High-Jacobian approximation in the free surface dynamics of an ideal fluid

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Abstract

Using a combination of the canonical formalism for free-surface hydrodynamics and conformal mapping to a horizontal strip we obtain a simple system of pseudo-differential equations for the surface shape and hydrodynamic velocity potential. The system is well-suited for numerical simulation. It can be effectively studied in the case when the Jacobian of the conformal mapping takes very high values in the vicinity of some point on the surface. At first order in an expansion in inverse powers of the Jacobian one can reduce the whole system of equations to a single equation which coincides with the well-known Laplacian Growth Equation (LGE). In the framework of this model one can construct remarkable special solutions of the system describing such physical phenomena as formation of finger-type configurations or changing of the surface topology – generation of separate droplets.

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1. Introduction

Two-dimensional irrotational motion of an ideal fluid in a domain confined between a free surface and a flat bottom is one of the classical subjects of investigation in hydrodynamics. The method of conformal mapping is the traditional approach to its study. The first important result in this area dates from the middle of the last century and belongs to Stokes [1]. Since the classic works of Nekrasov [2] and Levi-Civita [3] performed in the 1920s, many publications have been devoted to this subject. (See, for instance, the beautiful book of Stoker [4] and references therein.) The mathematical aspects of these works gave a powerful impulse to the development of certain branches in the theory of integral equations and in functional analysis.

For the nonstationary surface phenomena studied in the 1960s and later, the Lagrangian description was more common [5,6]. Some authors (see [7] and the review [8] and references therein) tried to perform an analytical continuation with respect to Lagrangian coordinates. However, these coordinates do not allow a proper conformal

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mapping, since their analytical continuation has singularities in both half-planes. Recently Tanveer [9,10] suggested applying the conformal mapping to the nonstationary problem directly in the Euler description. He applied for the periodic deep water case the mapping of the fluid region into the interior of the unit circle. The equations obtained are quite complicated and therefore are difficult for both analytical and numerical analysis.

A convenient approach to the description of the potential flow of a fluid with a free surface in any dimension may be obtained by using of the Canonical Formalism known since 1968 (see [11]). For 2D geometry a combination of the Canonical Formalism and the conformal mapping appears to be the most natural approach to the problem. This approach was implemented for the deep fluid in the recent paper [12]. Both gravity and surface tension were taken into consideration. The equations obtained in the paper [12] can be written in two different forms – implicit (not resolved with respect to time derivatives of surface shape and surface potential) and explicit (resolved with respect to time derivatives). The implicit equations are simple and symmetric. In the absence of surface tension they contain only quadratic nonlinearity. The explicit equations, though not as simple as the implicit ones, are perfectly suited for numerical simulation.

The most interesting unresolved problems in free-surface hydrodynamics are associated with formation of singularities (wave breaking) and essential modification of the surface geometry – wave generation, sprays, plumes, and droplets. Only in very special cases [13] they can be solved by using a traditional perturbation technique against the background of the flat surface. In these cases the Jacobian J of the conformal mapping remains close to unity. However, in many typical cases the Jacobian takes very large values in some piece of the surface. In this situation one can treat the inverse Jacobian $1/J$ as a small parameter and expand the solution in its powers. The first step in this direction was done in our paper [12]. Due to the essential nonlocality of the basic equation, the whole procedure of expansion in powers of $1/J$ is tricky, but a first approximation can be found easily. It is interesting that in this case the system of two equations for surface shape and surface potential reduces to a single equation, coinciding for some mysterious reason with the well-known Lagrangian Growth Equation (LGE). This equation is completely integrable, it has an infinite set of special solutions expressed in elementary functions. Among them are the solutions describing formation of finger-type configurations (quite similar to the Saffman–Taylor fingers [14]) as well as the solutions describing formation of droplets.

In the present paper we shall present a more detailed description of the results, briefly announced in the short letter [12]. We also extend most of our previous results to the case of a fluid of a finite depth.

