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Is free-surface hydrodynamics an integrable system?

A.I. Dyachenko, V.E. Zakharov

Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA,
Landau Institute for Theoretical Physics, 2 Kosygina Street, 117334 Moscow, Russian Federation

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Abstract

A strong argument is found in support of the integrability of free-surface hydrodynamics in the one-dimensional case. It is shown that the first term in the perturbation series in powers of nonlinearity is identically equal to zero, the consequences of which are discussed as well.

1. It is well known that the equations describing an ideal fluid with a free surface in a gravity field are completely integrable in several important limiting cases. Integrability occurs for long waves in shallow water (KdV [1] and KP [2] equations, the Boussinesq approximation [3], Kaup's approximation [4], the Holm–Camassa approximation [5]) and for spectrally narrow wave trains in a fluid of arbitrary depth (nonlinear Schrödinger equation [6]). The weakly nonlinear motion of the fluid in the absence of a gravity field is integrable as well [7].

It is very natural to formulate a conjecture that an arbitrary one-dimensional motion of an ideal fluid in a gravity field is integrable. In this article we give arguments in support of this conjecture. We will consider weakly nonlinear waves on the surface of a fluid of infinite depth and study their simplest resonant interactions, and we will show that the amplitude of this process is zero.

Given the current stage of mathematical physics there are no effective general methods for checking and proving integrability for the nonlinear wave Hamiltonian systems. Proving nonintegrability is a much easier problem. Following Poincaré, one can do that by analysing the perturbation series in powers of the nonlinearity [8]. Terms of this series being limited on their resonant manifolds are identified with the "amplitudes of the nonlinear interactions" in the wave system.

Nonintegrability is a quite evident fact. To have nonintegrability, it is enough to prove that at least one of these amplitudes is nonzero. As the complexity of calculations increases significantly with the order of nonlinearity, much information can be extracted from the consideration of the first (lowest order) nontrivial nonlinear process. For instance, nonintegrability of the nonlinear Schrödinger equation for $d \geqslant 2$ is a trivial fact due to the nonzero amplitude for the process $2 \rightarrow 2$ wave scattering. This scattering is trivial for the integrable case $d = 1$. One may verify (albeit with much effort) that the amplitude of the first nontrivial scattering $3 \rightarrow 3$ is identically zero in this case.

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2. A one-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is described by the following set of equations,

$$\phi_{xx} + \phi_{zz} = 0 \qquad (\phi_z \to 0, z \to -\infty),$$

$$\eta_t + \eta_x\phi_x = \phi_z|_{z=\eta}, \qquad \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0|_{z=\eta}. \tag{1}$$

Here $\eta(x,t)$ is the shape of the surface, $\phi(x,z,t)$ is the stream function and g is the gravitational constant. As was shown in Ref. [9], the variables $\eta(x,t)$ and $\psi(x,t) = \phi(x,z,t)|_{z=\eta}$ are canonically conjugated, and their Fourier transforms satisfy the equations

$$\frac{\partial \psi_k}{\partial t} = -\frac{\delta H}{\delta \eta_k^*}, \qquad \frac{\partial \eta_k}{\partial t} = \frac{\delta H}{\delta \psi_k^*}.$$

Here $H = K + U$ is the total energy of the fluid with the following kinetic and potential energy terms,

$$K = \tfrac{1}{2}\int \mathrm{d}x \int_{-\infty}^{\eta} v^2 \mathrm{d}z, \qquad U = \tfrac{1}{2}g\int \eta^2 \mathrm{d}x.$$

A Hamiltonian can be expanded in an infinite series in powers of the characteristic wave steepness $k\eta_k \ll 1$ (see Refs. [9,10]),

$$H = H_0 + H_1 + H_2 + \ldots\,. \tag{2}$$

It is convenient to introduce a normal complex variable a_k,

$$\eta_k = \sqrt{\omega_k/2g}\,(a_k + a^*_{-k}), \qquad \psi_k = -\mathrm{i}\sqrt{2g/\omega_k}\,(a_k - a^*_{-k}). \tag{3}$$

Here $\omega_k = \sqrt{gk}$ is the dispersion law for the gravity waves. This variable satisfies the equation

$$\frac{\partial a_k}{\partial t} + \mathrm{i}\frac{\delta H}{\delta a_k^*} = 0, \tag{4}$$

where

$$H_0 = \int \omega_k a_k a_k^* \,\mathrm{d}k,$$

$$H_1 = \int V_{kk_1k_2}(a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^*)\delta_{k-k_1-k_2} \mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2 + \tfrac{1}{3}\int U_{kk_1k_2}(a_k a_{k_1} a_{k_2} + a_k^* a_{k_1}^* a_{k_2}^*)\delta_{k+k_1+k_2} \mathrm{d}k\,\mathrm{d}k_1\,\mathrm{d}k_2, \tag{5}$$

