langle a_k a_{k'}^* \rangle = n_k \delta(\boldsymbol{k} - \boldsymbol{k}'). \tag{4.23}$$

It is easy to find from (4.19) that

$$\epsilon = \frac{\langle \delta n^2 \rangle}{2n_0^2} = \frac{1}{n_0} \int \frac{k^2}{\Omega_k} n_k \, \mathrm{d}\boldsymbol{k}. \tag{4.24}$$

If the condition $\epsilon \ll 1$ is satisfied, it is reasonable to use the three-wave kinetic equation to describe the weak turbulence in the presence of a condensate. We find

$$\frac{\partial n_k}{\partial t} + 2\gamma_k n_k = \mathrm{st}(n, n), \tag{4.25}$$

where

$$\begin{aligned} \mathrm{st}(n,n) = {} & 2\pi \int |V_{kk_1k_2}|^2 \, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \, \delta(\Omega_k - \Omega_{k_1} - \Omega_{k_2}) \, (n_{k_1} n_{k_2} - n_k n_{k_1} - n_k n_{k_2}) \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \\ & - 2\pi \int |V_{k_1kk_2}|^2 \, \delta(\boldsymbol{k}_1 - \boldsymbol{k} - \boldsymbol{k}_2) \, \delta(\Omega_{k_1} - \Omega_k - \Omega_{k_2}) \, (n_{k_1} n_{k_2} + n_k n_{k_1} - n_k n_{k_2}) \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \\ & - 2\pi \int |V_{k_2kk_1}|^2 \, \delta(\boldsymbol{k}_2 - \boldsymbol{k} - \boldsymbol{k}_1) \, \delta(\Omega_{k_2} - \Omega_k - \Omega_{k_1}) \, (n_{k_1} n_{k_2} + n_k n_{k_2} - n_k n_{k_1}) \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2. \end{aligned} \tag{4.26}$$

We are only interested in the value of $|V_{kk_1k_2}|^2$ on the resonant manifold,

$$\Omega_k = \Omega_{k_1} + \Omega_{k_2}, \quad \boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2. \tag{4.27}$$

In particular, we find that it depends only on the moduli of the wavevectors $\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2$. For a range of wavenumbers in the vicinity of k_0, the condition $\epsilon \ll 1$ is sufficient for the applicability of eq. (4.25). It is interesting, however, to note the limiting cases $k \ll k_0$ and $k \gg k_0$. If $k \ll k_0$, Ω_k is almost a linear function of k. In this case (the case of weak dispersion) the resonant conditions (4.27) are satisfied only for almost parallel wavevectors. It allows us to simplify the expression for $V_{kk_1k_2}$ to

$$V_{kk_1k_2} = \frac{3}{4(2\pi)^{d/2}} (k_0/n_0)^{1/2} (kk_1k_2)^{1/2}. \tag{4.28}$$

In the opposite case $k \gg k_0$, the resonance conditions (4.27) are satisfied for almost orthogonal vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$, so $(\boldsymbol{k}_1 \cdot \boldsymbol{k}_2) \simeq \frac{1}{4}k_0^2$. As a result, the leading term in (4.22) cancels and, after some calculation, we obtain

$$V_{k,k_1k_2} = \frac{k_0^2}{2n_0^{1/2}(2\pi)^{d/2}} = \frac{n_0^{1/2}}{(2\pi)^{d/2}}. \tag{4.29}$$

In both cases, $V_{kk_1k_2}$ is homogeneous and we can apply the theory of Kolmogorov spectra for the three-wave resonance weak turbulence described by eq. (4.25).

After averaging on angles in (4.25), we obtain

$$\frac{\partial N_\omega}{\partial t} + 2\gamma_\omega N_\omega = T(n), \tag{4.30}$$

$$T(n) = \int_{\Delta_1} S(\omega,\omega_1,\omega_2)\,(n_{\omega_1}n_{\omega_2} - n_\omega n_{\omega_1} - n_\omega n_{\omega_2})\,\delta(\omega-\omega_1-\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2$$
$$+ \int_{\Delta_2} S(\omega_1,\omega,\omega_2)\,(n_{\omega_1}n_{\omega_2} + n_\omega n_{\omega_1} - n_\omega n_{\omega_2})\,\delta(\omega_1-\omega-\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2$$
$$+ \int_{\Delta_3} S(\omega_2,\omega,\omega_2)\,(n_{\omega_1}n_{\omega_2} + n_\omega n_{\omega_2} - n_\omega n_{\omega_1})\,\delta(\omega_2-\omega-\omega_1)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2. \tag{4.31}$$

The regions of integration Δ_1, Δ_2 and Δ_3 are as follows. Δ_1 is all that region in the ω_1, ω_2 plane where $0<\omega_1<\infty$, $0<\omega_2<\infty$ where $\omega_1+\omega_2=\omega$, which is simply the line $\omega_2=\omega-\omega_1$ from $\omega_1=0$ to $\omega_1=1$. As an integral in ω_1 it should be thought of as a directed integral in the direction that $\mathrm{d}\omega_1$ is a positive increment because $\mathrm{d}\omega_1$ means the length of the incremental interval. Δ_2 is the line $\omega_2=\omega_1-\omega$ which joins $\omega_1=\omega$ to $\omega_1=\infty$ and Δ_3 is the line segment $\omega_2=\omega+\omega_1$ from $\omega_1=0$ to $\omega_1=\infty$. The coefficient $S(\omega,\omega_1,\omega_2)$ is given by

$$S(\omega,\omega_1,\omega_2) = 2\pi |V_{kk_1k_2}|^2 \langle \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\rangle (kk_1k_2)^{d-1} \frac{\mathrm{d}k}{\mathrm{d}\omega}\frac{\mathrm{d}k_1}{\mathrm{d}\omega_1}\frac{\mathrm{d}k_2}{\mathrm{d}\omega_2} \tag{4.32}$$

and N_ω is defined as before. In averaging we took into account that, due to resonance conditions (4.27), the function V_{k,k_1,k_2} depends only on the moduli of $\boldsymbol{k}$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. The value of $\langle \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\rangle$ depends on dimension d. For $d=2$, we have (see appendix A):

$$\langle \delta(\boldsymbol{k}-k_1-k_2)\rangle = \frac{\pi}{\Delta_{kk_1k_2}}. \tag{4.33}$$

Here $\Delta_{kk_1k_2}$ is an area of a triangle with sides $\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2$,

$$\Delta_{kk_1k_2} = \tfrac{1}{4}\sqrt{2k_1^2k_2^2 + 2k^2k_1^2 + 2k^2k_2^2 - k^4 - k_1^4 - k_2^4}. \tag{4.34}$$

In the weak dispersion case, $k \ll k_0$, the vectors $\boldsymbol{k}$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are almost parallel, and $\Delta_{k,k_1,k_2} \to 0$ as $k/k_0 \to 0$. To leading order,

$$\Delta_{k,k_1,k_2} \simeq (\sqrt{3}/2k_0)kk_1k_2,$$

and

$$S(\omega,\omega_1,\omega_2) = (3\sqrt{3}/16\pi)\omega\omega_1\omega_2. \tag{4.35}$$

In the opposite case $k \gg k_0$, $\Delta \simeq \frac{1}{2}k_1k_2$ and

$$S(\omega,\omega_1,\omega_2) \simeq \tfrac{1}{8}n_0(\omega_1\omega_2)^{-1/2}. \tag{4.36}$$

We now look at several classes of solutions, (i) thermodynamic equilibria, (ii) Kolmogorov spectra and (iii) time-dependent, self-similar relaxation to equilibrium states.

(i) Thermodynamic equilibria

The theory of weak turbulence is simpler for three-wave interactions than for four-wave case because the equation

$$\mathrm{st}(n,n)=0 \tag{4.37}$$

has only a one-parameter thermodynamic solution,

$$n_k = T/\omega_k. \tag{4.38}$$

(ii) Kolmogorov spectra

Eq. (4.25) has only one integral of motion $E=\int \omega_k n_k \,\mathrm{d}\boldsymbol{k}$. It can have, therefore, only one Kolmogorov spectrum. Suppose that $\omega = k^\alpha$ is a power function and $V_{kk_1k_2}$ is a homogeneous function of degree β invariant with respect to rotations. Then

$$V_{\epsilon k,\epsilon k_1,\epsilon k_2} = \epsilon^\beta V_{k,k_1,k_2}. \tag{4.39}$$

The function $S(\omega,\omega_1,\omega_2)$ is also homogeneous,

$$S(\epsilon\omega,\epsilon\omega_1,\epsilon\omega_2) = \epsilon^\gamma S(\omega,\omega_1,\omega_2), \tag{4.40}$$

where, from (4.32),

$$\gamma = 2(\beta + d)/\alpha - 3 - \delta, \tag{4.41}$$

where δ is the extra contribution coming from the fact that $\Delta_{kk_1k_2}$ may be zero as in the weak dispersion case when the three vectors $\boldsymbol{k}$, $\boldsymbol{k}_1$, and $\boldsymbol{k}_2$ are almost parallel. In that case $\delta=\frac{1}{2}$. For the strong dispersion case $\delta = 0$.

We now proceed to find the pure Kolmogorov solution in a manner similar to that used in section 3. Assuming that $n_\omega = \omega^{-x}$, we make the change of variables,

$$\omega_1 = \omega_2/\omega_1', \quad \omega_2 = \omega\omega_2'/\omega_1' \tag{4.42}$$

for the second integral in (4.31) which maps Δ_2 onto Δ_1 and the change of variables

$$\omega_1 = \omega\omega_1'/\omega_2', \quad \omega_2 = \omega^2/\omega_2' \tag{4.43}$$

for the third integral in (4.31) which maps Δ_3 onto Δ_1. As a result, we obtain

$$T(n) = \int_{\Delta_1} S(\omega,\omega_1,\omega_2)\,(\omega\omega_1\omega_2)^{-x}(\omega^x - \omega_1^x - \omega_2^x)\left[1 - \left(\frac{\omega_1}{\omega}\right)^y - \left(\frac{\omega_2}{\omega}\right)^y\right]\delta(\omega-\omega_1-\omega_2)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2, \tag{4.44}$$

where $y = 2x - \gamma - 2$. The expression (4.44) is zero if either $x = 1$ (thermodynamic spectrum) or $y = 1$

(Kolmogorov spectrum). The second solution,

$$x_2 = (\beta + d)/\alpha = \tfrac{1}{2}(\gamma + 3), \tag{4.45}$$

is a Kolmogorov spectrum. In terms of wavenumber it has the form

$$n_k = \frac{aP^{1/2}}{k^{\beta+d}}. \tag{4.46}$$

Here P is the flux of energy, defined by the relation

$$P = -\int_0^{\omega} \omega T(n)\,\mathrm{d}\omega, \tag{4.47}$$

and a is a constant given by

$$\frac{1}{a^2} = 2\int_0^{1/2} \frac{S(1,\xi,1-\xi)}{\xi^{x_2}(1-\xi)^{x_2}} \left[1 - \xi^{x_2} - (1-\xi)^{x_2}\right]\left[\xi \log\left(\frac{1}{\xi}\right) + (1-\xi)\log\left(\frac{1}{1-\xi}\right)\right]\mathrm{d}\xi. \tag{4.48}$$

The Kolmogorov spectrum will only exist if $a^2 > 0$ or if $x_2 > 1$. The turbulence is local if the integral (4.44) converges.

Let us consider the case of turbulence on the NLS equation with a condensate. In the limiting case $k \ll k_0$, we have from (4.2) $\gamma = 3$. Hence $x_2 = 3$, and so

$$n_k = \frac{aP^{1/2}}{k^3}. \tag{4.49}$$

In the opposite limiting case $k \gg k_0$, eq. (4.26) has no power Kolmogorov solution because in this case from (4.36), $\gamma = -1$, $x_2 = 1$ which means $a^2 = \infty$. What this means is that the Kolmogorov solution has now the more complicated form, analogous to the corrections found in section 3,

$$n_k \sim \frac{P^{1/2}}{k^2} f(\log k), \tag{4.50}$$

where $f(\log k)$ changes slowly with k. However, we wish to make one further point. Considering the integral for total energy

$$E = \int \omega_k n_k \,\mathrm{d}\boldsymbol{k} = aP^{1/2} \int 1/(k^{\beta-\alpha+d})\,\mathrm{d}\boldsymbol{k}, \tag{4.51}$$

we observe two different possibilities. (1) The integral (4.51) *converges* at $k \to \infty$. This occurs if $\beta > \alpha$. In this case the region of large k has a "finite capacity", and a weak turbulence is qualitatively similar to a turbulence in an incompressible fluid. (2) The integral (4.51) *diverges* at $k \to \infty$. This situation takes place if $\alpha \geq \beta$ and in particular for the special case (4.49). Now the region $k \to \infty$ can contain an arbitrary quantity of energy. This situation has no direct analogies in the theory of hydrodynamic turbulence and is, in some sense, simpler. As in section 3, the nonstationary behavior consisting of a front in whose wake lies the Kolmogorov spectrum can be found as follows.