The paper is organized as follows. In Section 2 we introduce the Lagrangian description of a free-surface fluid of a finite depth, combining Canonical Formalism and conformal mapping, and derive the implicit equation as the corresponding Euler–Lagrange equation. In Section 3 we find the explicit equation. In section 4 we consider stationary waves and calculate the dependence of their dispersion relation on the wave amplitude. The first approximation in the high-Jacobian expansion is introduced in Section 5. In Sections 6 and 7 we study finger-type and droplet-type solutions, respectively.

2. Lagrangian description of a finite depth fluid

Let an incompressible fluid fill a domain on the (x, y)-plane bounded by the free surface,

$$y = \eta(x, t), \quad -\infty < x < \infty, \tag{2.1}$$

and the bottom,

$$y = -\bar{h}.$$

The fluid flow in $-\bar{h} < y < \eta$ is potential,

$$V = \nabla\Phi, \quad \Delta\Phi = 0. \tag{2.2}$$

The Laplace equation (2.2) must be considered together with the following boundary conditions.

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Phi}{\partial y}\bigg|_{y=\eta}, \tag{2.3}$$

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2\bigg|_{y=\eta} + g\eta = 0, \tag{2.4}$$

$$\frac{\partial\Phi}{\partial y}\bigg|_{y=\tilde{h}} = 0. \tag{2.5}$$

In (2.4) g is gravity acceleration. Let us introduce the quantity

$$\Psi(x,t) = \Phi(x,y,t)|_{y=\eta} = \Phi(x,\eta(x,t),t). \tag{2.6}$$

As found in [11], $\eta(x,t)$ and $\Psi(x,t)$ are canonically conjugate variables,

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi}, \qquad \frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta}, \tag{2.7}$$

where the Hamiltonian H is the total energy of the fluid,

$$H = T + U, \quad T = \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\eta}^{\eta} (\nabla\Phi)^2 \, dy, \tag{2.8}$$

$$U = \frac{g}{2}\int \eta^2 \, dx. \tag{2.9}$$

Eqs. (2.7) and (2.8) extremize the action,

$$\delta S = 0, \quad S = \int L \, dt, \tag{2.10}$$

$$L = \int \Psi\frac{\partial\eta}{\partial t}\, dx - H. \tag{2.11}$$

Let us apply the conformal mapping of the domain on the plane $z = x + iy$,

$$-\infty < x < \infty, \quad -\bar{h} \leq y \leq \eta(x,t), \tag{2.12}$$

to the strip,

$$-\infty < u < \infty, \quad -h \leq v \leq 0, \tag{2.13}$$

on the plane $\omega = u + iv$ [1]. After this transformation, the shape of the surface is given parametrically by

$$y = y(u,t), \qquad x = u + \tilde{x}(u,t). \tag{2.14}$$

Functions y and $\tilde{x}$ are connected by the relation,

$$y = \hat{R}\tilde{x}. \tag{2.15}$$

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[1] If $\eta \to 0$ at $|x| \to \infty$, then $\bar{h} = h$. In the periodical case $\bar{h}$ and h are different.

Here $\hat{R}$ is the operator,

$$\hat{R} f(x) = \frac{1}{2h} P.V. \int\limits_{-\infty}^{\infty} \frac{f(x')}{\sinh \pi/2h(x'-x)} \, dx'. \tag{2.16}$$

Taking Fourier transforms gives

$$y_k = R_k x_k \quad R_k = i \tanh kh. \tag{2.17}$$

In the limiting case of infinitely deep water $h \to \infty$ and $\hat{R}$ goes over to the Hilbert transformation,

$$\lim_{h \to \infty} \hat{R} = \hat{H}, \tag{2.18}$$

$$\hat{H} f = \frac{1}{\pi} P.V. \int\limits_{-\infty}^{\infty} \frac{f(u') \, du'}{u' - u}. \tag{2.19}$$