$$V_{kk_1k_2} = \frac{g^{1/4}}{8\pi\sqrt{2}}\left[\left(\frac{k}{k_1k_2}\right)^{1/4} L_{k_1k_2} - \left(\frac{k_2}{kk_1}\right)^{1/4} L_{-kk_1} - \left(\frac{k_1}{kk_2}\right)^{1/4} L_{-kk_2}\right],$$

$$U_{kk_1k_2} = \frac{g^{1/4}}{8\pi\sqrt{2}}\left[\left(\frac{k}{k_1k_2}\right)^{1/4} L_{k_1k_2} + \left(\frac{k_2}{kk_1}\right)^{1/4} L_{kk_1} + \left(\frac{k_1}{kk_2}\right)^{1/4} L_{kk_2}\right], \tag{6}$$

$$L_{kk_1} = \boldsymbol{k}\cdot\boldsymbol{k}_1 + |k||k_1|. \tag{7}$$

Among the various components of H_2 only one is important,

$$H_2 = \tfrac{1}{2}\int W_{kk_1k_2k_3} a_k^* a_{k_1}^* a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3, \tag{8}$$

$$W_{k_1k_2k_3k_4} = -\frac{1}{64\pi^2}(M_{-k_1-k_2k_3k_3} + M_{k_3k_4-k_1-k_2} - M_{-k_1k_3-k_2k_4}$$
$$- M_{-k_1k_4-k_2k_3} - M_{-k_2k_3-k_1k_4} - M_{-k_2k_4-k_1k_3}),$$
$$M_{k_1k_2k_3k_4} = |k_1k_2|\left(\frac{k_3k_4}{k_1k_2}\right)^{1/4}(|k_1+k_3|+|k_1+k_4|+|k_2+k_3|+|k_2+k_4|-2|k_1|-2|k_2|). \tag{9}$$

The variable a_k is not appropriate for gravity waves because the Hamiltonian contains cubic terms, while there are no three-wave frequency resonances. The following canonical (up to $O(a^5)$) transformation from a_k to b_k (see Refs. [9,11]) (note: in Ref. [9] there is a misprint – the negative sign is missing for W_{kk_1,k_2k_3}),

$$a_k = b_k + \int \Gamma^{(1)}_{kk_1k_2} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} - 2\int \Gamma^{(1)}_{k_2kk_1} b^*_{k_1} b_{k_2} \delta_{k+k_1-k_2}$$
$$+ \int \Gamma^{(2)}_{kk_1k_2} b^*_{k_1} b^*_{k_2} \delta_{k+k_1+k_2} + \int B_{kk_1k_2k_3} b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3},$$
$$B_{kk_1k_2k_3} = \Gamma^{(1)}_{k_1k_2k_1-k_2}\Gamma^{(1)}_{k_3kk_3-k} + \Gamma^{(1)}_{k_1k_3k_1-k_3}\Gamma^{(1)}_{k_2kk_2-k} - \Gamma^{(1)}_{kk_2k-k_2}\Gamma^{(1)}_{k_3k_1k_3-k_1}$$
$$- \Gamma^{(1)}_{k_1k_3k_1-k_3}\Gamma^{(1)}_{k_2k_1k_2-k_1} - \Gamma^{(1)}_{k+k_1kk_1}\Gamma^{(1)}_{k_2+k_3k_2k_3} + \Gamma^{(2)}_{-k-k_1kk_1}\Gamma^{(2)}_{-k_2-k_3k_2k_3},$$
$$\Gamma^{(1)}_{kk_1k_2} = -\frac{V_{kk_1k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}}, \qquad \Gamma^{(2)}_{kk_1k_2} = -\frac{U_{kk_1k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}, \tag{10}$$

transforms the Hamiltonian to a form not containing cubic terms,

$$H = \int \omega_k b_k b^*_k \mathrm{d}k + \tfrac{1}{2}\int T_{kk_1,k_2k_3} b^*_k b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3}\, \mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 + \dots . \tag{11}$$

Here T_{kk_1,k_2k_3} satisfies the symmetry conditions $T_{kk_1,k_2k_3} = T_{k_1k,k_2k_3} = T_{kk_1,k_3k_2} = T_{k_2k_3kk_1}$ and has the form

