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# The role of the convective modes and sheared variables in the Hamiltonian dynamics of uniform-shear-flow perturbations

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**Abstract**

The convective modes in shear flows are associated with the algebraic time evolution of perturbations. The sheared variables, corresponding to the amplitudes of the convective modes, are shown to diagonalize the quadratic Hamiltonian. In other words, the sheared variables are the normal variables of the Hamilton equations, and the convective modes are the normal modes in the Hamiltonian dynamics.

## 1. Introduction

The fundamental fact in the theory of the shear flow stability is that the spectral continuum is as important as the discrete eigenmodes and can be responsible for the algebraic growth of perturbations. In the coordinate space, the continuous spectrum corresponds to a convective-type evolution of two scalar fields, called the sheared variables, having the form of a translation with the velocity of the mean flow. This motion is known as the convective modes or the ballistic waves, the latter usually in the context of plasma physics. Convective modes were discovered by Case [1] for the 2D fluid dynamics governed by the Euler equations. The convective modes can be found analytically only for a few basic flow profiles. Goldstein [2] has shown, nevertheless, that the convective modes do exist for an arbitrary transversely sheared parallel flow and provide a complete set of functions for the description of the linear dynamics of the perturbations.

There is a general understanding that classical spectral analysis is not quite appropriate for the theoretical description of the shear flow perturbations associated with the convective modes because the linear operator of the dynamical equations is not self-conjugated. In this Letter we will show that the convective modes appear in the most natural way in the framework of the Hamiltonian theory as providing the normal canonical variables for the Hamilton equations.

## 2. Hamiltonian dynamics in Clebsh variables

Our basis model will be the Euler equations for an incompressible inviscid fluid written in terms of the Clebsh variables. The Clebsh variables are convenient for our purposes because written in these variables the Euler equations have a canonical Hamiltonian structure, which is very helpful for finding the normal modes. The Clebsh variables for an incompressible inviscid fluid, $\lambda$ and $\mu$, are introduced by the Lamb formula

$$v=\lambda\nabla\mu+\nabla\phi\,, \tag{1}$$


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where

$$\boldsymbol{v}=(u, v, w) \tag{2}$$

is the velocity field. The potential $\phi$ can be expressed in terms of $\lambda$ and $\mu$ using the incompressibility condition

$$\operatorname{div} \boldsymbol{v}=0 . \tag{3}$$

The fundamental fact is that the Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-\nabla P/\rho \tag{4}$$

is satisfied if $\lambda$ and $\mu$ obey the equations

$$[\partial/\partial t+(\boldsymbol{v}\cdot\nabla)]\lambda=\frac{\partial F}{\partial \mu}, \tag{5}$$

$$[\partial/\partial t+(\boldsymbol{v}\cdot\nabla)]\mu=-\frac{\partial F}{\partial \lambda}, \tag{6}$$

where $F=F(\lambda, \mu, t)$ is an arbitrary function of $\lambda$, $\mu$ and $t$. It is remarkable that Eqs. (5), (6) are Hamiltonian,

$$\frac{\partial \lambda}{\partial t}=\frac{\delta H}{\delta \mu}, \tag{7}$$

$$\frac{\partial \mu}{\partial t}=-\frac{\delta H}{\delta \lambda}, \tag{8}$$

with the following Hamiltonian functional,

$$H=\tfrac{1}{2}\int \boldsymbol{v}^2\,\mathrm{d}\boldsymbol{r}+\int F(\lambda, \mu, t)\,\mathrm{d}\boldsymbol{r} . \tag{9}$$

The Clebsh variables are defined with accuracy up to an arbitrary canonical transformation in the plane $(\lambda, \mu)$. Time dependent transformations change the function $F$. For a reason which will be explained below we choose

$$F=-\tfrac{1}{2}\lambda^2 , \tag{10}$$

so that Eqs. (5), (6) now become

$$[\partial/\partial t+(\boldsymbol{v}\cdot\nabla)]\lambda=0 , \tag{11}$$

$$[\partial/\partial t+(\boldsymbol{v}\cdot\nabla)]\mu=\lambda . \tag{12}$$

Consider a stationary shear flow with the velocity field directed along the $x$-axis,

$$\boldsymbol{v}_0=(\beta y, 0, 0), \quad \beta=\text{const} . \tag{13}$$

We use the following representation of the shear flow (13),

$$\lambda_0=\beta y, \quad \mu_0=x, \quad \phi_0=0 . \tag{14}$$

In this case $H=0$. One can see that Eqs. (11), (12) are satisfied. The motivation of the choice (10) is that in such a representation $\lambda_0$, $\mu_0$ and $\phi_0$ are time independent, which makes the formulae look more elegant.