One can introduce also the inverse operator $\hat{T}$,

$$\tilde{x} = \hat{T} y, \quad \text{with } \hat{R}\hat{T} = \hat{T}\hat{R} = 1, \tag{2.20}$$

in which

$$\hat{T} f = \frac{1}{h} P.V. \int\limits_{-\infty}^{\infty} \frac{f(u')}{1 - e^{-\pi/h(u-u')}} \, du'. \tag{2.21}$$

Asymptotically as $h \to \infty$,

$$\hat{T} \to -\hat{R}, \quad \text{but } \hat{T}^{-1} \neq -\hat{R}. $$

Both operators $\hat{R}, \hat{T}$ are anti-self-adjoint,

$$\hat{R}^+ = -\hat{R} \quad \text{and} \quad \hat{T}^+ = -\hat{T} \tag{2.22}$$

with Fourier transforms

$$\tilde{x}_k = T_k y_k \quad \text{and} \quad T_k = -i \coth kh. \tag{2.23}$$

After the conformal transformation, the velocity potential $\Phi = \Phi(u, v)$ obeys the Laplace equation

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0 \tag{2.24}$$

with boundary conditions

$$\frac{\partial \Phi}{\partial v}\bigg|_{v=-h} = 0 \quad \text{and} \quad \Phi|_{v=0} = \Psi(u, t). \tag{2.25}$$

One can check that

$$\frac{\partial \Phi}{\partial v}\bigg|_{v=0} = -\frac{\partial}{\partial u} \hat{R} \Psi = -\hat{R} \Psi_u, \tag{2.26}$$

so the kinetic energy is

$$T = -\frac{1}{2} \int_{-\infty}^{\infty} \Psi \hat{R} \Psi_u . \tag{2.27}$$

Then

$$\mathrm{d}x = x_u \, \mathrm{d}u, \qquad \eta_t \, \mathrm{d}x = (y_t x_u - x_t y_u) \, \mathrm{d}u, \tag{2.28}$$

and the Lagrangian can be expressed as follows,

$$L = \int_{-\infty}^{\infty} \Psi (y_t x_u - x_t y_u) \, \mathrm{d}u + \frac{1}{2} \int_{-\infty}^{\infty} \Psi \hat{R} \Psi_u \, \mathrm{d}u - \frac{g}{2} \int_{-\infty}^{\infty} y^2 x_u \, \mathrm{d}u + \int_{-\infty}^{\infty} f(y - \hat{R}\tilde{x}) \, \mathrm{d}u. \tag{2.29}$$

Here f is the Lagrange multiplier which imposes relation (2.20). Hamilton's principle,

$$\frac{\delta S}{\delta \Psi} = 0,$$

gives the following equations,

$$y_t x_u - x_t y_u = -\hat{R}\Psi_u, \qquad (1 + \tilde{x}_u) y_t - \tilde{x}_t y_u = -\hat{R}\Psi_u. \tag{2.30}$$

The mean level of fluid is constant, so

$$\int_{-\infty}^{\infty} (y_t x_u - x_t y_u) \, \mathrm{d}u = 0. \tag{2.31}$$

From (2.30) one finds

$$\Psi = -\hat{T}\delta^{-1}(y_t x_u - x_t y_u) + C(t). \tag{2.32}$$

Substituting (2.29) into (2.32) and using the identity (2.31) implies

$$L = T - U + \int_{-\infty}^{\infty} f(y - \hat{R}\tilde{x}) \, \mathrm{d}u. \tag{2.33}$$

If $y = y(u, y)$ is chosen as the set of coordinates, one can drop the last term in (2.33). Consequently,

$$L = T - U. \tag{2.34}$$

Consider now the conditions,

$$\frac{\delta S}{\delta y} = 0 \quad \text{and} \quad \frac{\delta S}{\delta x} = 0.$$

These conditions imply the equations

$$\Psi_t x_u - \Psi_u x_t + g y x_u = f, \tag{2.35}$$

$$\Psi_t y_u - \Psi_u x_t + g y y_u = -\hat{R} f, \tag{2.36}$$

which are equivalent to the equation

$$\Psi_t y_u - \Psi_u y_t + g y y_u + \hat{R}(\Psi_t x_u - \Psi_u x_t + g y x_u) = 0. \tag{2.37}$$

Eqs. (2.30) and (2.37) constitute the complete system of equations describing the potential flow of a free-surface fluid. They are not resolved with respect to time-derivatives, rather they are written in implicit form.

