Optical solitons and quasisolitons

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Optical solitons and quasisolitons are investigated in reference to Cherenkov radiation. It is shown that both solitons and quasisolitons can exist, if the linear operator specifying their asymptotic behavior at infinity is sign-definite. In particular, the application of this criterion to stationary optical solitons shifts the soliton carrier frequency at which the first derivative of the dielectric constant with respect to the frequency vanishes. At that point the phase and group velocities coincide. Solitons and quasisolitons are absent, if the third-order dispersion is taken into account. The stability of a soliton is proved for fourth order dispersion using the sign-definiteness of the operator and integral estimates of the Sobolev type. This proof is based on the boundedness of the Hamiltonian for a fixed value of the pulse energy. [S1063-7761(98)02405-6]

1. INTRODUCTION

Solitons in nonlinear optical fibers have been very popular objects of investigation since the early nineteen seventies, i.e., since the structural stability of the solitons for the Korteweg-de Vries (KdV) equation[1] and the nonlinear Schrödinger equation[2] was demonstrated and since Hasegawa and Tappert[3] subsequently proposed the use of optical solitons as data bits in fiber communications. The interest in optical solitons has increased dramatically in the last decade due to the practical achievements from the use of solitons in modern optical communication systems.[4,5] However, despite the great practical significance of optical solitons, the theory for them is far from complete.

When reference is made to optical solitons, it is assumed that their spectrum is concentrated within a certain transparency window, where the linear damping is small and dispersion effects dominate. The width of the soliton spectrum $\delta\omega$ is assumed to be fairly small compared with the frequency band $\Delta\omega$ of that window, i.e., $\delta\omega \ll \Delta\omega$. In real systems, however, the band $\Delta\omega$ is always narrower than the mean frequency of the window $\overline{\omega}$, i.e., $\Delta\omega \ll \overline{\omega}$. Thus, we have the following hierarchy of inverse characteristic times:

$$\delta\omega \ll \Delta\omega \ll \overline{\omega}. \tag{1.1}$$

These criteria permit consideration of the slow ($\tau^{-1} \sim \delta\omega$) dynamics of soliton propagation in terms of amplitude envelopes. In particular, to derive a nonlinear Schrödinger equation (NLSE), i.e., the basic model for describing optical envelope solitons, the wave number is approximated by a quadratic polynomial

$$\delta k = \frac{1}{v_{\rm gr}}\delta\omega - \frac{1}{2}\frac{\omega''}{v_{\rm gr}^3}(\delta\omega)^2. \tag{1.2}$$

Here $\delta k = k - k_0$, $\delta\omega = \omega - \omega_0$, $v_{\rm gr} = \partial\omega/\partial k$ is the group velocity, and k_0 and ω_0 are the wave number and frequency of the soliton carrier wave. However, in the frequency interval $\Delta\omega$ the dispersion of the wave can differ significantly from the quadratic approximation (1.2), although it remains small in the sense of the criterion (1.1). It is noteworthy that the existing experimental possibilities (see, for example, Ref. 6) make it possible to obtain very short pulses, for which $\delta\omega/\omega_0 < 1$. On the other hand, the efficiency of optical fibers as media for transmitting information is inversely proportional to the soliton width. Thus, practical considerations call for reducing the soliton width as much as possible.

In this paper we show that the properties of ''short'' and ''long'' solitons can be very different. For short solitons the expansion (1.2) is largely incorrect and should be replaced by the more general formula

$$\delta k - \frac{1}{v_{\rm gr}}\delta\omega = -F(\delta\omega). \tag{1.3}$$

Here $F(\zeta)$ is a certain function, which should be taken from a microscopic treatment or extracted from experimental data. Although $F(\zeta)$ can be far from the parabolic dependence (1.2), averaging over the fast time $1/\omega_0$ can be performed, providing a description of slow soliton dynamics by means of a generalized nonlinear Schrödinger equation (GNLSE). This averaging also leads to the appearance of an additional integral of motion, viz., an adiabatic invariant, which has the meaning of the pulse energy. Accordingly, owing to this invariant, the GNLSE allows a soliton solution for the envelope of the electromagnetic field $E(x,t)$ in the form a propagating pulse with the additional phase multiplier $e^{i\lambda x}$:

$$E(x, t - x/v_{\rm gr}) = e^{i\lambda x}\psi(t - x/v_{\rm gr} + \beta x), \quad v_{\rm gr}^{-1} \gg \beta.$$

The main result of this paper is as follows. Solitons can exist, if $L(\zeta) = \lambda - \beta\zeta + F(\zeta)$ is a positive (or negative) definite function for all ζ. This criterion is the basic selection rule for solitons. If this criterion is not satisfied, the soliton loses its energy through Cherenkov radiation and ceases to exist after a certain time. This occurs, for example, if $F(\zeta)$ is a third-degree polynomial.