(iii) Relaxation to equilibrium states

Eq. (4.25) at $\hat{\gamma}=0$ has a family of self-similar solutions:

$$n=\frac{1}{t^{a}}n_0\left(\frac{k}{t^{b}}\right). \tag{4.52}$$

Here the indices a and b are connected by one relation found by comparing the two sides of eq. (4.25),

$$a=1+(2\beta-\alpha+d)b. \tag{4.53}$$

To find an additional relation between a and b, we must specify the time dependence of the total energy.

First, suppose that we have a constant source of energy at small k. Then, the energy grows linearly in time. Consequently,

$$a+1=(\alpha+d)b \tag{4.54}$$

and, therefore,

$$b=\frac{1}{\alpha-\beta},\quad a=\frac{\beta+d}{\alpha-\beta}. \tag{4.55}$$

The solution (4.52) has the asymptotic behavior $n\to k^{1/(\beta+d)}$ the weak turbulence Kolmogorov spectrum (4.51) in the limit $k/[1/(\alpha-\beta)]\to 0$ and therefore the advancing front in k space leaves the Kolmogorov spectrum (4.51) in its wake. If $\beta\to\alpha$, the speed of this front tends to infinity, and when $\beta>0$, the energy leaks "through infinity" in a finite time. In this case, the energy is only a "formal" integral of eq. (4.25).

We now look for a second self-similar solution valid in the absence of energy sources. For $\alpha>\beta$, this solution conserves energy. Instead of (4.54), we get

$$a=(\alpha+d)b \tag{4.56}$$

so that

$$b=\frac{1}{2(\alpha-\beta)},\quad a=\frac{\alpha+d}{2(\alpha-\beta)}.$$

The maximum of the wave spectrum goes to infinity like $k\sim t^{1/2(\alpha-\beta)}$. The total number of particles $N=\int n_k\,\mathrm{d}k$ tends to zero according to the law

$$N\sim t^{-\alpha/2(\alpha-\beta)}. \tag{4.57}$$

In the particular case of NLS equation, in the limit $k\gg k_0$, we have $\alpha=2$, $\beta=0$ and

$$n(k,t)=\frac{1}{t}n_0\left(\frac{k}{t^{1/4}}\right). \tag{4.58}$$

This solution describes how the wave field relaxes to a thermodynamic equilibrium consisting of the pure condensate $n = n_0\,\delta(k)$. For this solution $N = n_0$ and $E = 0$. The total number of particles in (4.58) tends to zero as $t \to \infty$ like $N \sim t^{-1/2}$. It means that all the particles at $t = \infty$ will be concentrated in the condensate, all the energy will go to infinity in wavenumber space, and the energy density in any finite domain tends to zero. These results were also obtained by Pomeau [18].

In the case of a positive nonlinearity $\alpha = 1$, the dispersion relation (4.12) is

$$\omega_k = \pm\left(-k_0^2 k^2 + k^4\right)^{1/2}. \tag{4.59}$$

Perturbations with wavenumbers $k^2 < k_0^2$ are unstable and grow exponentially. As was mentioned in section 1, the formation of collapses is the result of this instability. It is important to stress that this instability is a very strong effect. It is incorrect to think that it just moderately changes the theory of a condensate. As will be shown in our numerical experiment (see section 6), the modulation instability demolishes the condensate so effectively that the resulting wave spectrum has no growth in the vicinity of $k = 0$. The resulting strongly nonlinear behavior in the unstable case is the occurrence of collapse events. In section 5 we study a collapsing filament and derive a formula for the number of particles lost per event.

5. Theory of the individual collapse event

We consider

$$\psi_t - \mathrm{i}\nabla^2\psi - \mathrm{i}|\psi|^{4/d}\psi = -\epsilon|\psi|^{4(1+s)/d}\psi - \int\gamma(k)\,\hat{\psi}(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}, \tag{5.1}$$

with ϵ, $s > 0$ and $\gamma(k) = k^2 h(k/k_\mathrm{d})$, where h is a cutoff function which makes a smooth transition from zero to a constant value as k increases through k_d. Our aim is to determine the influence of damping on the collapsing solution. The effect is not small because the number of particles $\int|\psi|^2\,\mathrm{d}\boldsymbol{r}$ contained in a collapsing filament is precisely the minimum number of particles required to sustain the collapse towards its singular state. Therefore any damping causes some loss of power and arrests the collapse. We will show that in the limit of small nonlinear damping, i.e. $0 < \epsilon \ll 1$ and large $k_\mathrm{d} \gg 1$ that the number of particles $\Delta N = \int_{-\infty}^{\infty}\mathrm{d}t\,(\partial|\psi|^2/\partial t)\,\mathrm{d}\boldsymbol{r}$ lost in a collapse event is given by a calcuable quantity whose ϵ and k_d dependence is given by

$$\Delta N \sim (\ln\ln\epsilon^{-1})^{-2} \quad \text{or} \quad (\ln\ln k_\mathrm{d})^{-1}. \tag{5.2}$$

This result strictly means that in the limits $\epsilon \to 0$ and $k_\mathrm{d} \to \infty$ no particles are lost but in practice for large ranges of ϵ and k_d, $(\ln\ln\epsilon^{-1})^{-2}$ and $(\ln\ln k_\mathrm{d})^{-1}$ are so slowly varying as to be almost constant. The reason for this strange loss law is the anomalous behavior of the collapsing filament in the zero viscosity limit. Near the point $r = 0$, $t = t_0$ of collapse,

$$|\psi| \to \frac{1}{f(t)}R\left(\frac{r}{f(t)}\right), \tag{5.3}$$

where

$$f(t) \sim (t_0 - t)^{1/2}\left[\ln\ln(t_0 - t)^{-1}\right], \tag{5.4}$$

as $t \to t_0$. We shall now derive (5.2) in several stages.

5.1. Preliminaries

The equation

$$i\psi_t + \Delta\psi + |\psi|^{4/d}\psi = 0, \tag{5.5}$$

where d is dimension, has two main integrals of motion, number of particles

$$N = \int |\psi|^2 \, d\boldsymbol{r}, \tag{5.6}$$

and energy

$$H = \int \left(|\nabla\psi|^2 - \frac{d}{d+2}|\psi|^{4/d+2} \right) d\boldsymbol{r}. \tag{5.7}$$

The last one is the Hamiltonian for eq. (5.5), which can be rewritten in the form

$$i\psi_t = \frac{\delta H}{\delta\psi^*}. \tag{5.8}$$

Eq. (5.5) admits a set of special solutions of a form

$$\psi = \lambda^{2/d} R(\lambda \boldsymbol{r})\, e^{i\lambda^2 t}. \tag{5.9}$$

Here $\lambda > 0$ is a parameter, $R(\boldsymbol{\eta})$ is a real function, obeying the equation

$$\Delta R - R + R^{4/d+1} = 0. \tag{5.10}$$

Regular and localized, i.e. $R \to 0$, at $|\boldsymbol{\eta}| \to \infty$, solutions of (5.10) are solitons. In the one-dimensional case, the equation

$$R_{xx} - R + R^5 = 0 \tag{5.11}$$

has a unique (up to a shift of x) solution,

$$R = \frac{3^{1/4}}{(\cosh 2x)^{1/2}}. \tag{5.12}$$

If $d > 1$, eq. (5.10) has an infinite set of solutions. The simplest one, $R_0(\boldsymbol{\eta})$, has the maximal symmetry and has no zeros. The parameter λ is a characteristic inverse width of a soliton. Substituting (5.9) into

(5.6) we can find that N does not depend on λ for any soliton solution. It depends only on the type of soliton and it is minimal for the simplest soliton $R_0(\eta)$. The corresponding value of N_0 is the so-called *critical particle number* depending only on the dimension d. It is also easy to show that for any soliton solution,

$$H \equiv 0. \tag{5.13}$$

Let us introduce the self-adjoint linear operator

$$L\psi = \Delta\psi - \psi + \left(\frac{4}{d} + 1\right) R^{4/d} \psi. \tag{5.14}$$

Substituting (5.9) into (5.5), differentiating by λ and putting $\lambda = 1$, we get

$$LR_1 = 2R_0. \tag{5.15}$$

Here

$$R_1 = \frac{\partial}{\partial\lambda} \lambda^{2/d} R_0(\lambda r)\bigg|_{\lambda=1} = \frac{2}{d} \frac{1}{r^{d/2-1}} \frac{\partial}{\partial r} r^{d/2} R_0. \tag{5.16}$$

Define, for two real functions, the scalar product

$$\langle A|B\rangle = \int AB \,\mathrm{d}\boldsymbol{r}. \tag{5.17}$$

It follows that

$$\langle R_0|R_1\rangle = 0. \tag{5.18}$$

It is simply another form of the identity $\partial N/\partial\lambda = 0$. Let two localized functions be connected by

$$LX = Y. \tag{5.19}$$

Multiplying (5.19) by R_1 and using (5.15) we find

$$\langle R_0|X\rangle = \tfrac{1}{2}\langle R_1|Y\rangle. \tag{5.20}$$

The Hamiltonian integral (5.7) is not in general positive definite. It is positive only for field of small enough amplitude. It is very easy to prove that $H > 0$ if $N < N_0$.

5.2. *The lens transformation*

Eq. (5.5) has global solutions to the Cauchy initial value problem if $N < N_0$. If $N > N_0$, a singularity, local in both space and time is formed. We will assume that this singularity (collapse) takes place at $r = 0$ and at $t = t_0$. In formulating a theory of collapse, we will take advantage of the fact that neither N nor H depend on the size of the soliton $(1/\lambda)$. This fact leads us to the conjecture that the leading term in the asymptotic behavior of the solutions as $t \to t_0$ in the vicinity of collapse is a *compressing simple soliton*, a conjecture confirmed by numerical experiments [16–18]. We therefore introduce the change of variables called the lens transformation,

$$\psi = g^{d/2}\phi(\xi,\tau)\,e^{i\tau}, \quad \xi = gr. \tag{5.21}$$

Here g is an as yet unknown function of t, while τ is a new time variable defined by

$$d\tau/dt = g^2, \tag{5.22}$$

so that

$$\tau = \int g^2\,dt. \tag{5.23}$$

As $t \to t_0$, g will tend to infinity so fast that the integral (5.23) is divergent and therefore the collapse occurs at $\tau = \infty$. To avoid any ambiguity we will assume

$$\operatorname{Im}\phi(0,\tau) \equiv 0. \tag{5.24}$$

The function $\phi(\xi,\tau)$ obeys the equation

$$i\phi_\tau + \Delta\phi - \phi + |\phi|^{4/d}\phi + i\alpha(\tau)\left(\tfrac{1}{2}d\phi + \xi\phi_\xi\right) = 0, \tag{5.25}$$

where

$$\alpha = g_\tau/g = g_t/g^3. \tag{5.26}$$

Next we assume the collapse is symmetric so that $\Delta\phi = \phi_{\xi\xi} + [(d-1)/\xi]\phi_\xi$. Now introduce a new change of variables,

$$\phi = \chi\exp\left(-\tfrac{1}{4}i\alpha\xi^2\right) \tag{5.27}$$

and we obtain

$$i\chi_\tau + \Delta\chi - \chi + |\chi|^{4/d}\chi + \beta(\tau)\,\xi^2\chi = 0, \tag{5.28}$$

where

$$\beta = \tfrac{1}{4}\left(\alpha^2 + \alpha_\tau\right) = \tfrac{1}{4}\frac{g_{\tau\tau}}{g} = -\frac{1}{4g}\left(\frac{1}{g}\right)_{tt}. \tag{5.29}$$

If β is constant, we can put $\chi_\tau = 0$ and get a self-similar solution $\chi(\xi)$ of eq. (5.5) satisfying

$$\Delta\chi - \chi + |\chi|^{4/d}\chi + \beta\xi^2\chi = 0. \tag{5.30}$$

For $\beta < 0$, eq. (5.30) has a localized solution. From (5.29), we see that $g(t)$ is bounded and so we have no collapse in this case. For $\beta > 0$, $g \sim e^{2\sqrt{\beta}\tau}$ as $\tau \to \infty$, so $g(t) \simeq c/(t-t_0)^{1/2}$. But eq. (5.30) has no localized solution. The special case $\beta = 0$ is a particularly interesting one. In this case $g \simeq c/(t - t_0)$, $\alpha = p/(t - t_0)$, and we have a collapse. But it is a very peculiar type of collapse because it is unstable with respect to small perturbations. It is important to note in this case, eq. (5.30) for χ coincides (up to trivial transformation) with the initial equation (5.5). It means that the transformation from ψ to χ (for $\beta = 0$) is an intrinsic symmetry of the nonlinear Schrödinger equation (5.5). This transformation exists only in the critical case $s = 4/d$ and is called the lens transformation in the literature.