3. Explicit form of the motion equations

We resolve the system (2.30), (2.35) and (2.36) with respect to time-derivatives only in the simplest case, $h = \infty$. In this case, the equations take the following form,

$$y_t x_u - x_t y_u = -\hat{H}\Psi_u, \tag{3.1}$$

$$\Psi_t x_u - \Psi_u x_t + g y x_u = f, \tag{3.2}$$

$$\Psi_t y_u - \Psi_u y_t + g y y_u = -\hat{H} f. \tag{3.3}$$

Since $z = x + iy$, we have $x = \frac{1}{2}(z + z^*)$ and $y = \frac{1}{2}i(z - z^*)$. So Equation (3.1) can be rewritten as

$$z_t z_u^* - z_t^* z_u = -2i\hat{H}\Psi_u \quad \text{or} \quad \frac{z_t}{z_u} - \frac{z_t^*}{z_u^*} = -\frac{2i\hat{H}\Psi_u}{|z_u|^2}. \tag{3.4}$$

We introduce projection operators,

$$p^+ = \frac{1}{2}(1 - i\hat{H}), \quad p^- = \frac{1}{2}(1 + i\hat{H}), \quad (p^+)^2 = p^+, \quad (p^-)^2 = p^-, \tag{3.5}$$

which satisfy,

$$p^- \frac{z_t}{z_u} = \frac{z_t}{z_u}, \qquad p^- \frac{z_t^*}{z_u^*} = 0. \tag{3.6}$$

Hence, one obtains from (3.4)

$$z_t = z_u(\hat{H} - i)\frac{\hat{H}\Psi_u}{|z_u|^2}, \tag{3.7}$$

or,

$$y_t = (y_u \hat{H} - x_u)\frac{\hat{H}\Psi_u}{|z_u|^2}, \tag{3.8}$$

$$x_t = (y_u + x_u \hat{H})\frac{\hat{H}\Psi}{|z_u|^2}. \tag{3.9}$$

Subtracting (3.3) multiplied by x_u from (3.2) multiplied by y_u and using (3.1) yields

$$-\Psi_u \hat{H}\Psi_u = y_u f + x_u \hat{H} f. \tag{3.10}$$

Introducing

$$\Psi_u^{\pm} = p^{\pm}\Psi_u \quad \text{and} \quad f^{\pm} = p^{\pm} f$$

leads to the relation

$$(\Psi_u^+)^2 - (\Psi_u^-)^2 = -i\Psi_u \hat{H}\Psi_u. \tag{3.11}$$

Consequently, the identity (3.10) can be rewritten as follows,

$$(\Psi_u^+)^2 - (\Psi_u^-)^2 = f^+ z_u^* - f^- z_u. \tag{3.12}$$

From (3.12) one finds immediately that

$$f^+ = \frac{(\Psi_u^+)^2}{z_u^*} \quad \text{and} \quad f^- = \frac{(\Psi_u^-)^2}{z_u}. \tag{3.13}$$

We now subtract Eq. (3.3) multiplied by x_t from Eq (3.2) multiplied by y_t to find

$$(\Psi_t + gy)(y_t x_u - x_t y_u) = y_t f_u + x_t \hat{H} f = f^+ z_t^* - f^- z_t. \tag{3.14}$$