Even if $L(\zeta)$ is positive definite and a soliton exists, the question of its stability is far from trivial. In this paper we establish that a soliton is stable if $L(\zeta)$ is a positive definite fourth-order polynomial. The proof of its stability is based on the boundedness of the Hamiltonian for a fixed adiabatic invariant. We assume that the same conclusion regarding the stability will be valid for any positive definite polynomial $L(\zeta)>0$ of even degree. However, if we have

$$|F(\zeta)|<C|\zeta|^{\alpha} \quad \text{for } |\zeta|\to\infty \tag{1.4}$$

and $\alpha\leqslant 1/2$, stability of the soliton is doubtful, and it is more likely unstable.

There is one more important point on which we would like to focus attention in this article. The objects which have traditionally been called solitons in nonlinear optics are not such in the strict sense of the word. They are quasisolitons, i.e., approximate solutions of Maxwell's equations, which depend on four parameters. Real stationary solitons, which propagate with a constant velocity without changing their form, are exact solutions of Maxwell's equations, which depend on two parameters. The latter exist, if the dielectric constant $\varepsilon(\omega)$ has a maximum in the frequency range under consideration for a focusing nonlinearity or a minimum, if the medium is defocusing. In a purely conservative medium quasisolitons exist for a finite time owing to radiation as a result of multiphoton processes. In reality, however, this time is much greater than the lifetime resulting from the linear damping, and the difference between solitons and quasisolitons is insignificant.

2. STATIONARY SOLITONS

In this section we demonstrate how to find a soliton solution directly from Maxwell's equations. We consider a very simple model of the simultaneous propagation of pulses, assuming that the polarization is linear and that the electric field $E(x,t)$ is perpendicular to the propagation direction. In this case Maxwell's equations can be reduced to the wave equation for the field $E(x,t)$:

$$\frac{\partial^2 D}{\partial t^2}-c^2\frac{\partial^2 E}{\partial x^2}=0, \tag{2.1}$$

where the electric displacement D is assumed to be related to the electric field by the expression

$$D(x,t)=\hat{\varepsilon}(t)E(x,t)+\chi E^3(x,t). \tag{2.2}$$

In this expression $\hat{\varepsilon}$ is an integral operator; the Fourier transform of its kernel is $\varepsilon(\omega)$, i.e., the dielectric constant. The second term in (2.2) corresponds to the Kerr effect, and χ is the Kerr constant.

The function $\varepsilon(\omega)$ is analytically continuable into the upper half-plane of ω (see, for example, Ref. 7). For real values of ω the magnitude of $\varepsilon(\omega)$ obeys the Kramers–Kronig relations. It particular, it follows from these relations that on the real axis the imaginary part of the dielectric constant ε'', which is responsible for the dissipation of electromagnetic waves, cannot be equal to zero at all frequencies. Below we shall assume that there is a certain frequency band $\Delta\omega$, within which the imaginary part of the dielectric constant is small enough that it can be neglected.

Let us consider the propagation of a wave packet with a spectrum lying within this transparency window, assuming that the frequency width of the pulse spectrum is small compared with $\Delta\omega$. A solution in the form of an isolated pulse, i.e., a soliton, can be expected only under such conditions.

As was noted in the Introduction, two types of solitons are possible. The solitons of the first type are stationary in a moving frame. They propagate with a constant velocity without changing their form. A classical example of solitons of this type is provided by the solitons for the KdV equation, which, in particular, describe solitary waves in shallow water. The solitons of the other type are called quasisolitons. They have internal dynamics and propagate with a constant velocity only on the average. The classical quasisolitons include breezers, which are described by the sine-Gordon equation (for further information, see, for example, Refs. 8–10).