Now consider perturbations of the stationary shear flow (13),

$$\lambda=\lambda_0+\tilde{\lambda}, \quad \mu=\mu_0+\tilde{\mu}, \quad \phi=\tilde{\phi}, \tag{15}$$

so that

$$\boldsymbol{v}=\boldsymbol{v}_0+\tilde{\boldsymbol{v}}, \tag{16}$$

$$\tilde{\boldsymbol{v}}=\lambda_0\nabla\tilde{\mu}+\tilde{\lambda}\nabla\mu_0+\tilde{\lambda}\nabla\tilde{\mu} . \tag{17}$$

The potential $\tilde{\phi}$ is related to $\tilde{\lambda}$, $\tilde{\mu}$ by virtue of continuity equation (3).

Hereafter, it will be convenient to work in Fourier space. According to (17), (14) we have the following expressions for the components of the velocity perturbations,

$$u_k=\lambda_k+\mathrm{i}p\chi_k+(\tilde{\lambda}\partial\tilde{\mu}/\partial x)_k , \tag{18}$$

$$v_k=-\beta\mu_k+\mathrm{i}q\chi_k+(\tilde{\lambda}\partial\tilde{\mu}/\partial y)_k , \tag{19}$$

$$w_k=\mathrm{i}s\chi_k+(\tilde{\lambda}\partial\tilde{\mu}/\partial z)_k , \tag{20}$$

where

$$\chi_k=(\beta y\tilde{\mu}+\tilde{\phi})_k ,$$

and the subscript $k$ denotes the 3D Fourier transformation of the corresponding perturbed function,

$$u_k=\int \tilde{u}(x, y, z)\exp(\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z ,$$

$$\boldsymbol{k}=(p, q, s), \quad \text{etc} . \tag{21}$$

Incompressibility condition (3) can be rewritten as

$$\boldsymbol{k}\cdot\tilde{\boldsymbol{v}}=0 . \tag{22}$$

Taking into account expressions (18)–(20) this condition gives

$$\chi_k=\frac{1}{k^2}[\mathrm{i}p\lambda_k-\mathrm{i}\beta q\mu_k+\boldsymbol{k}(\tilde{\lambda}\nabla\tilde{\mu})_k] . \tag{23}$$

Substituting (23) into Eqs. (18)–(20) we get

$$u_k = \frac{q^2+s^2}{k^2}\lambda_k + \frac{pq}{k^2}\beta\mu_k + u_k^{\mathrm{N}}, \quad (24)$$

$$v_k = -\frac{pq}{k^2}\lambda_k - \frac{k_\perp^2}{k^2}\beta\mu_k + v_k^{\mathrm{N}}, \quad (25)$$

$$w_k = -\frac{sp}{k^2}\lambda_k + \frac{sq}{k^2}\beta\mu_k + w_k^{\mathrm{N}}, \quad (26)$$

where

$$\boldsymbol{v}_k^{\mathrm{N}} = (u_k^N, v_k^{\mathrm{N}}, w_k^{\mathrm{N}}) = (\tilde{\lambda}\nabla\tilde{\mu})_k - \frac{\boldsymbol{k}}{k^2}(\boldsymbol{k}(\tilde{\lambda}\nabla\tilde{\mu})_k) \quad (27)$$

is the nonlinear part of the velocity perturbation, and $k_\perp = (p^2+s^2)^{1/2}$.