5.3. The structure of the collapse

We conclude therefore that the collapse is almost, but not quite, self-similar. It is "quasi-self-similar". Both α and β are slowly varying functions of τ, so that

$$\alpha^2 \gg \alpha_\tau \tag{5.31}$$

and the term χ_τ in (5.28) is very small. We will assume also that for small $\beta(\tau)$ which occurs as $\tau \to \infty$, the solution of the equation has series form

$$\chi = \chi_0 + \chi_1 + \dots . \tag{5.32}$$

A difficulty is that it is impossible to use for the zeroth-order term χ_0, the naive stationary equation (5.25), because any solution of this equation, satisfying the condition (5.24), is a real function with the following asymptotic behavior at $\xi \to \infty$,

$$\chi \to \frac{1}{\xi^{d/2}} \cos\left(\tfrac{1}{4}\alpha\xi^2 + c\right),$$

where we have assumed that $\beta \simeq \frac{1}{4}\alpha^2$. Thus, ϕ has in its asymptotic expansion a fast-oscillating term $1/\xi^{d/2} c^* \, e^{-\xi^2 i\alpha/2}$, obviously having no physical sense because we expect only outgoing waves to emanate from $\xi = 0$. To avoid the difficulty we assume that χ_0 satisfies the equation

$$\Delta\chi_0 - \chi_0 + |\chi_0|^{4/d}\chi_0 + \beta\xi^2\chi_0 - iv(\beta)\chi_0 = 0 \tag{5.33}$$

with an additional term $-iv(\beta)\chi_0$, added to its left-hand side, where $v(\beta)$ is chosen so that the asymptotic behavior of χ_0 at $\xi \to 0$ contains only one exponent,

$$\chi \to A(\xi) \exp\left(\tfrac{1}{4} i\alpha\xi^2\right), \tag{5.34}$$

and $A(\xi)$ is some real function. At $\xi = 0$, we have the natural boundary condition

$$\chi_\xi|_{\xi=0} = 0. \tag{5.35}$$

Eq. (5.28) can be rewritten as a Schrödinger equation,

$$-\Delta\chi + U\chi = E\chi, \tag{5.36}$$

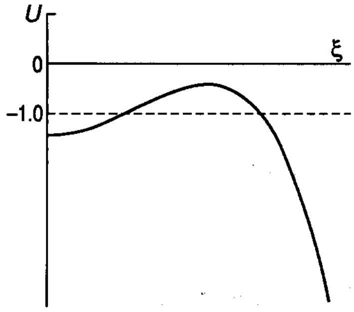

Fig. 2. The effective potential $U(\xi)$ for eq. (5.31b).

with an effective potential

$$U = -|\chi|^{4/d} - \beta\xi^2 \tag{5.37}$$

and eigenvalue

$$E = -1 - \mathrm{i}v(\beta).$$

The boundary problem (5.36), (5.37) is nonself-ajoint and the eigenvalue has an imaginary part. Similar types of nonself-ajoint problems for linear equations are common in nuclear physics. They were introduced first by Gamov in 1936 in connection with the theory of α-decay. The only difference is that in the linear case both (real and imaginary) parts of the eigenvalue are unknown, while in the nonlinear case, the real part of the eigenvalue can be chosen in an arbitrary way. The potential U is plotted in fig. 2. If β is not too small, $v(\beta)$ can be computed numerically. But, as $\beta \to 0$, we can use the WKB approach. In this case for $\xi \gg 1$,

$$\chi \simeq \frac{c}{\xi^{(d-1)/2}} \exp\left(-\int_{\xi_1}^{\xi} [1 + U(\xi)]^{1/2}\, \mathrm{d}\xi \right)$$

and

$$v(\beta) = c \exp\left(-2\int_{\xi_1}^{\xi_2} [1 + U(\xi)]^{1/2}\, \mathrm{d}\xi \right), \tag{5.38}$$

where c is of order 1. Now from the graph of $v(\beta)$ in fig. 2, $1 + U(\xi)$ is zero at $\xi = \xi_1, \xi_2$. At $\beta \to 0$ we can write with exponential accuracy, $U \simeq -\beta\xi^2$, $\xi_1 = 0$, $\xi_2 = 1/\sqrt{\beta}$. Then

$$v(\beta) \simeq \mathrm{e}^{-2\int_0^{1/\beta^{1/2}}} (1 - \beta\xi^2)^{1/2}\, \mathrm{d}\xi \simeq \exp(-\pi/2\beta^{1/2}) \simeq \exp(-\pi/\alpha). \tag{5.39}$$

In (5.39) we omitted an unknown pre-exponent factor. One can see that as far as $\beta(\tau)$ tends to zero, $v(\beta)$ tends to very fast and this justifies including the term $-v\chi_0$ in the zeroth-order equation.

5.4. *The slow time dependence of $\beta(\tau)$*

The equation for the first correction χ_1 has the form

$$\Delta\chi_1 - \chi_1 + \left(\frac{2}{d}+1\right)|\chi_0|^{2/d}\chi_1 + \frac{2}{d}|\chi_0|^{2/d-2}\chi_0^2\chi_1^* + \beta\xi^2\chi_1 - \mathrm{i}v(\beta)\chi_1 = -\mathrm{i}\left(\frac{\partial\chi_0}{\partial\beta}\beta_\tau + v(\beta)\chi_0\right). \tag{5.40}$$

It is sufficient to study this equation in the region where ξ is order 1, where it can be simplified. In this region one can neglect small terms $\beta\xi^2\chi_1$ and $v\chi_1$ and consider χ_0 as a real function. As a result $\chi_1 = -\mathrm{i}p$ is a pure imaginary function satisfying the equation

$$\Delta p - p + \chi_0^{4/d}p = \frac{\partial\chi_0}{\partial\beta}\beta_\tau + v(\beta)\chi_0. \tag{5.41}$$

Without loss of accuracy, we can put $\chi_0 = R_0$. After multiplying (5.41) by R_0 and integrating one gets the solvability condition

$$\beta_t\langle R_0|\partial_{\chi_0}/\partial\beta\rangle = -v(\beta)\langle R_0^2\rangle. \tag{5.42}$$

Now, for $\partial\chi_0/\partial\beta$ we have the equation

$$L\,\partial\chi_0/\partial\beta = -\xi^2 R_0, \tag{5.43}$$

where the operator L is defined in (5.14). Eq. (5.43) belongs to the class (5.19), and we can use (5.20) to obtain

$$\langle R_0|\partial\chi_0/\partial\beta\rangle = -\tfrac{1}{2}\langle\xi^2 R_0|R_1\rangle = (1/d)\langle\xi^2|R_0^2\rangle. \tag{5.44}$$

We obtain finally

$$\frac{\partial\beta}{\partial\tau} + av(\beta) = 0, \quad a = \frac{d\langle R_0^2\rangle}{\langle\xi^2 R_0^2\rangle}, \tag{4.45}$$

or to within our limits of accuracy,

$$\frac{\partial\alpha}{\partial\tau} \simeq -\frac{a}{\alpha}\exp(-\pi/\alpha). \tag{5.46}$$

The function $\alpha(\tau)$ has sophisticated asymptotics $\tau\to\infty$. It turns out that the first term of its asymptotic behavior does not depend on pre-exponent factor in $v(\beta)$. Actually as $\tau\to\infty$,

$$\alpha \simeq \frac{1}{\pi\ln\tau}\cdots. \tag{5.47}$$

Note $\alpha_\tau \ll \alpha^2$ in agreement with our previous assumptions. From (5.26), we find

$$g = g_0 \exp\left(\frac{1}{\pi}\int^\tau \frac{ds}{\ln s}\right),$$

$$g \simeq g_0 \exp\left(1/\pi \int \frac{dt}{\ln t}\right) \sim g_0 \exp\left(\frac{\tau}{\pi \ln \tau}\right). \tag{5.48}$$

As $\tau \to \infty$,

$$\ln \tau \sim \ln \ln g + \ldots .$$

Now, from (5.22),

$$t_0 - t = \int_\tau^\infty d\tau\, g^{-2} \simeq \tfrac{1}{2}\pi g^{-2} \ln(\ln g), \tag{5.49}$$

so that

$$g^2 \simeq \frac{2}{\pi(t_0 - t)} \ln\left[\ln\left(\frac{1}{t_0 - t}\right)\right]. \tag{5.50}$$

Eq. (5.50) was independently derived by Fraiman [19] and by Papanicolau, Sulem, Sulem, Landman and LeMesurier [20, 21]. This was truly an impressive result, which had defied resolution for more than seventeen years. Numerical experiments give a satisfactory confirmation of the behavior (5.50).

So we have shown that structure of collapse in the region $\xi \sim 1$ is close to a compressing soliton. This region is the most inner zone of the collapse domain. The next zone occurs for $1 \gg \xi \gg 1/\beta$ and is described by WKB formula (5.38). In the next region,

$$\frac{1}{\sqrt{\beta}} \gg \xi \gg \xi_{\max}(\tau) \gg g, \tag{5.51}$$

and the WKB approach could be used again. The result should be formulated more conveniently for the function ϕ, where

$$\phi \sim v^{1/2} \xi^{-d/2 + v/\alpha - i/\alpha}. \tag{5.52}$$

The formula (5.52) is correct up to some $\xi_{\max}(\tau)$ that is still unknown. We would conjecture that in the outer region

$$\xi_{\max} < \xi < g$$

the collapse will create some kind of integrable singularity.

5.5. The dependence of the number of particles absorbed per collapse on damping

The central question of interest to us here is the amount of power absorbed in collapsing event. Let us first work this out in the case of the nonlinear dissipation. We have seen that the equation determining the evolution of $\beta = g_{\tau\tau}/4g$ is (5.45). We now ask how this equation is modified with the inclusion of nonlinear damping. The easiest way to derive this is to use the exact conservation law

$$\frac{\partial}{\partial t}\int |\psi|^2\,\mathrm{d}\boldsymbol{r} = -2\epsilon\int |\psi|^{(4+4s)/d+2}\,\mathrm{d}\boldsymbol{r}. \tag{5.53}$$

Now, assuming that the collapse has attained its self-similar shape before the damping sets in and thereby replacing $\psi(\boldsymbol{x},t)$ by (5.21) and (5.27) with $\chi_0(\beta)$ as $\chi_0(0) + \beta\,\partial\chi_0/\partial\beta$ where $\chi_0(0) = R_0$, we find

$$\frac{\partial}{\partial t}\int_0^\infty \left(R_0^2 + 2\beta R_0 \frac{\partial \chi_0}{\partial \beta}\right)\xi^{d-1}\,\mathrm{d}\xi = -2\epsilon g^{2s+2}\int_0^\infty R_0^{4(1+s)/d+2}\xi^{d-1}\,\mathrm{d}\xi. \tag{5.54}$$

But

$$\int R_0 \frac{\partial\chi_0}{\partial\beta}\xi^{d-1}\,\mathrm{d}\xi = \tfrac{1}{2}\int_0^\infty R_0 R_1 \xi^{d+1}\,\mathrm{d}\xi = \frac{1}{d}\int_0^\infty R_0^2 \xi^{d+1}\,\mathrm{d}\xi$$

and defining

$$b = \frac{\int_0^\infty R_0^{4(1+s)/d+2}\xi^{d-1}\,\mathrm{d}\xi}{\int_0^\infty R_0^2\xi^{d+1}\,\mathrm{d}\xi}, \tag{5.55}$$

we obtain, using $g^{-2}\,\partial/\partial t = \partial/\partial\tau$, and adding the combination for $v(\beta)$,

$$\frac{\partial\beta}{\partial\tau} + av(\beta) + \epsilon b g^{2s} = 0, \quad \beta = \frac{1}{4}\frac{g_{\tau\tau}}{g}. \tag{5.56}$$