Time-derivatives x_t, y_t, z_t, z_t^* can be excluded by using (3.1), (3.8) and (3.9). Hence,

$$-\hat{H}\Psi_u(\Psi_t + gy) = [(\Psi_u^+)^2 + (\Psi_u^-)^2]\frac{\hat{H}\Psi_u}{|z_u|^2} - [(\Psi_u^+)^2 - (\Psi_u^-)^2]\hat{H}\frac{\hat{H}\Psi_u}{|z_u|^2}. \tag{3.15}$$

By means of the identity (3.11) one can divide this equation by $H\Psi_u$ and finally obtain

$$\Psi_t + gy = -\frac{(\Psi_u^+)^2 + (\Psi_u^-)^2}{J} + \Psi_u \hat{H}(\hat{H}\Psi_u/J). \tag{3.16}$$

The last equation can be rewritten as

$$\Psi_t + gy = \frac{\hat{H}(\Psi_u \hat{H}\Psi_u)}{|z_u|^2} + \Psi_u \hat{H}\left(\frac{\hat{H}\Psi_u}{|z_u|^2}\right). \tag{3.17}$$

4. Stationary waves

We rewrite the implicit equations for surface shape and hydrodynamic potential as

$$y_t(1 + \tilde{x}_u) - x_t y_u = -\hat{R}\Psi_u, \tag{4.1}$$

$$\Psi_t y_u - \Psi_u y_t + gy y_u + \hat{R}(\Psi_t 91 + \tilde{x}_u) - (x_t \Psi_u + gy \tilde{x}_u) = 0. \tag{4.2}$$

The last equation has a particular solution,

$$y = \hat{y}(u - ct), \qquad \tilde{x} = \hat{x}(u - ct), \qquad \Psi = \hat{\Psi}(x - ct), \tag{4.3}$$

which describes stationary waves propagating with a constant velocity c.

Substituting (4.3) into (4.1) yields

$$cy_u = \hat{R}\Psi_u. \tag{4.4}$$

Substituting (4.3) into (4.2) and using relation (4.4) leads to

$$-(c^2 + 2gb_0)y_u + gy_u + g\hat{R}(y(1 + \tilde{x}_u)) = 0. \tag{4.5}$$

Hereafter, we assume that all functions are periodic with period $L = 2\pi/k$. We assume further that the total amount of fluid is conserved,

$$\langle y(1 + \hat{x}_u) \rangle = 0. \tag{4.6}$$

One can then find a solution of Eq. (4.5) in the form

$$y = a_0 + \sum_{n=1}^{\infty} a_n \cos nku. \tag{4.7}$$

It is convenient to introduce the operator

$$\hat{S} = \hat{R}\partial^{-1},$$

satisfying

$$\hat{S}^{-1}y = \sum_{n=1}^{\infty} \frac{a_n}{S_n} \cos nku,$$

where

$$S_n = \frac{\tanh knh}{kn}.$$

From (4.6) one can define a_0 as

$$a_0 = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{a_n^2}{S_n}. \tag{4.8}$$

For b_0 there is the relation following from (3.17),

$$b_0 = a_0 - \frac{c^2}{g}\left\langle 1 - \frac{1}{|z_u|^2} \right\rangle \tag{4.9}$$

The other coefficients a_n satisfy the following system of equations:

$$(\tilde{c}^2 - gS_1)a_1 = \frac{g}{2}\sum_{m=1}^{\infty}\left(1 + \frac{S_1(S_m + S_{m+1})}{S_m S_{m+1}}\right)a_m a_{m+1}, \tag{4.10}$$

$$(\tilde{c}^2 - gS_n)a_n = \frac{g}{2}\sum_{m=1}^{\infty}\left(1 + \frac{S_n(S_m + S_{n+m})}{S_m S_{n+m}}\right)a_m a_{n+m}$$

$$+ \frac{g}{4}\sum_{m=1}^{n-1}\left(1 + \frac{S_n(S_m + S_{n-m})}{S_m S_{n-m}}\right)a_m a_{n-m}. \tag{4.11}$$

