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Toward an integrable model of deep water

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Abstract

Using the combination of the canonical formalism and conformal mapping, a theory of the free surface of deep water in the approximation of a high curvature is developed. It is shown that the numerical simulation is in excellent agreement with the analytical description.

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1. Introduction

The prospect of finding an integrable model in the theory of potential flow of an incompressible, infinitely deep fluid with a free surface is one of the most intriguing in mathematical physics. We will discuss in this article only a 2-D fluid (one dimension is the depth). So far, the only integrable model for deep water is the 1-D nonlinear Schrödinger equation [1], which describes an envelope of a quasi-monochromatic wave train of a small amplitude. But this model is not specific for surface waves. It is a universal model for generic nonlinear dispersive Hamiltonian wave systems.

Some information about the integrability can be extracted from an analysis of small-angle motion when the free surface is just slightly different from flat. Such an analysis can be made by expanding the Hamiltonian in powers of the nonlinearity and by then using Poincaré's normal form technique.

Several essential results were achieved recently in this direction. It was shown [2,3] that in the simplest case of a zero-density fluid with no capillarity, the first two terms in the expansion of the Hamiltonian give a very simple integrable system, but very little is known about the possible role of higher order terms. Nevertheless, the conjecture of integrability of the full system does not seem improbable.

In the presence of gravity one has to expand the Hamiltonian at least to the fourth order. Then, by applying a proper canonical transformation, third-order terms can be eliminated. It is remarkable that, for some unclear reasons, the resulting fourth-order Hamiltonian is equal identically to zero on the resonant manifold [4,5]. So in this approximation the system is integrable, but cumbersome calculations, performed by using an analytical computer program, showed that the effective fifth-order term does not vanish on the corre-

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sponding resonant manifold [6]. So it is proved that a free surface deep fluid in the presence of gravity is a nonintegrable system. It is interesting that in spite of a very complicated intermediate calculation the final expression for the effective fifth-order Hamiltonian is very simple [6].

As the non-zero-density free surface deep fluid is nonintegrable, we have no hope for integrability in the most realistic case, when both gravity and capillarity occur, but the case of pure capillarity is not so clear. Nothing is known about the nonstationary behavior of one-dimensional capillary waves, but the shape of stationary capillary waves (unlike gravity waves) can be found explicitly and expressed in elementary functions [7].

In the present article we offer a new approach to this problem. Instead of considering an almost flat surface, we now study an extremely curved one, close to the state of wave breaking. We will show that in this case there is a possibility to find a wide class of fluid motion, which can be described approximately by a certain integrable model, known in mathematical physics since 1945 and widely used nowadays in the theory of interface dynamics. In this article we essentially exploit our previous results, obtained in Ref. [8].

2. Approximation of a high curvature

Let us consider a non-zero-density incompressible fluid filling the domain $-\infty < x < \infty$, $-\infty < y < \eta(x,t)$. The surface of the fluid $\eta(x,t)$ is free, and the flow is the potential

$$v = \nabla\phi, \quad \triangle\phi = 0. \tag{2.1}$$

One can perform the conformal mapping of the domain filled by the fluid to the lower half-plane of a complex variable $w = u + \mathrm{i}v$

$$-\infty < u < \infty, \quad -\infty < v < 0.$$

Now the shape of the surface is given parametrically,

$$y = y(u,t), \quad x = x(u,t) = u + \tilde{x}(u,t), \tag{2.2}$$

where

$$y = \hat{H}\tilde{x}, \quad \tilde{x} = -\hat{H}y, \quad \hat{H}^2 = -1,$$

where $\hat{H}$ is the Hilbert transformation. After the conformal mapping $\phi(x,y,t) \to \phi(u,v,t)$. Let $\Psi(u,t) = \phi(u,0,t)$. It was shown in Ref. [8] that $y(u,t)$ and $\Psi(u,t)$ obey the following system of equations,

$$y_t = (y_u\hat{H} - x_u)\frac{\hat{H}\Psi_u}{J}, \tag{2.3}$$

$$\Psi_t = \frac{\hat{H}(\Psi_u\hat{H}\Psi_u)}{J} + \Psi_u\hat{H}\left(\frac{\hat{H}\Psi_u}{J}\right) - gy, \tag{2.4}$$