The total loss of particle number per collapse is given by

$$\Delta N = \int_{-\infty}^{\infty} \Gamma\,\mathrm{d}t, \tag{5.57}$$

where the dissipation rate

$$\Gamma = -\frac{\partial}{\partial t}\int |\psi|^2\,\mathrm{d}\boldsymbol{r} = 2\epsilon\int |\psi|^{4(1+s)/d+2}\,\mathrm{d}\boldsymbol{r} = c\epsilon b g^{2s+2}, \tag{5.58}$$

where $c = (2\Omega_0/d)\int_0^\infty R_0^2\xi^{d+1}\,\mathrm{d}\xi$, with $\Omega_0 = 1$ if $d=1$, $\Omega_0 = 2\pi$ if $d=2$, and $\Omega_0 = 4\pi$ if $d=3$. Therefore

$$\Delta N = c\int_{-\infty}^{\infty} \epsilon b g^{2s}\,\mathrm{d}\tau, \tag{5.59}$$

where g is obtained by solving (5.56). Now, recall that the relation between t and τ is given by (5.22). Observe that if $g = [2\beta_0^{1/4}(t_0 - t)^{1/2}]^{-1}$, that in terms of τ,

$$g = g_0\frac{\exp(2\beta_0^{1/2}\tau)}{\beta_0^{1/4}}, \quad g_0 = \frac{1}{2t_0^{1/2}}, \tag{5.60}$$

if we choose to have $t = 0$ correspond to $\tau = 0$. To calculate the dissipation ΔN, we must calculate $g(\tau)$ from (5.56). However, no significant dissipation occurs until a time τ_0 when ϵg^{2s} becomes of equal size to the other terms in (5.56). Up to that time, the balance in (5.56) involves the first two terms only and this solution we have already discussed. A very important feature is that after a long time, the rate of change of β and α is very small. Indeed for (5.47),

$$\alpha = \frac{g_\tau}{g} \sim \frac{1}{\pi \ln \tau}. \tag{5.61}$$

Therefore, if we wait until the time τ_0 when eg^{2s} is of order β_τ, the subsequent rates of change of α and β occur on the time scale given by

$$\mu = \pi \ln \tau_0. \tag{5.62}$$

Let

$$\tau = \tau_0 + \mu\sigma. \tag{5.63}$$

Then, set

$$\beta = (1/\mu^2)B, \quad \alpha = (1/\mu)A \tag{5.64}$$

and find

$$\frac{1}{\mu^3}B_\sigma + a\exp\left(-\frac{\pi\mu}{2B^{1/2}}\right) + \mu^{1/2}\epsilon g_0^{2s}\frac{\exp\int_0^{\tau_0/\mu+\sigma}4sB^{1/2}\,\mathrm{d}\sigma}{B^{s/2}} = 0, \tag{5.65}$$

and

$$\Delta N = \mu^{3/2}c\int_{-\infty}^{\infty}\epsilon g_0^{2s}\frac{\exp\int_0^{\tau_0/\mu+\sigma}(4sB^{1/2}\,\mathrm{d}\sigma')}{B^{s/2}}\,\mathrm{d}\sigma. \tag{5.66}$$

Now, until $\exp(\int_0^{\tau_0}4s\beta^{1/2}\,\mathrm{d}\tau)$ is sufficiently large to balance $(1/\mu^3)B_\sigma$ and $a\exp(\pi\mu/2B^{1/2})$, no significant dissipation occurs. Therefore we choose $\beta(\tau)$ and $B(\sigma)$ to solve

$$\beta_\tau + av(\beta) = 0, \quad 0 < \tau < \tau_0, \tag{5.67}$$

and then choose τ_0 such that

$$\mu^{1/2}\epsilon\exp\left(\int_0^{\tau_0}4s\sqrt{\beta}\,\mathrm{d}t\right) = \frac{1}{\mu^3}, \tag{5.68}$$

in which case (we can now start calculating dissipation from $\sigma = 0$)

$$\Delta N = \frac{1}{\mu^2}c\int_0^\infty \frac{g_0^{2s}}{B^{s/2}}\exp\left(\int_0^\sigma 4sB^{1/2}\,\mathrm{d}\sigma'\right)\mathrm{d}\sigma$$

and

$$B_\sigma + \mu^3 a \exp\left(-\frac{\pi}{2B^{1/2}}\right) + bg^{2s} = 0, \tag{5.69}$$

where

$$g_{\sigma\sigma} - 4Bg = 0 \tag{5.70}$$

and g behaves as $g_0 \exp(\int_0^\sigma 4sB^{1/2}\,d\sigma)/B^{1/4}$ near $\sigma = 0$. We can verify numerically that B never goes through zero but tends asymptotically to a new constant which is almost independent of μ and ϵ since $\mu^3 \exp(-\pi\mu/2B^{1/2}) \to 0$ as $\mu \to \infty$. From (5.68), we obtain to leading order that

$$\tau_0 = \frac{1}{4s\beta_0^{1/2}} \ln \epsilon^{-1} + \dots \tag{5.71}$$

and therefore

$$\frac{1}{\mu} = \frac{1}{\pi \ln\ln \epsilon^{-1}} + \dots, \tag{5.72}$$

and

$$\Delta N \sim \frac{1}{\mu^2} \sim \frac{1}{\pi^2(\ln\ln\epsilon^{-1})^2}. \tag{5.73}$$

Observe that if the term $v(\beta)$, which causes the $\ln\ln$ behavior, is absent, then there is no need to introduce the time scale as we did in (5.62) and μ is arbitrary and independent of ϵ. In that case, ΔN is

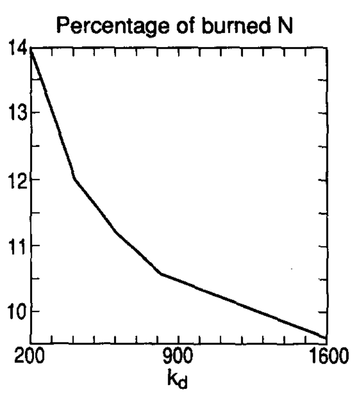

Fig. 3. Percentage of burned out energy $\Delta N(k_d)/N_0$ in 1D single collapse event for damping $\gamma_k = \frac{1}{2}k^2 h(k/k_d)$.

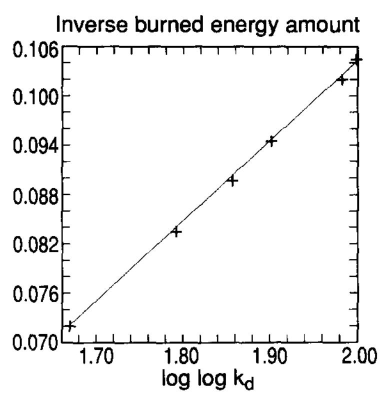

Fig. 4. Inverse amount of burned out energy $1/\Delta N(k_d)$ in 1D single collapse event as a function of $\log(\log k_d)$ for damping $\gamma_k = \frac{1}{2}k^2 h(k/k_d)$.

ϵ independent but of course, as we have emphasized, it is the presence of $v(\beta)$ that allows the localized solutions exist in the first place.

Similar arguments give us the parallel result that in the case where we choose the damping to be linear and concentrated wavenumbers $k > k_d$,

$$\Delta N \sim \frac{1}{\ln \ln k_d}. \tag{5.74}$$

This result is directly confirmed by numerical experiment (see figs. 3 and 4).

6. Numerical integration scheme

We now describe the procedure used to simulate the solution of the NLS equation (6.1) in the presence of linear amplification and damping, nonlinear damping (due to muliphoton absorption) and parametric forcing. The results we report in this paper do not include the last two. The equation is

$$\mathrm{i}\Psi_t + \nabla^2\Psi + \alpha|\Psi|^{4/d}\Psi + \mathrm{i}\hat{\gamma}\Psi + \mathrm{i}\beta|\Psi|^{2m}\Psi = A\Psi^* \,\mathrm{e}^{-2\mathrm{i}\lambda t}. \tag{6.1}$$

The two most important considerations when choosing an integration algorithm are that, in the absence of external forces, the constants of motion N, H and P are conserved and that we obtain a reliable description of the collapsing cavity when it is in its self-similar regime and its width is very small. The last point is very important because it is in the small scales where dissipation takes place that the greatest errors can occur. The source of these errors is aliasing. Power is transferred to higher harmonics through the cubic nonlinearity and thus mistakenly deposited in lower harmonics due to the inability of the grid to distinguish between the wavenumbers above and below $\frac{1}{2}k_m$, where k_m is equal to the number of points used to resolve either of the spatial directions x and y. We take $-\pi < x < \pi$, $-\pi < y < \pi$ and stepsize Δ so that $k_m = \pi/\Delta$ and we usually took this to be 128. To overcome the aliasing error, we divide our wavenumber grid into two regimes: R, $-\frac{1}{2}k_m \le k_x,\ k_y \le \frac{1}{2}k_m$, and R_a, $|k_x| \ge \frac{1}{2}k_m$, $|k_y| > \frac{1}{2}k_m$. At each time step t, we calculate $\hat{\Psi}(\boldsymbol{k},t)$ and $\Psi(\boldsymbol{r},t)$ and then $|\Psi|^2\Psi$ in real space and then its value $(\widehat{|\Psi|^2\Psi})$ in wavenumber space on $R \cup R_a$. The region R contains the true harmonics and the region R_a the aliasing harmonics. In calculating the updated value of $\Psi(\boldsymbol{k},t)$, we exclude the region R_a. Of course, ignoring three-quarters of the information on the grid reduces accuracy but it is necessary. Otherwise, we overestimate the amount of power lost in each collapse event.

The algorithm we use for the pure NLS equation ($\hat{\gamma} = \beta = A = 0$) is

$$\mathrm{i}\frac{\Psi_j^{n+1} - \Psi_j^n}{\tau} + \nabla^2\frac{\Psi_j^{n+1} + \Psi_j^n}{2} + \alpha\frac{|\Psi_j^{n+1}|^2 + |\Psi_j^n|^2}{2}\,\frac{\Psi_j^{n+1} + \Psi_j^n}{2} = 0, \tag{6.2}$$

where τ is the time step, the index $j = (j_x j_y)$ refers to the grid point $x = j_x\Delta$, $y = j_y\Delta$, $-k_m < j_x j_y < k_m$, and the index n refers to the time step. $\nabla^2\Psi_j^n$ means the inverse Fourier transform of $-k^2\hat{\Psi}_k$ where $k(k_x, k_y)$ is the index denoting points on the grid in wavevector space. It is easy to show that

$$N = \Delta^2 \sum_j |\Psi_j^{n+1}|^2 = \Delta^2 \sum_j |\Psi_j^n|^2 \tag{6.3}$$

and

$$H = 4\pi^2 \sum_k k^2 |\hat{\Psi}_k^{n+1}|^2 - \tfrac{1}{2}\alpha\Delta^2 \sum_j |\Psi_j^{n+1}|^4 = 4\pi^2 \sum_k k^2 |\hat{\Psi}_k^n|^2 - \tfrac{1}{2}\alpha\Delta^2 \sum_j |\Psi_j^n|^4. \tag{6.4}$$

To prove (6.3), multiply (6.2) by $(\Psi_j^{n+1} + \Psi_j^n)^*$ and subtract from the expression obtained its complex conjugate. To prove (6.4), multiply (6.2) by $(\Psi^{n+1} - \Psi^n)^*$ and add to this expression its complex conjugate. The linear damping and amplification terms $i\hat{\gamma} \cdot \Psi$ are written in wavevector space as $i\gamma_k \frac{1}{2}(\hat{\Psi}_k^{n+1} + \hat{\Psi}_k^n)$ where $\gamma_k = \gamma_0 + \gamma_p + \gamma_d$, where γ_0, γ_d are damping terms and γ_p is an amplification. Formulas are given in the next sections by eqs. (7.4)–(7.6). The nonlinear damping term which represents multiphoton absorption in the optical context is written as

$$\frac{i\beta}{m+1} \frac{|\Psi_j^{n+1}|^{2m+2} - |\Psi_j^n|^{2m+2}}{|\Psi_j^{n+1}|^2 - |\Psi_j^n|^2} \frac{\Psi_j^{n+1} + \Psi_j^n}{2}$$

and the parametric forcing term is written as $\frac{1}{2}A(\Psi_j^{*n+1} + \Psi_j^{*n})$. One finds (for $m = 2$)

$$\frac{N^{n+1} - N^n}{\tau} + \tfrac{1}{2}A\,\Delta^2 \sum_j \mathrm{Re}\left(\Psi_j^{n+1} + \Psi_j^n\right) \mathrm{Im}\left(\Psi_j^{n+1} + \Psi_j^n\right)$$
$$+ \tfrac{1}{6}\beta\Delta^2 \sum_j \left(|\Psi_j^{n+1}|^4 + |\Psi_j^{n+1}|^2 |\Psi_j^n|^2 + |\Psi_j^n|^4\right) |\Psi_j^{n+1} + \Psi_j^n|^2$$
$$+ 2\pi^2 \sum_k \gamma_k |\hat{\Psi}_k^{n+1} + \hat{\Psi}_k^n|^2 = 0, \tag{6.5}$$

and a similar but much more complicated expression for $H^{n+1} - H^n$. In the limits of small A, β and $\hat{\gamma}$, both quantities are conserved.

We solve the nonlinear implicit algorithm for Ψ_j^{n+1} iteratively as

$$i\frac{\Psi^{n+1,s+1} - \Psi^n}{\tau} + \nabla^2 \frac{\Psi^{n+1,s+1} + \Psi^n}{2} + \alpha \frac{|\Psi^{n+1,s}|^2 + |\Psi^n|^2}{2} \frac{\Psi^{n+1,s} + \Psi^n}{2} + D(\Psi^{n+1,s}, \Psi^n) = 0 \tag{6.6}$$

for $s = 0, 1, \ldots, \sigma$ where σ is that value of s for which $|N^{n+1,s+1} - N^{n+1,s}|$ is first less than $10^{-7}N^n$. We call $\Psi^{n+1,0} = \Psi^n$ and $\Psi^{n+1,\sigma} = \Psi^{n+1}$. The quantity D in (6.6) stands for all the damping, amplification and parametric forcing terms. Notice that both D and the nonlinear term $\alpha|\Psi|^2\Psi$ are evaluated at the previous iterate. To solve (6.6) we proceed as follows. Start with $\hat{\Psi}_k^n$ and $\hat{\Psi}^{n+1,0}$ with k belonging to R. Find Ψ_j^n and $\Psi_j^{n+1,0}$ and compute D and the nonlinear terms. Take their Fourier transform. Because of nonlinearity, there will be nonzero contributions for k belonging to R_a. Ignore them and solve the linear equation (6.6) for $\Psi_k^{n+1,1}$ for k belonging to R only. Repeat a until $s = \sigma$. We now have $\hat{\Psi}_k^{n+1}$ and Ψ_j^{n+1}.