## 3. Linear approximation

Suppose that perturbation $\tilde{v}$ is so small that in first approximation one can consider Eqs. (11), (12) linearized with respect to $\tilde{\lambda}$, $\tilde{\mu}$. In terms of the Fourier transforms we have

$$\hat{D}_k\lambda_k + \beta v_k^{\mathrm{L}} = 0, \quad (28)$$

$$\hat{D}_k\mu_k - \lambda_k + u_k^{\mathrm{L}} = 0, \quad (29)$$

where the operator $\hat{D}_k$ is defined as

$$\hat{D}_k = \frac{\partial}{\partial t} - \beta p\frac{\partial}{\partial q}$$

and $u_k^{\mathrm{L}}$, $v_k^{\mathrm{L}}$ are the linear parts of the velocity components (24), (25),

$$u_k^{\mathrm{L}} = \frac{q^2+s^2}{k^2}\lambda_k + \frac{pq}{k^2}\beta\mu_k, \quad (30)$$

$$v_k^{\mathrm{L}} = -\frac{pq}{k^2}\lambda_k - \frac{k_\perp^2}{k^2}\beta\mu_k. \quad (31)$$

Multiplying Eq. (28) by $qp/k_\perp^2$ and Eq. (29) by $\beta$ and adding the resulting equations (also, substituting $u_k^{\mathrm{L}}$, $v_k^{\mathrm{L}}$ from (30), (31)), we get

$$\hat{D}_k b_k = 0, \quad (32)$$

where

$$b_k = \beta\mu_k + \frac{qp}{k_\perp^2}\lambda_k, \quad (33)$$

One can see that

$$b_k = -\frac{k^2}{k_\perp^2}v_k^{\mathrm{L}}. \quad (34)$$

Eq. (32) has the form of a conservation law; its general solution is

$$b_k = B(p, q+\beta pt, s), \quad (35)$$

where $B(p,q,s)$ is an arbitrary function determined by the initial conditions. In other words, $b(p, q-\beta pt, s)$ is an integral of the linear equations (28), (29).

To find the second integral of motion, let us, taking into account (34), rewrite the equation for $\lambda_k$ as follows,

$$\hat{D}_k\lambda_k = \frac{k_\perp^2}{k^2}\beta b_k. \quad (36)$$

Eq. (36) means that $\lambda_k$ is an adjoint vector to the eigenvector $b_k$. Taking into account that $\int(k_\perp^2/k^2)\,\mathrm{d}q = k_\perp\arctan(k_\perp/q)$, one can rewrite Eq. (36) in the divergent form

$$\hat{D}_k c_k = 0, \quad (37)$$

where

$$c_k = \lambda_k - \frac{k_\perp}{p}b_k\arctan(k_\perp/q). \quad (38)$$

The general solution of Eq. (37) is

$$c_k = C(p, q+\beta pt, s), \quad (39)$$

where $C(p,q,s)$ is an arbitrary function. Therefore, $c(p, q-\beta pt, s)$ is an integral of motion.

Using (33), (38), one can write the expressions for $\lambda_k$, $\mu_k$ in terms of $b_k$, $c_k$,

$$\lambda_k = c_k + b_k\frac{k_\perp}{p}\arctan(k_\perp/q), \quad (40)$$

$$\mu_k = -\frac{qp}{\beta k_\perp^2}c_k + \frac{1}{\beta}\left(1 - \frac{q}{k_\perp}\arctan(k_\perp/q)\right)b_k. \quad (41)$$

Further, using (24)–(26), one can express the velocity components in terms of $b_k$ and $c_k$,

$$u_k = \left(\frac{pq}{k^2} + \frac{s^2}{k_\perp p}\arctan(k_\perp/q)\right)b_k + \frac{s^2}{k_\perp^2}c_k, \quad (42)$$

$$v_k = -\frac{k_\perp^2}{k^2} b_k\,, \tag{43}$$

$$w_k = \left(\frac{sq}{k^2} - \frac{s}{k_\perp}\arctan(k_\perp/q)\right) b_k - \frac{ps}{k_\perp^2} c_k\,. \tag{44}$$