where g is the gravity acceleration and

$$J = x_u^2 + y_u^2 = 1 + 2\tilde{x}_u + \tilde{x}_u^2 + y_u^2. \tag{2.5}$$

Another form of Eq. (2.3) (see Ref. [8]) is

$$y_t x_u - x_t y_u = -\hat{H}\Psi_u. \tag{2.6}$$

Let us present Ψ in the following form,

$$\Psi = a(t) + \lambda(t)y + \tilde{\Psi}. \tag{2.7}$$

Here

$$\dot{\lambda} = -g, \quad \dot{a} = -\tfrac{1}{2}\lambda^2. \tag{2.8}$$

One can show that y and $\tilde{\Psi}$ satisfy the following system of equations,

$$y_t = -\lambda(y_u\hat{H} - x_u)\frac{\tilde{x}_u}{J} + (y_u\hat{H} - x_u)\frac{\hat{H}\tilde{\Psi}_u}{J}, \tag{2.9}$$

$$\tilde{\Psi}_t = \frac{\lambda^2}{2J} + \lambda\left(\frac{1}{J}\hat{H}\tilde{\Psi}_u + \tilde{\Psi}_u\hat{H}\frac{1}{J}\right) + \frac{1}{2J}[(\hat{H}\tilde{\Psi}_u)^2 - \tilde{\Psi}_u^2] + \tilde{\Psi}_u\hat{H}\frac{\hat{H}\tilde{\Psi}_u}{J}. \tag{2.10}$$

Our key assumption is the following: in some small vicinity of the point $u = 0$

$$|\tilde{x}_u| \gg |\hat{H}\tilde{\Psi}_u|. \tag{2.11}$$

So one can neglect in (2.9) the last term and get a closed form equation for y,

$$y_t = -\lambda(y_u\hat{H} - x_u)\frac{\hat{H}\tilde{x}_u}{J}. \tag{2.12}$$

Substituting (2.7) into (2.6) and casting off $\hat{H}\tilde{\Psi}_u$ one can find another form of Eq. (2.12),

$$y_t(1 + \tilde{x}_u) - \tilde{x}_t y_u = \lambda\tilde{x}_u. \tag{2.13}$$

Condition (2.11) holds when the Jacobian J together with the curvature of the surface tends to infinity. So we call it the approximation of a high curvature.

3. Finger-type solutions

Let us denote $\tilde{z} = \tilde{x} + iy$. In terms of $\tilde{z}$ Eq. (2.13) can be rewritten as follows,

$$\tilde{z}_t - \tilde{z}_t^* + \tilde{z}_t\tilde{z}_u^* - \tilde{z}_t^*\tilde{z}_u = i\lambda(t)(\tilde{z}_u + \tilde{z}_u^*). \tag{3.1}$$

This is an integrable system. One can find a general solution of this equation starting from a very wide class of special solutions (N-finger solutions)

$$\tilde{z} = \sum_{n=1}^{N} q_n \log[u - a_n(t)]. \tag{3.2}$$

Here N is any positive integer (including $N = \infty$), q_n are some complex constants and $\operatorname{Im} a_n > 0$. Strictly speaking, to satisfy the condition $\tilde{z} \to 0$ at $|u| \to \infty$, one has to demand

$$\sum_{n=1}^{N} q_n = 0, \tag{3.3}$$

but constraint (3.3) is not significant from a physical point of view. For an arbitrary choice of q_n it can be satisfied by adding to (3.2) one more term

$$-\left(\sum_{n=1}^{N} q_n\right) \log[u - a_{N+1}(t)], \quad \operatorname{Im} a_{N+1} \to +\infty.$$

Substituting (3.2) into (3.1) and using the expansion to the sum of elementary fraction, one can obtain the following system of ODEs for a_n,

$$\dot{a}_n + \sum_m q_m^* \frac{\dot{a}_n - \dot{a}_m^*}{a_n - a_m^*} = -i\lambda(t). \tag{3.4}$$

Integration by t gives the system of transcendental equations

$$a_n + \sum_m q_m^* \log(a_n - a_m^*) = -\int \lambda(t)\,dt + C_n, \tag{3.5}$$

where C_n are arbitrary complex constants. The simplest possible solution (one-finger solution) of this type is

$$\tilde{z} = -i\log[u - ib(t)], \quad b \text{ is real}. \tag{3.6}$$

Now

$$b + \log b = -\int \lambda\,dt + \log r$$

r is some real constant. (3.7)