Convergence of the iterative procedure requires an upper bound on the time step τ. For the moment, ignore D and subtract (6.6) from (6.2) and expand the nonlinear term near Ψ^n. We obtain

$$\left(\frac{2i}{\tau} + \nabla^2\right)(\Psi^{n+1} - \Psi^{n+1,s+1}) + 2|\Psi^n|^2(\Psi^{n+1} - \Psi^{n+1,s}) + \Psi^{n\,2}(\Psi^{n+1} - \Psi^{n+1,s})^* = 0,$$

from which we can show that the L_2 norm of $(\Psi^{n+1} - \Psi^{n+1,s+1})$ obeys

$$\|\Psi^{n+1} - \Psi^{n+1,s+1}\| \le \tfrac{3}{2}\max|\Psi^n|^2\tau\|\Psi^{n+1} - \Psi^{n+1,s}\|.$$

Convergence requires

$$\tau < \frac{2}{3\max|\Psi^n|^2} = \tau_c. \tag{6.7}$$

When the extra terms are added, this criterion must be slightly modified; for example, the inclusion of parametric forcing requires us to add $|A|$ to the denominator of τ_c. We chose $\tau = \frac{1}{3}\tau$ and found that it takes about six to eight iterations to converge to the chosen accuracy. Observe that near a collapse, τ_c becomes very small indeed.

7. Discussion of results

We now present the results of numerical simulations in the two-dimensional case. As we have pointed out already, this case is difficult because none of the simpler kinds of stationary solutions obtain and in particular we cannot obtain a pure Kolmogorov spectrum in either the left or right transparency window. On the left, the sign of the flux Q is negative for the pure Kolmogorov spectrum (3.21) because $\gamma = -1$ lies in the interval for which $-\frac{1}{3}I'(x_1)$ in (3.23) is negative. To remedy this problem, we then introduced the finite temperature Kolmogorov solution (3.39) and $\phi(\omega)$ given by (3.46), so that

$$n = \frac{T}{\mu + \omega + aQT^{-3}\omega^2[\ln(\omega/\omega_c)]^2}. \tag{7.1}$$

We emphasize that this solution cannot hold everywhere on the frequency axis $0 < \omega < \infty$ because the correction is not everywhere small. Nevertheless, as we show in figs. 5, 6, and 7, it does seem to hold very well in the left transparency window $\omega_2 < \omega < \omega_0$, where the correction is small with respect to the thermodynamic background.

Such a solution is not unreasonable. Recall that the reason for the failure of the thermodynamic equilibrium solution is that it does not allow the energy to tend to zero, as it must because of finite damping, as $\omega \to \infty$. However, the left transparency window is shielded from the energy sink by the source of new particles and energy at $\omega = \omega_0$. However, we cannot expect a pure thermodynamic equilibrium because of the necessity for a finite flux of particles towards $\omega = 0$. Hence the compromise solution (7.1). We find that (7.1) fits the data very well for both the focusing and defocusing values of α when $\mu = 400$, $T = 10$, and $aQ = 0.25$ and $\omega_c = 1$ over the broad frequency range $60 < \omega < 4100$.

In our simulations, the amplification and damping components of the coefficient $\gamma(k) = \gamma_p(k) + \gamma_d(k) + \gamma_0(k)$ were given by

$$\begin{aligned} \gamma_p(k) &= -\tfrac{1}{30}\left[(k^2 - 60^2)(62^2 - k^2)\right]^{1/2}, && 60 \le k \le 62, \\ &= 0, && 0 \le k < 60,\ k > 62, \end{aligned} \tag{7.2}$$

$$\gamma_d(k) = 0.5k^2 h(k/k_d), \tag{7.3}$$

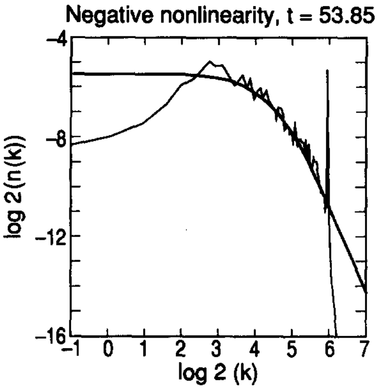

Fig. 5. Time averaged stationary spectrum $|\psi_k|^2$ for run A ($\alpha = -1$ with strong damping at $k = 0$) and theoretical prediction for 2D spectrum (7.1) with $T = 10$, $\mu = 400$, $\omega_c = 1$, $aQ = 0.25$.

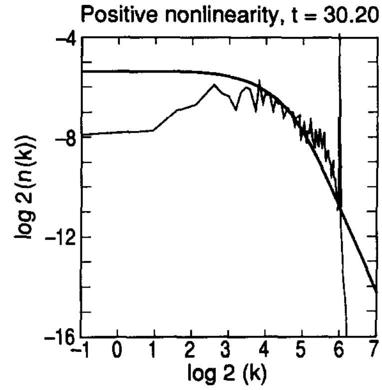

Fig. 6. The same as in fig. 5, but for run B ($\alpha = +1$ with strong damping at $k = 0$).

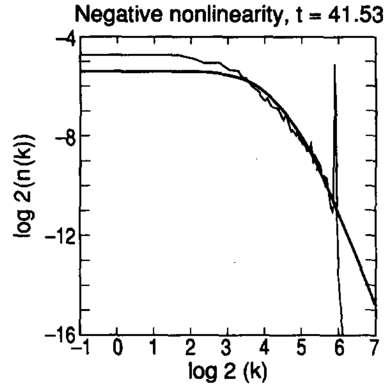

Fig. 7. The same as in fig. 5, but for run C ($\alpha = +1$ with no damping at $k = 0$).

where

$$h(x) = \frac{1}{6}\frac{1}{x^5}\, e^{5[1-(1/x^2)]}, \quad x \le 1,$$
$$= 1 - \tfrac{5}{6}\, e^{1/2(1-x^2)}, \quad x > 1,$$

and

$$\gamma_0(k) = 20(\tfrac{1}{6}k - 1)^2, \quad k \le 6,$$
$$= 0, \qquad k > 6. \tag{7.4}$$

The motivation for these choices was as follows: $\gamma_p(k)$ is the linear growth rate of the parametric instability for spin waves in ferromagnetics, $\gamma_d(k)$ models the linear Landau damping in plasmas, and $\gamma_0(k)$ is chosen to control the growth of the condensate at $k = 0$. The study was carried out in the case of weak instability, i.e. $\gamma_p/k_0^2 \sim 10^{-3}$.

All simulations were initialized by a field $\hat{\Psi}_k(0)$ of small random noise. In figs. 5, 6, 7, we show the left transparency window $60 < \omega < 4100$, for three cases, $\alpha = -1$, $\alpha = +1$ with strong damping at the origin and $\alpha = +1$ with no damping at the origin, with the spectrum (7.1) superposed. These graphs were obtained by averaging the calculated data Ψ_k for a fixed value of $k = |\boldsymbol{k}|$ first over angle by taking Ψ_k at $\boldsymbol{k}_j = k[\cos \frac{1}{4}n\pi, \sin \frac{1}{4}n\pi]$, $n = 0, \ldots, 7$ and then over a time interval approximately ten times the period of the lowest undamped mode $k = 6$. Observe that all spectra are approximately the same and fit the theoretical curve (7.1) very well. What is evident is that the presence of intermittency and collapses does not affect the weak turbulence spectrum in any substantial way. Nevertheless their presence is seen vividly when we examine the particle number dissipation

$$\Gamma(t) = \int \gamma_k |\Psi_k|^2 \, d\boldsymbol{k} \tag{7.5}$$

for the same three cases $\alpha = -1$ (defocusing), $\alpha = +1$ (γ_0 present), $\alpha = +1$ ($\gamma_0 = 0$) (see figs. 8, 9 and

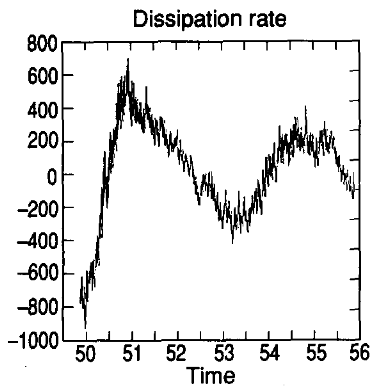

Fig. 8. $\Gamma(t)=\int \gamma_k |\psi_k|^2 \,\mathrm{d}\boldsymbol{k}$ in the stationary state for run A.

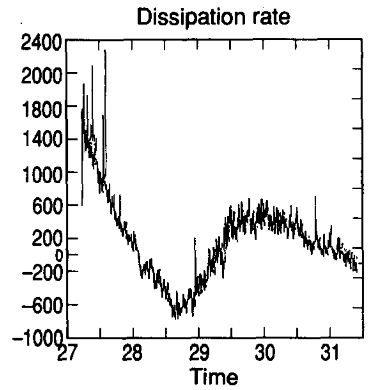

Fig. 9. The same as in fig. 8, but for run B.

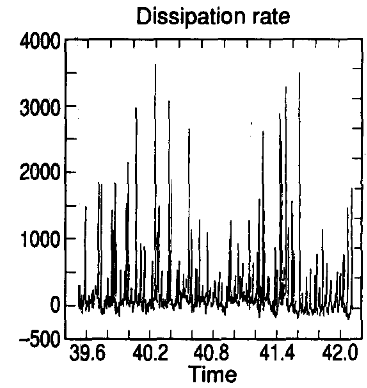

Fig. 10. The same as in fig. 8, but for run C. Note the difference in scale from fig. 8.

10). Because we include contributions from all k values including the neighborhood of $k=k_0$ where γ_k is negative, $\Gamma(t)$ takes on positive and negative values. In the last case, because of no damping at $k=0$, the unstable condensate tries but is unable to form due to the modulational instability and a sequence of collapsing filaments occur at random points in space and time. Although they carry little energy to high wavenumbers (the value of H for a collapsing filament is approximately zero; see section 5), they do carry particle number and deposit it as shown in fig. 10. Notice the difference in scales from figs. 8 and 9. The dissipation rate measures the flux of particles towards high wavenumbers and frequencies. Remember this does not violate our argument that a flux of energy towards $\omega=\infty$ must be accompanied by a flux of particles towards $\omega=0$. This argument is valid for the case of weak turbulence when $E_k=\omega_k n_k$, namely when the principal contribution to H is from its kinetic energy part $\int |\nabla\Psi|^2 \,\mathrm{d}\boldsymbol{r}$. But for the strongly nonlinear collapse, the potential energy component $-\int \frac{1}{2}|\Psi|^4 \,\mathrm{d}\boldsymbol{r}$ is large and balances the kinetic energy so that the energy carried by the nonlinear pulse is zero. In such a situation, there is no contradiction in having a negative Q or a rightward flux of particles.

When there is strong damping at the origin, the number of particles fluctuates weakly as is shown in figs. 11 and 12. The reason for this is that the gain-loss history of a particle in the left transparency window is out of phase with the gain-loss of particles in the damping window near $k=0$. One can

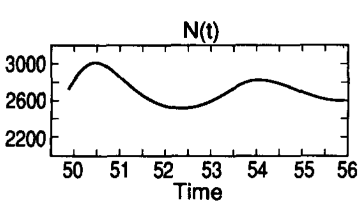

Fig. 11. Time-evolution of $N(t)=\int |\psi|^2 \,\mathrm{d}\boldsymbol{r}$ in the stationary state for run A.

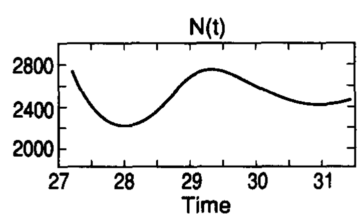

Fig. 12. The same as in fig. 11, but for run B.

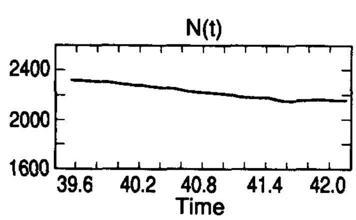

Fig. 13. The same as in fig. 11, but for run C.