Here $\tilde{c}^2 = c^2 - 2g(a_0 - b_0)$. In these equations the amplitude of the first harmonic, a_1, is arbitrary. It must be treated as small. In a zero approximation,

$$a_0 = 0, \quad b_0 = 0 \quad \text{and} \quad c^2 = \tilde{c}^2 = \frac{g}{k}\tanh kh, \tag{4.12}$$

in accordance with the linear theory. In the first nontrivial approximation,

$$a_2 = a_1^2 k \frac{3 + \tanh^2 kh}{4 \tanh^3 kh},$$

$$c^2 = \frac{g}{k} \tanh kh \left(1 + \left[\frac{9 - 6 \tanh^2 kh + 5 \tanh^4 kh}{8 \tanh^4 kh} \right] (ka_1)^2 \right). \tag{4.13}$$

If we get back to the physical depth $\bar{h}$ the dispersion relation (4.13) exactly coincides with that of in [15],

$$c^2 = \frac{g}{k} \tanh k\bar{h} \left(1 + \left[\frac{9 - 10 \tanh^2 k\bar{h} + 9 \tanh^4 k\bar{h}}{8 \tanh^4 k\bar{h}} \right] (ka_1)^2 \right). \tag{4.14}$$

In the limiting case of deep water one gets

$$c^2 = (g/k)(1 + (ka_1)^2). \tag{4.15}$$

This result was first obtained by Stokes [1]. For shallow water,

$$\hat{S} \simeq h(1 - \tfrac{1}{3} h^2 \partial^2 / \partial x^2),$$

and Eq. (4.5) goes over to a differential equation describing KdV-type solitons.

5. High-Jacobian approximations

Let us suppose that the function $z = z(\omega)$ has a singularity in the upper half-plane on the imaginary axis close to zero. In the vicinity of zero the quantity $J = J(u)$ is a very singular function. But it might happen that $1/J$ is in this region a smooth function close locally to zero. In this case one can develop a new type of an approximate theory. We will discuss only the case of infinite depth. We will seek a solution of Eq. (3.10) in the form

$$\Psi = -\frac{1}{2} \int \lambda^2 \, dt + \lambda(t) y + \tilde{\Psi}. \tag{5.1}$$

Here

$$\dot{\lambda} = -g \tag{5.2}$$

and $\lambda = \lambda(t) = \lambda_0 - gt$ is a linear function of time.

The idea of separating $\tilde{\Psi}$ from Ψ is the following. Let the singularity in Ψ be posed at a distance $\delta \ll 1$ from the real axis. Then the functions $\Psi(u)$ and $y(u)$ are sharp, they change their value essentially in a region $u \simeq \delta$. Our central assumption is that in (5.1) $\tilde{\Psi}$ is a "smooth" function. It varies on a scale of order of unity.

To justify this assumption we must write the equation for $\tilde{\Psi}$. To obtain this equation we exploit the identities,

$$(1 - i\hat{H})(\tilde{x}_u + iy_u)^2 = 0, \tag{5.3}$$

$$(1 + i\hat{H})(1/z^* - 1) = 0. \tag{5.4}$$

These imply

$$\hat{H}(y_u \hat{H} y_u) = \tfrac{1}{2} \left[(\tilde{x}_u)^2 - y_u^2 \right] \tag{5.5}$$

and

$$\frac{y_u}{J} + \hat{H}(x_n/J - 1) = 0. \tag{5.6}$$

After a simple transformation one finds

$$\frac{\partial \tilde{\Psi}}{\partial t} = \frac{\lambda^2}{2J} + \lambda \left(\frac{1}{J} \hat{H} \tilde{\Psi}_u + \tilde{\Psi}_u \hat{H} (1/J - 1) \right) + \frac{1}{2J} \left((\hat{H} \tilde{\Psi}_u)^2 - \tilde{\Psi}_u^2 \right) + \tilde{\Psi}_u \hat{H} \frac{\hat{H} \tilde{\Psi}_u}{J}. \tag{5.7}$$

Eq. (5.7) is an *exact* relation. All information about the shape of surface is hidden in the function $1/J$, which according to our assumption is smooth and "large scale". Therefore, we may consider $\tilde{\Psi}$ smooth and large scale as well.