If $\int \lambda\,dt \to +\infty$, one obtains the asymptotic behavior of b,

$$b \simeq r\exp\left(-\int \lambda\,dt\right) \to 0, \quad t \to \infty. \tag{3.8}$$

In our case

$$y = -\log\sqrt{u^2 + b^2(t)}, \tag{3.9}$$

$$\tilde{x} = \arctan\frac{u}{b(t)}, \tag{3.10}$$

$$\frac{1}{J} = \frac{u^2 + b^2(t)}{[1 + b(t)]^2 + u^2}. \tag{3.11}$$

Let $\int \lambda\,dt$ be positive and large. Then

$$\frac{1}{J} \to \frac{u^2}{1 + u^2}. \tag{3.12}$$

This expression is small for $u \simeq b$, and we can justify our approximation (2.11). From (2.8) one obtains

$$\int \lambda\,dt = -\tfrac{1}{2}gt^2 + Ct. \tag{3.13}$$

For positive g (stable case) $\int \lambda\,dt$ can be large only during a finite time (if C is positive and large). For $g \leqslant 0$ (neutrally stable or unstable cases) the approximation improves at $t \to \infty$.

In one-finger solution

$$y(0,t) = -2\log b(t) \simeq \int \lambda\,dt,$$

at $t \to \infty$. (3.14)

At the same time the curvature

$$1/R = \eta_{xx} \to -\text{const}, \quad \text{at } t \to \infty \tag{3.15}$$

By introducing a new function,

$$Z = \tilde{z} + u - i\int \lambda\,dt, \tag{3.16}$$

we can simplify Eq. (2.13) to

$$\operatorname{Im}(Z_t Z_u^*) = -\lambda. \tag{3.17}$$

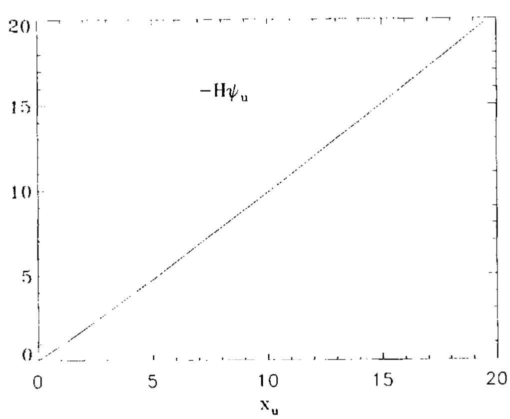

Fig. 1. $-\hat{H}\Psi_u$ as a function of $\tilde{x}_u$.

In the simplest case λ = const the equation has been known in the literature since 1945 [9,10]. The one-finger solution was found by Saffman and Taylor [11]. The N-finger solution was studied in Refs. [12,13]. A general solution of Eq. (3.1) can be obtained from (3.2) by a transition to the limit $N \to \infty$. We will not discuss this procedure here.

4. Numerical results

As it was reported in Ref. [8], we elaborated a numerical algorithm to integrate Eqs. (2.3), (2.4) in the case $g = 0$. We considered periodic boundary problem in the infinite half-strip with width 2π (both in real space and after a conformal transformation) in the absence of gravity and surface tension. Initial conditions were chosen symmetric with respect to the vertical axes: $\Psi(u) = \Psi(-u)$. This symmetry was used in the simulation, which allowed us to reduce the domain to $0 \leqslant u \leqslant \pi$. Initial conditions were chosen as follows,

$$\tilde{x} = y = 0, \quad \Phi = -A\log[1 - \exp(-iu - 1)],$$
$$A = 0.2\sinh(e). \tag{4.1}$$

The main goal of the numerical integration was to check the validity of the assumption (2.11), i.e. to verify that the potential Ψ does almost coincide with the shape of the surface.