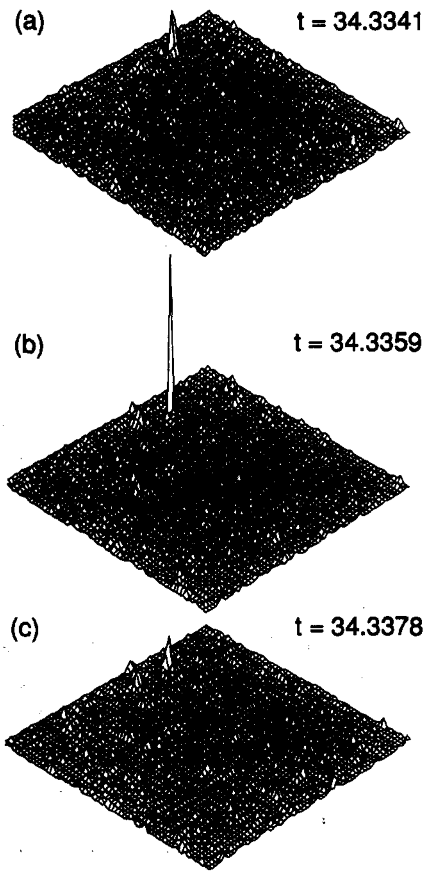

Fig. 14. Snapshots of one collapsing cavity on the weak-turbulent background. (a) Beginning of collapse, (b) peak, (c) end of collapse.

roughly model the situations as follows. Let n_w be a typical particle number density in the window and let n_2 be a typical particle density near $k = 0$. The former gains primarily through a nonlinear field from the neighborhood of k_0 proportional to $n_0^2 n_w$ and loses through a nonlinear interaction $-n_w n_2^2$. On the other hand, particles are lost at k_2 through linear damping γ_0 and gain through a nonlinear interaction $n_w^2 n_2$. Write $\dot{n}_2 = -\gamma_0 n_2 + n_w^2 n_2$, $\dot{n}_w = \gamma n_w - n_w n_2^2$ $\dot{}\,= \mathrm{d}/\mathrm{d}t$, and we obtain the classical predator–prey model. The equilibrium $n_w^2 = \gamma_0$, $n_1^2 = \gamma$ is a center about which there are periodic orbits with periods $2\pi\sqrt{\gamma_0 \gamma}$, which is the behavior we observe. On the other hand, in fig. 13, the presence of many collapses changes the picture so that there is a strong and immediate feedback from n_2 to n_w and, as a result, no oscillations are seen.

In fig. 14 we show a single collapse in three stages of its life. Observe how large the fluctuations are with respect to the weak turbulence background so that one can expect the tails of the almost Gaussian distributions to be lifted if there are many of these events. As a measure of the effects of the collapses on the weak turbulence background, we measured the fourth-order correlation in Fourier space

$$Q^{(4)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = \mathrm{Im}\, \frac{1}{T} \int_0^T \frac{\Psi_{k_1} \Psi_{k_2} \Psi_{k_3}^* \Psi_{k_4}^*}{(S_1 S_2 S_3 S_4)^{1/2}}\, \mathrm{d}t, \quad \boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4, \tag{7.6}$$

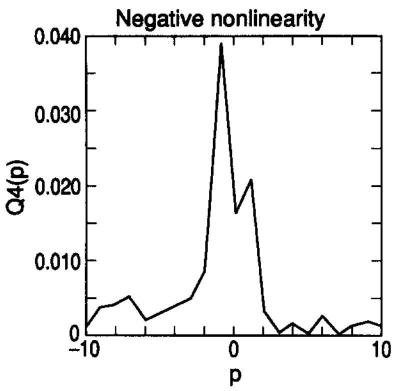

Fig. 15. Imaginary part of fourth-order correlation function for run A.

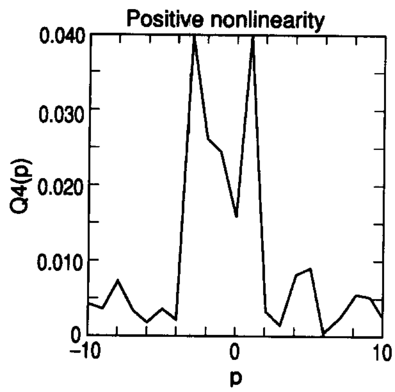

Fig. 16. Imaginary part of fourth-order correlation function for run C.

where

$$S_j = \frac{1}{T}\int_0^T \left|\hat{\Psi}_{k_j}\right|^2 \mathrm{d}t$$

and

$$\boldsymbol{k}_1 = (20,0), \quad \boldsymbol{k}_2 = (-20,0), \quad \boldsymbol{k}_3 = (0,20-p), \quad \boldsymbol{k}_4 = (0,-20+p).$$

If weak turbulence dominates, we expect this function to be nonzero. Only in the vicinity of a resonance $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4$, $\tilde{\omega}_1 + \tilde{\omega}_2 = \tilde{\omega}_3 + \tilde{\omega}_4$ where $\tilde{\omega}_k = \omega_k + \tilde{\Gamma}_k$ the (slightly) amplitude modified frequency. We note that when $\alpha = -1$, there is indeed a peak at $p = -1$ (see fig. 15). The presence of collapses acts like a perturbation of the double spectrum (a single frequency ω has two free travelling waves $e^{\pm i\sqrt{\omega}x - i\omega t}$) and splits the resonance to $p = -5$ and $+1$ (see fig. 16). In a manner we do not yet know how to interpret, the Gaussian nature of the weak turbulence theory is slightly modified by the occurrence of collapse events obeying a Poisson distribution.

8. Conclusion

In this paper, we have presented the weak turbulent theory of a class of Hamiltonian systems of nonlinear Schrödinger type and discussed the relevance of various equilibrium states. Our main attention has been directed to the stationary situation where a balance is achieved between the input of particle number and energy in a narrow window of intermediate wavenumbers $(k_0 - \Delta k,\ k_0 + \Delta k)$ and the losses of particle number and energy through damping at the origin $k = 0$ and infinity $k = \infty$ respectively. One of our main points has been that the conservation of energy and particle number in the ranges of wavenumbers $(0, k_0 - \Delta k)$ and $(k_0 + \Delta k, \infty)$ means that as energy density flows towards high wavenumbers, particle density flows towards low wavenumbers. The consequences of the buildup of particle number near $k = 0$ are nontrivial and lead either to the formation of condensates in the defocusing

($\alpha = -1$) case or to collapsing filaments in the focusing ($\alpha = +1$) case. In the former case, it is necessary to introduce damping at $k = 0$ in order to control the amplitude of the condensate. In the latter case, because the condensate is unstable it is never reached. Instead, collapsing filaments are formed which reverse the flux of particle number and induce a secondary flow which carries particles to high wavenumbers. No damping at $k = 0$ is required.

The primary flows of energy and particle number density were described by weak turbulence theory in which the transfer of spectral densities are achieved by four-wave resonant interactions. The interactions are local in the sense that the transfer functions $\mathrm{st}(n, n, n)$ exists for classes of solutions n_k in the neighborhood of the Kolmogorov spectra. Because all nonresonant interactions are ignored, statistical information is lost, and the solutions depend only on the values of the particle number and energy density fluxes and parameters T and μ, which we identify as temperature and chemical potential. For the primary fluxes, all the usual Kolmogorov assumptions made in the context of hydrodynamic turbulence, and listed below, are valid. In the secondary flux of particle number density, however, things are radically different. The flow in wavenumber space is simply the manifestation of a collapsing filament in physical space in which number density is squeezed from large scales to small scales in a highly organized and coherent manner. No statistical information is lost. Statistical considerations are introduced by the intermittent nature of these events, the uncertainty in time and space as to when and where they occur. They are in all likelihood, governed by a Poisson statistics whose parameters depend on the flux of particle number towards the origin. Because the events involve large amplitude fluctuations, their impact on the probability density function of the field $\Psi(r, t)$ is to cause an elevation in the tails of the distribution. In the remainder of this conclusion, we will argue that this mechanism for intermittency consisting of (i) an inverse cascade associated with the spectral density of an additional finite flux motion invariant, leading to (ii) a continuous formation of large-scale structures which (iii) are intrinsically unstable to a broadband spectrum of perturbing modes which directly and quickly transfer power or energy back to the small dissipative scales, in some cases through highly organized, collapsing solutions, may have broad applications. In particular, we will look at its possible relevance for three-dimensional hydrodynamics.

Before we begin, we want to note some important similarities and differences between two-dimensional optical turbulence and practical manifestations of three-dimensional hydrodynamical turbulence. In optical turbulence, there are situations in which it is reasonable to assume that number and energy density production occurs at scales intermediate between the dissipation scales and overall box size. As long as the ratio of the pumping rate γ_0 to the insertion frequency ω_0 is small, weak turbulence theory obtains to a good approximation and there is room for the resulting inverse cascade. Since collapsing filaments follow from a modulational instability which is most easily triggered when the spectral number density has condensed at a single wavenumber, and in an infinite geometry this wavenumber is $k = 0$, damping at small wavenumbers, as we have shown (cf. compare figs. 9 and 10), severely inhibits intermittency. If the box size were finite of linear dimension L and small enough that $2\pi L^{-1} \sim k_0$, intermittency would still be present because of the tendency of the spectral number density to condense at the smallest wavenumber available to the system. Collapses would form and there would be direct cascades of both energy and number density towards $k = \infty$. Indeed, even if $L \gg 2\pi k_0^{-1}$, as the pumping rate is increased so that the ratio γ_0/ω_0 approaches unity, the rapid growth of N, the average particle number, would mean collapses would form long before weak turbulence theory had a chance to come into play and build an inverse cascade. In that case, too, number density and energy would exhibit direct cascades. Intermittency effects would be large and in all likelihood overcome any relaxation to a Kolmogorov-like spectrum.

The appearance of a good approximation to the Kolmogorov spectrum at high wavenumbers means that, in some sense, three-dimensional hydrodynamics is more like the weak turbulence case with a small frequency of collapses. However, in that context, we are usually dealing with (a) decaying turbulence or (b) turbulence created, usually through instabilities, at scales $2\pi k_0^{-1}$ of the same order as the box size. For (b), the integral scale of the turbulence, the length scale corresponding to the wavenumber at which the spectral energy $E(k)$ is maximum, is the box size. In this case, there will not be any inverse flux from k_0 to smaller wavenumbers because there are none available to the system. Any intermittency which occurs comes from collapsing filaments which arise from large-scale structures at the scale $2\pi k_0^{-1}$ which are either directly forced or else are built through an inverse cascade of the spectral density of the second motion invariant from wavenumbers k such that $k_0 \ll k \ll k_d$. We remark that just because energy is inserted into the system at k_0 does not mean that the instabilities which introduce this energy also produce the type of large-scale structure susceptible to fast instabilities of the collapsing filament type. Furthermore, an increase in box size beyond the integral scale could leave room for an inverse cascade and an increase in the frequency of events which lead to intermittency. It is quite possible that this increase would never reach the point where one loses the Kolmogorov spectrum altogether and so, in this sense, one might say that the behavior at small scales is independent of box size. Nevertheless, the convergence to the Kolmogorov picture would be less uniform in the higher order moments.

Therefore, although the inverse cascade may not be the only mechanism to produce the structures which give rise to intermittent collapses, there is no doubt that it can play some role. We will show that if the spectral density of squared angular momentum is produced at a constant rate at some intermediate wavenumber, then only a finite amount can reside in small scales and the rest must be reassigned to large scales. Further, we suggest that this spectral density has the right character to produce the large-scale structures (the analog of the condensate in optical turbulence) which can lead to collapsing filaments. There is some experimental evidence that the incomplete burnout or dissipation of the filaments themselves also give rise to larger scale eddies in the hydrodynamic context. Douady, Couder and Brachet [22], using a new bubble visualization technique, observe that the short-lived high vorticity filaments, which appear to form spontaneously, disintegrate through helical instabilities which stir large eddies. In what follows, we simply discuss a plausible scenario, motivated by our observations in optical turbulence. Direct verification of the role of an inverse cascade will probably have to await a more complete understanding of the nature of the instabilities which result in intermittent events.