In the first approximation, we put

$$\tilde{\Psi} = 0. \tag{5.8}$$

The corresponding equation for the surface shape looks very simple in its implicit form,

$$y_t(1 + \tilde{x}_u) - \tilde{x}_t y_u = \lambda \tilde{x}_u. \tag{5.9}$$

Let us denote

$$\tilde{z} = \tilde{x} + \mathrm{i} y.$$

In terms of $\tilde{z}$ Eq. (5.9) can be rewritten as follows,

$$\tilde{z}_t - \tilde{z}_t^* + \tilde{z}_t \tilde{z}_u^* - \tilde{z}_t^* \tilde{z}_u = \mathrm{i}\lambda (\tilde{z}_u + \tilde{z}_u^*). \tag{5.10}$$

By introducing a new function

$$z = \tilde{z} + u - \mathrm{i} \int \lambda \, \mathrm{d}t,$$

one transforms (5.10) to the form

$$\Im(z_t z_u^*) = -\lambda. \tag{5.11}$$

In the simplest case $\lambda = \mathrm{const.}$, this equation has been known in the literature since 1945 [16,17]. It is usually called the LGE and is used widely in the theory of interfaces and dendrite growth.

6. Finger-type solutions

Eq. (5.10) is an integrable system. One can find a general solution of this equation starting from a very wide class of special solutions (N-finger solutions),

$$\tilde{z} = \sum_{n=1}^{N} q_n \log(u - a_n(t)). \tag{6.1}$$

Here N is any positive integer (including $N = \infty$), q_n are complex constants and $\Im a_n > 0$. Strictly speaking, to satisfy the condition $\tilde{z} \to 0$ at $|u| \to \infty$, one must demand

$$\sum_{n=1}^{N} q_n = 0. \tag{6.2}$$

However, the constraint (6.2) is not significant from the physical view-point. For arbitrary choice of q_n it can be satisfied by adding to (6.1) one more term,

$$-\left(\sum_{n=1}^{N} q_n\right) \log(u - a_{N+1}(t)), \quad \Im a_{N+1} \to +\infty.$$

Substituting (6.1) into (5.10) and using an expansion in the sum of elementary fractions, one obtains a system of ODE for a_n,

$$\dot{a}_n + \sum_m q_m^* \frac{\dot{a}_n - \dot{a}_m^*}{a_n - a_m^*} = -i\lambda(t). \tag{6.3}$$

Integration by t gives the following system of transcendental equations,

$$a_n + \sum_m q_m^* \log(a_n - a_m^*) = -\int \lambda(t)\, dt + C_n, \tag{6.4}$$

where C_n are arbitrary complex constants. The simplest possible solution (one-finger solution) of this type is

$$\tilde{z} = -i \log(u - ib(t)), \quad \text{where } b \text{ is real.} \tag{6.5}$$

Now

$$b + \log b = -\int \lambda\, dt + \log r, \quad \text{where } r \text{ is a real constant.} \tag{6.6}$$

If $\int \lambda\, dt \to +\infty$, one finds the asymptotic behavior of b,

$$b \simeq r \exp\left(-\int \lambda\, dt\right) \to 0, \quad t \to \infty. \tag{6.7}$$

In our case, we find

$$y = -\log\sqrt{u^2 + b^2(t)}, \quad \tilde{x} = \arctan\frac{u}{b(t)}, \quad \frac{1}{J} = \frac{u^2 + b^2(t)}{(1 + b(t))^2 + u^2}. \tag{6.8}$$

Let $\int \lambda\, dt$ be positive and large. Then

$$\frac{1}{J} \to \frac{u^2}{1 + u^2}. \tag{6.9}$$

This expression is small for $u \simeq b$, and is indeed smooth and large scale. From (5.2) one gets

$$\int \lambda\, dt = -\tfrac{1}{2}gt^2 + Ct. \tag{6.10}$$

For positive g (stable case), $\int \lambda\, dt$ can be large only during a finite time (if C is positive and large). For $g \le 0$ (neutrally stable or unstable cases) the approximation improves as $t \to \infty$.