Fig. 1 shows $-\hat{H}\Psi_u$ as a function of $\tilde{x}_u$. Its almost linear behavior clearly demonstrates that the term

$$\hat{H}\tilde{\Psi}_u = \lambda\tilde{x}_u + \hat{H}\Psi_u$$

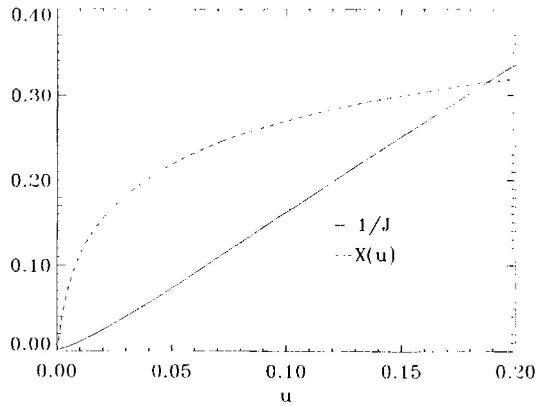

Fig. 2. Spatial behavior of the Jacobian of the conformal mapping. On the u-axis there are $\simeq 2000$ grid points.

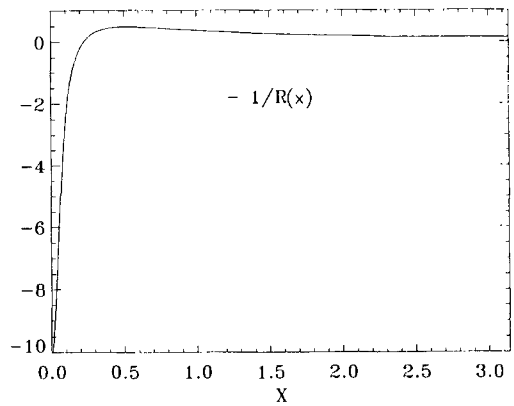

Fig. 3. Curvature of the surface in physical space.

is small with respect to $\tilde{x}_u$. We calculated from the numerical data $\lambda = 1.043$ here, and it practically does not depend upon time.

Another check of the approximation of high curvature theory is the spatial behavior of the Jacobian of the conformal mapping. It is shown in Fig. 2. To understand this picture better, one should realize that the surface at this moment evolves in the region $0 \leqslant |u| \simeq 0.05$, which corresponds to $0 \leqslant |x| \simeq 0.22$, and it decreases in time. We would like to emphasize here that the number of grid points in the interval $0 \leqslant u \leqslant 0.2$ (that is shown in the Fig. 2) is $\simeq 2000$. So in the vicinity of the origin the assumption $1/J \to 0$ improves when $t \to \infty$. The curvature of the surface in physical space is given in Fig. 3. Its value given in the origin grows faster in time than a linear function.

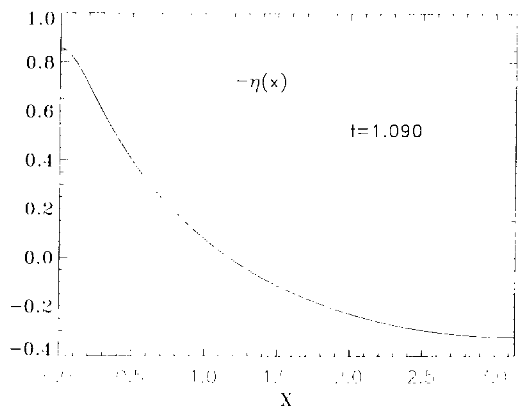

Fig. 4. Shape of the surface in physical space.

Thus, the high curvature near the origin indicates that a finger-type solution does not occur here. Instead of that, the shape of the surface in the physical space, $\eta(x)$ is very close to

$$\eta(x) \simeq \mathrm{Re}(-\alpha(t)\log\{1-\exp[-\mathrm{i}x-a(t)]\}), \tag{4.2}$$

where $\alpha(t)$ and $a(t)$ both are positive, and $\dot{\alpha}(t) > 0$, $\dot{a}(t) < 0$. The shape is shown in Fig. 4.

In conclusion we would like to stress that the approximation of a high curvature is in excellent agreement with numerical experiment, so that Eq. (2.13) or (3.17) describes the evolution of the free surface at large t. However, the simplest one-finger solution is not realized in the experiment. We have to use more general solutions of Eq. (2.1) to interpret the numerical results. The description of this solution will be published elsewhere. We will show also that on the infinite axis $-\infty < u < \infty$ the approximation of a high curvature generates a wide class of exact solutions of the system (2.3), (2.4). So far this fact was established only for the simplest case $g = 0$.

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