Kolmogorov [12] proposed a universal theory for small-scale eddies in high Reynolds number turbulent flows. It rests on several assumptions, that the transfer of spectral energy to large wavenumbers is local over a window of transparency (the inertial range) in wavenumber space, that statistical information is lost in the cascade so that average flow quantities are scale invariant and determined by the mean rate of energy flux ϵ, which is constant in a statistically steady state. If $E(k)$ is the spectral energy ($\overline{u^2} = \int E(k)\,\mathrm{d}k$) then, for isotropic turbulence, the equation for $E(k)$ is

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k) + f(k), \tag{8.1}$$

where $T(k)$ is the energy transfer integral given by a linear functional of the third-order moment in velocities. It is called a transfer integral because it neither produces nor dissipates energy and it has the property that $\int T(k)\,\mathrm{d}k = 0$. The terms $f(k)$ and $-2\nu k^2 E(k)$ represent the production and dissipation of energy. The former could be proportional to $E(k)$ if energy is introduced by an instability process, but we assume that its domain of support is confined to a narrow window. Between the range of wavenumbers $(k_0 - \Delta k, k_0 + \Delta k)$, at which energy is produced, and the dissipation range $R_d = (\epsilon \nu^{-3})^{1/4}$, where the $-2\nu k^2 E(k)$ term is important, we assume there exists a window of transparency, the inertial range, where both $f(k)$ and $-2\nu k^2 E(k)$ can be ignored. In this range, we assume the turbulence relaxes to a stationary state $E_t = T(k) = 0$, but we will also assume that, in the neighborhood of the solutions which realize this state, $T(k)$ exists. This means that the interactions are local in the sense that the integrand of $T(k)$ ($T(k)$ is an integral over a third-order moment) must decay sufficiently fast as $|k' - k|$; k', the integration variable, becomes large so that the integral exists. In the absence of dissipation and forcing, (8.1) has conservation law form $E_t = T(k) = -P'(k)$, where $P(k)$ is the energy flux, positive when the flow of energy is to small scales and large wavenumbers. We remind the reader that we call $\int_a^b E(k)\,dk$ (usually $a = 0$, $b = \infty$) a true constant of the motion if $P(k) = 0$ at both $k = a$ and $k = b$, so that the total energy is trapped in the interval (a, b) for all time because of zero flux through the boundaries. However, the presence of viscosity makes these thermodynamic equilibria uninteresting because there is a constant leakage of energy through to the dissipation scales. Therefore, in any interval (a, b) of the transparency window where $a > k_0$, $b < k_d$, $\partial(\int_a^b E(k)\,dk)/\partial t$ is zero by virtue of the fact that the fluxes $P(k)$ at $k = a$ and $k = b$ are not zero but the same. In this case, we call $\int_a^b E(k)\,dk$ a finite flux constant of the motion. In this paper, it is the finite flux constants which are important. Moreover, within the window of transparency, the interval (a, b) is arbitrary, so that then $P(k)$ is a constant and equal to the mean dissipation rate ε throughout the inertial range. From dimensional considerations $E(k)(l^3t^{-2})$, $k(l^{-1})$ and $\varepsilon(l^2t^{-3})$ are related by the well-known Kolmogorov law $E(k) = c_2\varepsilon^{2/3}k^{-5/3}$, where c_2 is a universal constant. The relevance of such solutions to turbulence rests on the Kolmogorov assumption that the energy dissipation rate $\varepsilon = -d\langle u^2\rangle/dt = 2\nu\int k^2 E(k)\,dk$ does indeed settle down to a steady state value in which the energy production rate $\int f(k)\,dk$ is balanced by the dissipation rate.

What quantity or quantities in hydrodynamics can play the role of the additional finite flux motion invariant? We will consider only the isotropic case and define the velocity correlations

$$\overline{u^2} f(r) = \langle u(x)\, u(x+r)\rangle = \int_0^\infty F(k) \cos(kr)\,dk \tag{8.2}$$

and

$$\overline{u^2}^{3/2} h(r) = \langle v^2(x)\, u(x+r)\rangle = \int kH(k) \sin(kr)\,dk, \tag{8.3}$$

where u and v are the velocity components parallel and perpendicular respectively to r and $\overline{u^2} = \frac{1}{3}\sum_{i=1}^3 \langle u_i(x)\rangle^2 = E$, where E is the total energy. The Von Karman–Kowarth equation for the velocity correlation $f(r)$ is

$$\frac{\partial}{\partial t}\overline{u^2} f(r) + 2\overline{u^2}^{3/2} \frac{1}{r^4}\frac{\partial}{\partial r} r^4 h(r) = 2\nu\overline{u^2}\frac{1}{r^4}\frac{\partial}{\partial r} r^4 \frac{\partial f}{\partial r}. \tag{8.4}$$

We have not included in (8.4) any forcing terms. From (8.4), we obtain formally

$$\frac{\partial}{\partial t}\overline{u^2} + \lim_{r\to 0} 2\overline{u^2}^{3/2} \frac{1}{r^4}\frac{\partial}{\partial r} r^4 h(r) = -2\nu\frac{E}{\lambda^2} = -\varepsilon, \tag{8.5}$$

$$\frac{\partial}{\partial t}\overline{u^2} \lim_{r\to\infty} r^3 f(r) + \lim_{r\to\infty} 2\overline{u^2}^{3/2} \frac{1}{r}\frac{\partial}{\partial r} r^4 h(r) = -2\nu\overline{u^2} \lim_{r\to\infty} \frac{1}{r}\frac{\partial}{\partial r} r^4 \frac{\partial f}{\partial r}, \tag{8.6}$$

and

$$\frac{\partial}{\partial t}\overline{u^2}\int_0^\infty r^4 f(r)\,\mathrm{d}r + 2\overline{u^2}^{3/2}\left[r^4 h(r)\right]_0^\infty = 2\nu\overline{u^2}\left[r^4\frac{\partial f}{\partial r}\right]_0^\infty. \tag{8.7}$$

In (8.5)–(8.7), we have used the facts that as $r \to 0$ (λ is the Taylor microscale $[-f''(0)]^{-1/2}$),

$$f(r) \to 1 - r^2/2\lambda^2, \quad h(r) = \mathcal{O}(r^3). \tag{8.8}$$

In the absence of viscosity, eq. (8.5) expresses the conservation of energy. In the presence of forcing at intermediate scales k_0^{-1} and under the assumption that viscosity acts only after the viscous scales k_d^{-1}, the Kolmogorov assumptions assert that the turbulence relaxes to a steady state for which the dissipation rate ε is constant, and moreover that it is the only relevant parameter besides the local scale k^{-1} in determining the statistical behavior in the wavenumber window $(k_0, k_d = (\varepsilon\nu^{-3})^{1/4})$. Eq. (8.6) could be trivial if

$$M = \lim_{r\to\infty} \overline{u^2} r^3 f(r) \tag{8.9}$$

is zero. If, however, M is nonzero, then it has the additional consequence that the quantity, whose time derivative is given by (8.7),

$$L = \int_0^\infty \overline{u^2} r^4 f(u)\,\mathrm{d}r \tag{8.10}$$

does not exist. In this case, eq. (8.7) requires interpretation. If M is nonzero, then in (8.6), the viscous term involving $(1/r)(\partial/\partial r)r^4\,\partial f/\partial r$ tends to zero as $r \to \infty$ and so if $h(r) = \mathcal{O}(r^{-3})$ as $r \to \infty$, M is a motion invariant,

$$\partial M/\partial t = 0. \tag{8.11}$$

On the other hand, if $M = 0$ and $h(r) \to cr^{-4}$ as $r \to \infty$ (see ref. [21]), L exists and is almost a motion invariant,

$$\partial L/\partial t = -2\overline{u^2}^{3/2} c = -\mu. \tag{8.12}$$

The quantity L is called Loitsyanskii's invariant and represents a measure of squared angular momentum

$$4\pi L = \int r^2 \langle u(x)\,u(x+r)\rangle\,\mathrm{d}r. \tag{8.13}$$

Its existence was established by Batchelor and Proudman [23] under the assumption that at some initial moment in time the turbulent field has convergent integral moments of the velocity distribution. These authors showed that under similar conditions, the third-order velocity condition $h(r)$ does not decay sufficiently fast so that L is constant but found that $h(r) \to cr^{-4}$ as $r \to \infty$, which leads to (8.12). Nevertheless, we shall argue that in the case where L exists, L is a finite flux motion invariant in exactly the same way that the energy E is and that its loss occurs at low wavenumbers near $k = 0$. We shall show that if the fluid is continuously stirred at intermediate scales k_0^{-1}, then the energy density $E(k)$ ($E = \int E(k)\,\mathrm{d}k$) flows to high wavenumbers at constant rate ε and the squared angular momentum density $J(k)$ ($L = \int J(k)\,\mathrm{d}k$) flows to low wavenumbers at the rate μ.

The quantity

$$M = \lim_{r\to\infty} (r^3 f(u)) = \int_0^\infty (r^3 f)' \,\mathrm{d}r = (4\pi)^{-1} \int \sum_{i=1}^{3} \langle u_i(x)\, u_i(x+r)\rangle \,\mathrm{d}r \tag{8.14}$$

corresponds to squared momentum. It is a true motion constant. Its existence and invariance was first noted by Saffman [16]. Saffman argues persuasively that M is not likely in general to be zero and supports his argument by showing that if a turbulent field is generated by a distribution of random impulsive forces with convergent integral moments of cumulants, equivalent to convergent integral moments of the vorticity distribution at the initial time, thus M is nonzero and invariant.

In what follows, we will keep both options open so that when we discuss L we are assuming it exists and that $M = 0$. To continue our arguments in wavenumber space, we now define spectral densities for L and M. Consider

$$j(r) = \int_r^\infty \overline{u^2} r^4 f(r)\,\mathrm{d}r = \int_0^\infty J(k)\cos(kr)\,\mathrm{d}k \tag{8.15}$$

and

$$m(r) = \int_r^\infty \overline{u^2}(r^3 f(r))'\,\mathrm{d}r = M - \overline{u^2} r^3 f(r) = \int_0^\infty M(k)\cos(kr)\,\mathrm{d}k. \tag{8.16}$$

A little analysis will show that $kJ(k)$ and $kM(k)$ are the Fourier integral sine transformations of $\overline{u^2} r^4 f(r)$ and $\overline{u^2}(r^3 f)'$. The Fourier integral cosine transforms of these two quantities are respectively $\partial^4 F(k)/\partial k^4$ and $k\,\partial^3 F/\partial k^3 - 3\,\partial^2 F/\partial k^2$ and, using the relation

$$E(k) = \frac{1}{3}\left(k^2 \frac{\partial^2 F}{\partial k^2} - k\frac{\partial F}{\partial k}\right) \tag{8.17}$$

between the one-dimensional Fourier integral cosine transforms $F(k)$ defined by (8.2) and $E(k) = \frac{4}{3}\pi k^2 \Sigma_{i=1}^{3} \Phi_{ii}(k)$, where

$$\Phi_{lm}(\boldsymbol{k}) = \frac{1}{(2\pi)^3} \int_{-n}^{\infty} \langle u_l(\boldsymbol{x})\, u_m(\boldsymbol{x}+\boldsymbol{r})\rangle \,\mathrm{d}\boldsymbol{r},$$

we have $k\,\partial^3 F/\partial k^3 - 3\,\partial^2 F/\partial k^2$ is $(3k^{-1}E(k))'$. Now Fourier integral cosine $C(k)$ and sine transforms $S(k)$ are related through Hilbert transforms. (Observe that $C(k)$ and $S(k)$ can be defined by even and odd extensions to negative k, that $C(k) + \mathrm{i}S(k)$ is analytic for $\operatorname{Im} k > 0$ and thus the assertion follows from Cauchy's theorem.) We find (note $M = 0$ does not imply $M(k) \equiv 0$)

$$\frac{\partial^4 F(k)}{\partial k^4} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{k' J(k')\,\mathrm{d}k'}{k' - k}, \quad kJ(k) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\left[\partial^4 F(k')/\partial k'^4\right]\mathrm{d}k'}{k' - k}, \tag{8.18}$$

and

$$\frac{\partial}{\partial k}\frac{3E(k)}{k} = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{k'M(k')\,\mathrm{d}k'}{k'-k}, \quad kM(k) = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{1}{k'-k}\frac{\partial}{\partial k'}\frac{3E(k')}{k'}\,\mathrm{d}k'. \tag{8.19}$$

From (8.18), we obtain that, for small k,

$$\frac{\partial^4 F}{\partial k^4} = \frac{2}{\pi}\int_0^{\infty} J(k')\,\mathrm{d}k' = \frac{2}{\pi}L \tag{8.20}$$

so that, integrating (8.20) and using (8.17),

$$E(k) = \frac{2L}{9\pi}k^4, \tag{8.21}$$

and for large k ($\int F(k)\,\mathrm{d}k = \int E(k)\,\mathrm{d}k = E$),

$$kJ(k) = \frac{24E}{\pi k^5}. \tag{8.22}$$

Eq. (8.21) is the well known but often disputed result, that the spectral energy at large scales behaves as k^4. The consequences of (8.22) are nontrivial because it says that the amount of squared angular momentum between k_0 and ∞

$$\int_{k_0}^{\infty} J(k)\,\mathrm{d}k = \frac{24E}{5\pi k_0^5} < \infty. \tag{8.23}$$

Therefore if squared angular momentum density $J(k)$ is introduced at a constant rate at k_0, only a finite amount can be absorbed in wavenumbers $k > k_0$ and since, as we will shortly show, there is no squared angular momentum density sink at large wavenumbers, *the flux of $J(k)$ must be to small wavenumbers.* From (8.19), we have for small k that

$$E(k) = (1/6\pi)Mk^2 \tag{8.24}$$

so that when M is nonzero, the spectrum of $E(k)$ near k is thermodynamic.