In the one-finger solution,

$$y(0, t) = -2 \log b(t) \simeq \int \lambda\, dt, \quad \text{as } t \to \infty. \tag{6.11}$$

At the same time, the curvature tends to a constant

$$1/R = \eta_{xx} \to \quad -\text{const} \quad \text{as } t \to \infty. \tag{6.12}$$

The one-finger solution was found by Saffman and Taylor [14]. N-finger solution were studied in the articles [18,19]. A general solution of Eq. (5.10) can be obtained from (6.1) by a kind of transition to the limit $N \to \infty$. We will discuss this procedure elsewhere.

7. Droplet-type solutions

Let us consider the simplest case of rational solution for LGE, namely

$$\tilde{z} = \alpha/(u - ib).$$ (7.1)

Here α and b both are positive. Substituting (7.1) into (5.11) and using an expansion in the sum of elementary fractions leads to a system of ODE for α and b,

$$\dot{\alpha}b + \alpha\dot{b} + \lambda\alpha = 0, \qquad \dot{\alpha}b^3 + \alpha\dot{\alpha}b - \alpha b^2 \dot{b} - \alpha^2 \dot{b} - \lambda\alpha b^2 = 0.$$ (7.2)

This set of equations has a solution which is valid until some final time t_0. Asymptotically, when $t \to t_0$ the solution approaches

$$\alpha \Rightarrow \delta b, \qquad b \Rightarrow \tfrac{1}{2}\lambda(t_0 - t),$$ (7.3)

where δ is a constant. This solution describes formation of a droplet which separates from the fluid at $t = t_0$. Again, this solution is valid provided $\hat{H}\Psi_u$ is small compared to $\tilde{x}_u$. To study the validity of this assumption one must estimate the solution of (5.7) where J is calculated from (7.1) and (7.3). Here we are able to solve only the linear (with respect to $\tilde{\Psi}$) part of (5.7),

$$\tilde{\Phi}_t + iU\tilde{\Phi}_u = \tfrac{1}{2}\lambda^2 U.$$ (7.4)

This is a linear complex transport equation which can be solved by the method of characteristics. Here,

$$U(u, t) = 1/J + i\hat{H}1/J, \qquad \tilde{\Phi}(u, t) = \tilde{\Psi} + i\hat{H}\tilde{\Psi}.$$ (7.5)

All functions in Eq. (7.4) are analytic in the lower half-plane. The equation for the characteristic is

$$\dot{u} = iU(u, t).$$ (7.6)

In the vicinity of t_0 the complex velocity U is given by

$$U \Rightarrow -\frac{i}{2}\frac{\delta u}{u^2 - \delta b}.$$ (7.7)

In this case the characteristic equation can be solved exactly, namely

$$u/(u^2 + \delta b) = C,$$ (7.8)

and the general solution of Eq. (7.4) is

$$\tilde{\Phi} = \tilde{\Phi}(u/(u^2 + \delta b)) - \tfrac{1}{2}iu,$$ (7.9)

where $\tilde{\Phi}$ is arbitrary function.

Let us impose zero initial condition for $\tilde{\Phi}$ at $\delta b = 1$. Then the solution acquires the form

$$\tilde{\Phi} = \frac{1}{4}\left[i\frac{u^2 + \delta b}{u} + \sqrt{4 - \left(\frac{u^2 + \delta b}{u}\right)^2} - i\frac{u}{2}\right]. \tag{7.10}$$

The behavior of $\hat{H}\tilde{\Psi}_u = \Re\tilde{\Phi}_u$ at $u = 0$ is proportional to C/b and is similar to that for $\tilde{x}_u$.

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