Substitution of (35), (39) into (42)–(44) gives the linear solution in terms of the velocity components for the inviscid case,

$$u_k = \left(\frac{pq}{k^2} + \frac{s^2}{k_\perp p}\arctan(k_\perp/q)\right) B(p, q+\beta pt, s) + \frac{s^2}{k_\perp^2} C(p; q+\beta pt; s)\,, \tag{45}$$

$$v_k = -\frac{k_\perp^2}{k^2} B(p, q+\beta pt, s)\,, \tag{46}$$

$$w_k = \left(\frac{sq}{k^2} - \frac{s}{k_\perp}\arctan(k_\perp/q)\right) B(p, q+\beta pt, s) - \frac{ps}{k_\perp^2} C(p; q+\beta pt; s)\,. \tag{47}$$

If we assume $B(\boldsymbol{k})=0$ in (45)–(47), then we will get $v_k=0$, while $u(p, q-\beta pt)$ and $w(p, q-\beta pt)$ will become integrals of motion (dependent). This means that there is no motion in the $y$ direction in the ballistic wave associated with the eigenvector $c_k$: the velocity perturbation lies in the $x$–$z$ plane and remains constant in the frame moving with the speed of the background shear flow.

In the case $s=0$ all the terms containing $C(\boldsymbol{k})$ vanish, which means that in the 2D case the ballistic wave associated with $c_k$ is fictive. This can be explained by the fact that the 2D dynamics can be expressed in terms of only the vorticity, which is related to $b_k$.

Taking the inverse Fourier transform of (35), (39) we have

$$b(\boldsymbol{r}, t) = B(x-\beta yt, y, z)\,, \tag{48}$$

$$c(\boldsymbol{r}, t) = C(x-\beta yt, y, z)\,. \tag{49}$$

As we see, in the coordinate space the solutions have the form of the convective modes. Correspondingly, we will call $b$ and $c$ sheared variables.

## 4. Normal variables of the Hamilton equations

The sheared variables (33), (38) could be obtained without using the Clebsh variables from the Rayleigh equation and the equation and the equation for the $y$-component of the vorticity,

$$\hat{D}_k(k^2 v_k) = 0\,, \tag{50}$$

$$\hat{D}_k(su_k - pw_k) = -\beta s v_k\,. \tag{51}$$

In fact, Eqs. (50), (51) are just another form of Eqs. (32) and (36) respectively. Also, to solve the linear problem we can simply work with velocity variables and do not need to calculate the sheared variables; the corresponding technique is known as rapid distortion theory [3,4]. Using the Clebsh variables is important in our case because we can show now that the sheared variables (33), (38) play the role of the normal variables in the Hamiltonian theory.

First, we note that the transformation $\lambda_k, \mu_k \to b_k, c_k$ is canonical because, by virtue of (33), (38), the following commutation relation holds,

$$\frac{\delta c_k}{\delta \lambda_k}\frac{\delta b_k}{\delta \mu_k} - \frac{\delta c_k}{\delta \mu_k}\frac{\delta b_k}{\delta \lambda_k} = \beta\,. \tag{52}$$

The Hamilton equations in terms of $b_k$, $c_k$ are

$$\frac{\partial b_k}{\partial t} = -\frac{\delta \Xi}{\delta c_{-k}}\,, \tag{53}$$

$$\frac{\partial c_k}{\partial t} = \frac{\delta \Xi}{\delta b_{-k}}\,, \tag{54}$$

where

$$\Xi = \beta H\,. \tag{55}$$

The new Hamiltonian $\Xi$ differs from the old one $H$ only by a constant due to the property (52).

Further, substituting (10) in (9) and using the spectral representation we obtain

$$\Xi = \tfrac{1}{2}\beta \int (\boldsymbol{v}_k \cdot \boldsymbol{v}_{-k} - \lambda_k \lambda_{-k})\, \mathrm{d}\boldsymbol{k}\,. \tag{56}$$

Expressing the integrand in (56) in terms of $b_k$ and $c_k$ after a straightforward algebra we get

$$\Xi = \Xi^{(2)} + \Xi^{(3)} + \Xi^{(4)}\,, \tag{57}$$

where $\Xi^{(2)}$, $\Xi^{(3)}$ and $\Xi^{(4)}$ are the quadratic, cubic and fourth order terms in the Hamiltonian,

$$\Xi^{(2)} = \frac{\beta}{2}\int \left(pb_{-k}\frac{\partial c_k}{\partial q} - pc_{-k}\frac{\partial b_k}{\partial q}\right) \mathrm{d}\boldsymbol{k}\,, \tag{58}$$