For large k,

$$M(k) = -6E/\pi k^4 \tag{8.25}$$

so that the amount of squared angular momentum between k_0 and ∞ is

$$\int_{k_0}^{\infty} M(k)\,\mathrm{d}k = -2E/\pi k_0^3. \tag{8.26}$$

Again, we see that if $M(k)$ is produced at a constant rate in $(k_0 - \Delta k,\ k_0 + \Delta k)$, most of it must drift to small wavenumbers and large scales. Unlike squared angular momentum density, however, there is no sink at low wavenumbers to absorb squared momentum density. Therefore, if $M \neq 0$, any large-scale structures produced by the buildup of $M(k)$ at low wavenumbers must be unstable and the instability must be fast enough to return squared momentum to high wavenumbers at its original production rate.

The equations for the spectral densities $F(k)$, $M(k)$ and $J(k)$, corresponding to (8.5), (8.6) and (8.7) are

$$\frac{\partial F(k)}{\partial t} + T_1(k) = -2\nu\left(k^2 F + 4\int_k^\infty kF\,\mathrm{d}k\right), \tag{8.27}$$

equivalent to (8.1) by applying the operator $\frac{1}{3}k^2\partial^3/\partial k^2 - \frac{1}{3}k\,\partial/\partial k$,

$$\frac{\partial M(k)}{\partial t} + T_2(k) = -\frac{2\nu\overline{u^2}}{\pi}\int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}r^4\frac{\partial f}{\partial r}\cos(kr)\,\mathrm{d}r, \tag{8.28}$$

and

$$\frac{\partial J(k)}{\partial t} + T_3(k) = \frac{4\nu\overline{u^2}}{\pi}\int_0^\infty \frac{\partial}{\partial r}r^4\frac{\partial f}{\partial r}\cos(kr)\,\mathrm{d}r, \tag{8.29}$$

where the transfer integrals T_j, $j = 1,2,3$, are

$$T_1(k) = \frac{4}{\pi}\overline{u^2}^{3/2}\int_0^\infty \frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4 h(r)\cos(kr)\right]\mathrm{d}r, \tag{8.30}$$

$$T_2(k) = -\frac{4}{\pi}\overline{u^2}^{3/2}\int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}\left[r^4 h(r)\right]\cos(kr)\,\mathrm{d}r, \tag{8.31}$$

and

$$T_3(k) = \frac{4}{\pi}\overline{u^2}^{3/2}\int_0^\infty \left[c - r^4 h(r)\right]\cos(kr)\,\mathrm{d}r. \tag{8.32}$$

When integrated over k, we obtain (8.10), (8.11), and (8.12). Observe that, in the absence of forcing, the decay of E is caused by the viscous term in (8.27) because $\int T_1(k)\mathrm{d}k = 0$, whereas this term gives no contribution to the decay of L. On the contrary, the decay of L is caused by the large-scale behavior of the third-order velocity correlation due to large-range pressure forces. For example, if we take $r^4 h(r) = c[1 - r_1^2/(r_1^2 + r^2) + \dots]$ then $T_3(k) = 2c\overline{u^2}^{3/2}\delta(k)$ as $r_1 \to \infty$, so that the dominant behavior in $\int T_3(k)\mathrm{d}k = -\mu$ arises from the neighborhood of $k = 0$. If we assert that μ is constant and that, just as the dissipation rate ε plays an important role in the window of transparency (k_0, k_d), the squared angular momentum flux rate μ plays an important role in the window of transparency $(k_1 = r_1^{-1}, k_0)$, then the dimensional considerations lead to the behavior

$$J(k) = c_1\mu^{2/3}k^{-10/3}. \tag{8.33}$$

We stress that there is no evidence to date that the requisite conditions of locality are satisfied for (8.33) to hold. We also stress, however, that this law is not crucial to our argument that $J(k)$ flows to small wavenumbers.

[At this point, we digress to mention several curious results, the ramifications of which we do yet attempt to build into the total picture. As we have mentioned in section 1, in the absence of viscosity and helicity, the use of Clebsch variables λ, μ ($\boldsymbol{u} = \lambda\nabla\mu + \nabla\Phi$, $(\mathrm{D}/\mathrm{D}t)(\lambda, \mu) = 0$, $a_k = \mathrm{FT}(\lambda + \mathrm{i}\mu)$) allows us to recast the Euler equations in the form (1.14). From this it follows $(\partial/\partial t)\int(a_k a_k^*)^n \mathrm{d}\boldsymbol{k}$ is zero, $n = 1,\ldots,\infty$. Further, if one uses a Kraichnan direct interaction approximation in (1.14), one obtains a kinetic equation for n_k, $n_k\delta(\boldsymbol{k}-\boldsymbol{k}_1) = \langle a_k a_{k_1}^*\rangle$, which formally possesses two Kolmogorov spectra $E_k \sim k^{-5/3}$ and $E_k \sim k^{-1}$. The latter corresponds to a constant flux of the density $F_n = (a_k a_k^*)^n$ for $n = 1$ and has the units of angular momentum. For general n, one obtains the spectrum $E_k \sim k^{\alpha(n)}$, $-1 \leq \alpha(n) \leq \frac{5}{3}$ where, in the limit $n \to \infty$, the exponent $\alpha(n)$ is $\frac{5}{3}$, which is exactly what obtains from (8.33), corresponding to a flux of squared angular momentum. It is interesting that the two limiting conservation laws both correspond to situations in which one assumes that the statistics are chiefly dominated by a flux of a functional of angular momentum.]

No matter whether M or L is invariant, we have shown that, if produced at a constant rate at k_0, the corresponding spectral density will flow to small scales. The question thus is: what is the fate of these "particles"? Do they condense into large scale structures as in the case of defocusing NLS or in two-dimensional hydrodynamics, where mean squared vorticity density flows to small scales and energy to large scales where it builds large vortices? Or do they behave as in the case of the focusing NLS, where instead of building condensates, they nucleate collapsing filaments which return the energy to high wavenumbers?

Our conjecture is that the inverse cascade of $J(k)$ should lead to the formation of large vortical structures just as the inverse cascade of particle number in NLS should lead to condensates. But in the focusing case, we have seen that because these condensates are unstable, they never get a chance to form. Instead, as soon as the particle number density reaches scales large enough to nucleate collapsing filaments, the latter are formed and the inverse cascade is reversed. We can picture this in phase space as follows. The phase point which represents the state of the system is attracted towards the unstable manifold of the condensate which is a saddle point. So although the condensate itself is never realized, its unstable manifold plays an important role in the dynamics. Likewise, in three-dimensional hydrodynamics, we should look at the instabilities of large vortical structures although these structures themselves will never get the chance to form. We know from the work of Bayly [24] that elliptical vortices are unstable to a subharmonic resonance between the inertial wave $\mathrm{e}^{i\boldsymbol{k}(t)\cdot\boldsymbol{x}}$ with frequency $\omega = 2\Omega\cos\theta$, where Ω is the rotation speed of the vortex and $\cos\theta = \boldsymbol{\Omega}\cdot\boldsymbol{k}/\Omega k$. The subharmonic resonance occurs at $\theta = \frac{1}{3}\pi$ and the window of instability depends on the amount of ellipicity ($\boldsymbol{u} = \Omega(-1-\alpha)y, \Omega(1-\alpha)x, 0$) in the original vortex. The rate of growth of the instability is proportional to α and independent of the wavenumber k and therefore the amount of energy which is inserted directly in short waves is largest because the volume between k and $k + \mathrm{d}k$ between the two cones $\frac{1}{3}\pi(1-\beta\alpha) < \theta < \frac{1}{3}\pi(1+\beta\alpha)$ is proportional to k. Therefore, although the lowest energy configuration for a given amount of angular momentum is a circular vortex, the distortion of a single vortex by its neighbors leads to three-dimensional instability. One would expect the net effect of the instability is to restore an isolated elliptical vortex to its circular shape, but the constant flux of squared angular momentum to low wavenumbers keeps producing distorting fields. Again we expect the rate of energy feedback to large wavenumbers through the instability process is directly proportional to the flux of squared angular momentum towards $k = 0$.

We remark that if $M = 0$, the existence of L means that squared angular momentum is $J^2 = \rho^2(\int(\boldsymbol{r}\times\boldsymbol{v})\mathrm{d}V)^2$ is proportional to the fluid volume V (see ref. [25]). If $M \neq 0$, then the divergence of L means that the ratio of J^2 to V tends to infinity as $V \to \infty$ and that large vortices may be even more important than in the case of finite L.

Finally, if $M \neq 0$, one should also ask about the fate of squared linear momentum as it reaches large scales. At this time, we have no reasonable scenario to suggest.

Appendix A

We show how to average the Dirac delta functions $\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)$ and $\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)$ over angles. First we consider $\langle\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\rangle$. We find

$$\int \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\theta\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 = 2\pi\int \delta(k-k_1\cos\theta_1-k_2\cos\theta_2)\,\delta(k_1\sin\theta_1+k_2\sin\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$$

$$= \frac{\pi}{S} = \frac{4\pi}{\left(2k^2k_1^2+2k^2k_2^2+2k_1^2k_2^2-k^4-k_1^4-k_2^4\right)^{1/2}}. \tag{A.1}$$

Here

$$S = \tfrac{1}{4}\left(2k^2k_1^2+2k^2k_2^2+2k_1^2k_2^2-k^4-k_1^4-k_2^4\right)^{1/2} \tag{A.2}$$

is the area of a triangle with given size lengths, k, k_1, k_2.

Next we consider $\langle\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\rangle$. Denote

$$\int\int\int\int \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\theta\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 = R.$$

Next, write

$$\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) = \int \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{\lambda})\,\delta(\boldsymbol{\lambda}-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{\lambda}$$

$$= 2\pi\int_{\lambda_{\min}}^{\lambda_{\max}} \delta(k\cos\theta+k_1\cos k_1-\lambda)\,\delta(k\sin\theta+k_1\sin\theta_1)$$

$$\times\,\delta(k_2\cos\theta_2+k_3\cos\theta_3-\lambda)\,(k_2\sin\theta_2+k_3\sin\theta_3)\,\lambda\,\mathrm{d}\lambda,$$

where

$$\lambda_{\min} = \max(|k-k_1|, |k_2-k_3|),$$
$$\lambda_{\max} = \min(k+k_1, k_2+k_3). \tag{A.3}$$

So,

$$R = 2\pi\int_{\lambda_{\min}}^{\lambda_{\max}} \lambda\,\mathrm{d}\lambda \int\int \delta(k\cos\theta+k_1\cos\theta_1-\lambda)\,\delta(k\sin\theta+k_1\sin\theta_1)\,\mathrm{d}\theta\,\mathrm{d}\theta_1$$

$$\times\int\int \delta(k_2\cos\theta_2+k_3\cos\theta_3-\lambda)\,\delta(k_2\sin\theta_2+k_3\sin\theta_3)\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3. \tag{A.4}$$

In each double integral in (A.4), one can change variables of integration and get

$$R = 2\pi \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{1}{S_1} \times \frac{1}{S_2} \lambda \, d\lambda, \tag{A.5}$$

where S_1 and S_2 are the squares of the triangle with sides k, k_1, λ and k_2, k_3, λ respectively:

$$S_1 = \tfrac{1}{4}\left\{\left[(k+k_1)^2 - \lambda^2\right]\left[\lambda^2 - (k-k_1)^2\right]\right\}^{1/2}, \quad S_2 = \tfrac{1}{4}\left\{\left[(k_2+k_3)^2 - \lambda^2\right]\left[\lambda^2 - (k_2-k_3)^2\right]\right\}^{1/2}.$$

Therefore R is a complete elliptic integral of the first kind,

$$R = 16\pi \int_{\lambda_{\min}}^{\lambda_{\max}} \frac{d\lambda^2}{\left\{\left[(k_1-k)^2 - \lambda^2\right]\left[(k_2-k_3)^2 - \lambda^2\right]\right\}^{1/2}} \frac{1}{\left\{\left[(k+k_1)^2 - \lambda^2\right]\left[(k_2+k_3)^2 - \lambda^2\right]\right\}^{1/2}}. \tag{A.6}$$

Now recall that $k^2 + k_1^2 = k_2^2 + k_3^2$. With this condition the limits of integration in R have two possibilities, $\lambda_{\min} = |k_2 - k_3|$, $\lambda_{\max} = k_2 + k_3$, or $\lambda_{\min} = |k - k_1|$, $\lambda_{\max} = k + k_1$. The result of integration is the same for both cases and we find

$$R = \frac{16\pi}{kk_1 + k_2k_3} F\left(\frac{2(kk_1k_2k_3)^{1/2}}{kk_1 + k_2k_3}\right), \tag{A.7}$$

where

$$F(q) = \int_0^{\pi/2} \frac{d\rho}{(1 - q^2 \sin^2\rho)^{1/2}}.$$

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