$$T_{kk_1,k_2k_3} = W_{k_1k,k_2k_3} - V_{kk_2k-k_2}V_{k_3k_1k_3-k_1}\left(\frac{1}{\omega_{k_2}+\omega_{k-k_2}-\omega_k} + \frac{1}{\omega_{k_1}+\omega_{k_3-k_1}-\omega_{k_3}}\right)$$
$$- V_{k_1k_2k_1-k_2}V_{k_3kk_3-k}\left(\frac{1}{\omega_{k_2}+\omega_{k_1-k_2}-\omega_{k_1}} + \frac{1}{\omega_k+\omega_{k_3-k}-\omega_{k_3}}\right)$$
$$- V_{kk_3k-k_3}V_{k_2k_1k_2-k_1}\left(\frac{1}{\omega_{k_3}+\omega_{k-k_3}-\omega_k} + \frac{1}{\omega_{k_1}+\omega_{k_2-k_1}-\omega_{k_2}}\right)$$
$$- V_{k_1k_3k_1-k_3}V_{k_2kk_2-k}\left(\frac{1}{\omega_{k_3}+\omega_{k_1-k_3}-\omega_{k_1}} + \frac{1}{\omega_k+\omega_{k_2-k}-\omega_{k_2}}\right)$$
$$- V_{k+k_1kk_1}V_{k_2+k_3k_2k_3}\left(\frac{1}{\omega_{k+k_1}-\omega_k-\omega_{k_1}} + \frac{1}{\omega_{k_2+k_3}-\omega_{k_2}-\omega_{k_3}}\right)$$
$$- U_{-k-k_1kk_1}U_{-k_2-k_3k_2k_3}\left(\frac{1}{\omega_{k+k_1}+\omega_k+\omega_{k_1}} + \frac{1}{\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}}\right). \tag{12}$$

The first nontrivial process is four-wave scattering, which is governed by the following resonant conditions (k_i are one-dimensional),

$$k + k_1 = k_2 + k_3, \qquad \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \tag{13}$$

and all frequencies ω_{k_i} are positive here. The system (13) describes a certain two-dimensional manifold in four-dimensional space (k, k_1, k_2, k_3). This manifold has a trivial component,

$$k_2 = k_1, \quad k_3 = k, \qquad \text{or} \qquad k_2 = k, \quad k_3 = k_1, \tag{14}$$

but it has also a nontrivial part. Let $k, k_1, k_3 > 0, k_2 < 0$. Now (13) describes the rational manifold, which can be parameterized in the following way,

$$k = a(1+\zeta)^2, \qquad k_1 = a(1+\zeta)^2\zeta^2, \qquad k_2 = -a\zeta^2, \qquad k_3 = a(1+\zeta+\zeta^2)^2. \tag{15}$$

Here $0 < \zeta < 1$ and $a > 0$. It is easy to see that these two manifolds, (14) and (15), represent the general solution for resonant interaction (except trivial permutations).

The main result of this article can be summed up as follows: *the amplitude T_{kk_1,k_2k_3} is identically equal to zero on the resonant surface* (15). This fact can be checked by direct calculation of T_{kk_1,k_2k_3} using expression (12) (we did it with the help of "Mathematica" [12]). The cancellation of dozens of terms in T_{kk_1,k_2k_3} apparently is not accidental; the cancellation would take place naturally if the system (1) were integrable, or had at least an additional integral of motion [8]. Of course, this is not a strict proof of integrability (we have no way of checking all higher order amplitudes in (11)), but there are other evidences, mentioned above, supporting our conjecture. Additionally, the integrability of gravity waves in the fluid of a finite depth can be checked by calculation of the appropriate T_{kk_1,k_2k_3} as well. Also, the recently published numerical simulation [13] of the evolution of a set of waves has shown that the wave number frequency spectrum is concentrated in *discrete* points near the curves $\omega_k = \sqrt{nk}$, $n = 1, 2, 3, \ldots$. In addition, the discrete spectrum is the direct consequence of the integrability of the system (1).

However, we can obtain the full proof of integrability by developing a new method of integration (e.g., inverse scattering method), or by finding an L–A pair, etc.

The vanishing T_{kk_1,k_2k_3} on the resonant surface leaves in effect only the trivial interaction (14), which corresponds to the nonlinear frequency shift of separate modes,

$$\widetilde{\omega_k} = \omega_k + \int T_{kk_1}|b_{k_1}|^2\,\mathrm{d}k_1,$$

where $T_{kk_1} = (1/4\pi^2)(\boldsymbol{k}\cdot\boldsymbol{k}_1)\min(|k|, |k_1|)$, and the Hamiltonian can be written (using new canonical variables c_k) as

$$H = \int \omega c_k c_k^*\,\mathrm{d}k + \tfrac{1}{2}\int T_{kk_1}|c_k|^2|c_{k_1}|^2\,\mathrm{d}k\,\mathrm{d}k_1 + \mathrm{O}(c_k{}^5).$$

Therefore, integrability occurs at least up to the fifth order of c_k (or steepness $k\eta_k$). Furthermore, any quantity of the form

$$I_4 = \int f(k)|c_k|^2\,\mathrm{d}k$$

is also an integral of motion up to the fifth order.

3. The integrability of the one-dimensional free-surface hydrodynamics results in a rather different view of the problem of sea waves. It is well known that the well-developed surface wave turbulent spectrum is very narrow in an angle (indeed, it is almost one-dimensional). Thus, there is a natural small parameter $k_\perp/k_\parallel$ permitting the development of a perturbation theory for sea waves. Wave breaking, which is also a nearly one-dimensional phenomenon, can be described by an integrable set of equations as well.

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