$$\Xi^{(3)} = \int (V^{ccc}_{123} c_1 c_2 c_3 + V^{ccb}_{123} c_1 c_2 b_3 + V^{bbc}_{123} b_1 b_2 c_3 + V^{bbb}_{123} b_1 b_2 b_3) \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2\, d\boldsymbol{k}_3 , \tag{59}$$

$$\Xi^{(4)} = \int (V^{cccc}_{1234} c_1 c_2 c_3 c_4 + V^{cccb}_{1234} c_1 c_2 c_3 b_4 + V^{ccbb}_{1234} c_1 c_2 b_3 b_4 + V^{cbbb}_{1234} c_1 b_2 b_3 b_4 + V^{bbbb}_{1234} b_1 b_2 b_3 b_4) \times \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4)\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2\, d\boldsymbol{k}_3\, d\boldsymbol{k}_4 . \tag{60}$$

The matrix elements $V^{ccc}_{123}$, $V^{ccb}_{123}$, $V^{bbc}_{123}$, $V^{bbb}_{123}$ and $V^{cccc}_{1234}$, $V^{cccb}_{1234}$, $V^{ccbb}_{1234}$, $V^{cbbb}_{1234}$, $V^{bbbb}_{1234}$ are functions of three and four wave numbers correspondingly. They are given by some lengthy though simple-structured expressions which we do not present in this Letter.

More importantly, the quadratic Hamiltonian (58) is diagonal in variables $b_k$, $c_k$. This condition may be considered a definition of the normal variables. Solutions (35), (39) of the linear approximation can be easily obtained after its substitution into the equations of motion (53), (54). Quadratic Hamiltonian (58) can be also written in the form

$$\Xi^{(2)} = \int \omega\, a a^*\, dk_\perp\, dy ,$$

where $\omega = \beta y p$ is the frequency of the ballistic waves, and

$$a \equiv a(k_\perp, y) = \int (c_k + i b_k) \exp(iqy) \frac{dq}{2\pi} .$$

This expression bears a resemblance to the standard form of the quadratic Hamiltonians in the theory of dispersive waves,

$$\Xi^{(2)} = \int \omega_k a_k a_k^*\, d\boldsymbol{k} ,$$

where $\omega_k$ is the wave frequency, $a_k$ is the normal variable. However, the differentiation in the integrand of the quadratic Hamiltonian (58) makes the non-linear behavior of the system under consideration very different from the Hamiltonian dynamics of the nonlinear waves in continuous dispersive media. For example, in linear approximation the Fourier harmonics of the dispersive waves are constant, whereas the Fourier amplitudes of the convective modes, (35), (39) translate in the spectral space. The latter results in the algebraic growth of the total energy of perturbations and condensation of the spectrum in the streamwise elongated structures known as "streaks" [3].

## 5. Conclusion

Above, we calculated the sheared variables, (33), (38), and proved that they are the normal variables of the Hamilton equations. Thence, the convective modes (48), (49) may be regarded as the normal modes of the Hamiltonian dynamics of the uniform-shear-flow perturbations.

The normal variables found in this Letter are also most natural variables for the nonlinear theory. Although the detailed discussion of the nonlinear theory is beyond the objectives of this Letter, we will outline briefly how to construct such a theory. For this one has to introduce an explicit time dependence of the amplitudes $B$ and $C$,

$$b_k = B(p, q + \beta pt, s, t) , \tag{61}$$

$$c_k = C(p, q + \beta pt, s, t) . \tag{62}$$

Observe that in the weakly nonlinear regime the explicit time dependence in $B$ and $C$ is much slower than the implicit one. Therefore, by making the change of variables (61), (62) we can eliminate the fast dynamics associated with the linear mechanisms. The corresponding approach in quantum field theory is known as the interaction representation. We would like to emphasize that the weakly nonlinear theories of the convective and dispersive waves are also very different. The implicit time dependence in (61), (62) will not disappear for large times and will result in the algebraic growth of the nonlinear interaction, whereas the time dependence of the nonlinear interaction of the dispersive waves disappears after averaging over many periods. The nonlinear theory of the uniform-shear-flow perturbations will be reported elsewhere.

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