Stationary solitons are exact solutions of Eq. (2.1). We shall seek these solutions in the form

$$E=E(x-vt), \tag{2.3}$$

where v is the constant velocity and E tends to zero at infinity. The substitution of (2.3) into (2.1) makes it possible to integrate the equation twice:

$$\hat{L}E(x)=\alpha E^3(x), \quad \alpha=\chi v^2/c^2, \tag{2.4}$$

where the operator $\hat{L}$ equals

$$\hat{L}=1-\frac{v^2}{c^2}\hat{\varepsilon}. \tag{2.5}$$

In the Fourier representation $\hat{L}$ is written in the form

$$L(\omega)=1-\frac{v^2\varepsilon(\omega)}{c^2}, \tag{2.6}$$

where the frequency ω and the wave number k are related by the equality $\omega=kv$. The second term in (2.6) is the square of the ratio between v and the phase velocity of an electromagnetic wave of small amplitude:

$$v_{\text{ph}}=c/\sqrt{\varepsilon(\omega)}. \tag{2.7}$$

Hence it is easily seen that the operator $\hat{L}$ becomes positive definite if and only if

$$v_{\text{ph}}^2(\omega)>v^2, \tag{2.8}$$

for all ω, and it accordingly becomes negative definite in the opposite case:

$$v_{\text{ph}}^2(\omega)<v^2. \tag{2.9}$$

We now show that a soliton solution is possible only when condition (2.8) or (2.9) is satisfied. Let us assume that the opposite is true, i.e., let the conditions (2.8) and (2.9) not be satisfied. In this case the equation

$$\frac{v^2\varepsilon(\omega)}{c^2}=1 \tag{2.10}$$

has a solution (for simplicity we assume that it is unique: $\omega=\omega_0$). Then Eq. (2.3) can be rewritten in the following manner:

$$E(x-vt)=E_0(x-vt)+\hat{L}^{-1}(1-\hat{P})\alpha E^3(x-vt). \tag{2.11}$$

Here

$$E_0(x-vt)=\mathrm{Re}(A\,\exp[-i\omega_0(t-x/v)])$$

is the solution of the homogeneous linear equation

$$\hat{L}E_0=0, \tag{2.12}$$

and $\hat{P}$ is a projector onto the state $E_0(x-vt)$, so that $(1-\hat{P})\chi E^3(x-vt)$ is orthogonal to E_0 and, therefore, the operator $\hat{L}$ is reversible in this class of functions. To find the explicit solution of Eq. (2.11), we can use, for example, an iterative scheme, taking E_0 as the zeroth approximation. It is of fundamental significance that, by proceeding in this manner, we must arrive at nonlocalized solutions, which depend on two parameters, viz., the imaginary and real parts of the complex amplitude A. Hence the following conclusion can be drawn: the stationary equation (2.3) can have a soliton solution if $\hat{L}$ is sign-definite. If Eq. (2.12) has a nontrivial solution, or, equivalently, if the phase velocity v_{ph} and the velocity v are equal, i.e., if

$$v_{\mathrm{ph}}=v, \tag{2.13}$$

there is no stationary soliton solution. We note that this conclusion relies heavily on the fact that the singularity on the right-hand side of Eq. (2.11) $(E^3)_\omega/L(\omega)$ is not removable. As will be shown below, singularities of this type can be removed, if the matrix element of the four-wave interaction (χ in the present case) has a frequency dependence.

Equation (2.13) can also be regarded as a condition for Cherenkov radiation by a moving object. The nature of the object itself is not important here. It can be a charged particle, a ship, or, for example, a soliton. In any case the moving object loses energy as a result of Cherenkov radiation. In the case under discussion this means that if the velocity of an electromagnetic soliton satisfies the conditions (2.9), it must emit waves, and, therefore, such a pulse cannot exist as a stationary object. Thus, we arrive at the following condition for the existence of solitons: a soliton solution can exist when the equation

$$\omega(k)=kv \tag{2.14}$$

does not have a (real) solution. Here $\omega=\omega(k)$ is the dispersion law. For electromagnetic waves $\omega(k)$ is determined from the equation

$$\omega^2=k^2c^2/\varepsilon(\omega). \tag{2.15}$$

The relation (2.14) has a simple interpretation in the $\omega-k$ plane. The right-hand side of (2.14) corresponds to a straight line emerging from the origin of coordinates, and, accordingly, the velocity v in this plane equals the slope tan ϕ:

$$v=\tan\,\phi.$$

The existence of a solution for Eq. (2.14) is indicated by the intersection of the $\omega=\omega(k)$ curve by the straight lines. This assigns a complete cone of angles Ω, where stationary soliton solutions are impossible. Cone $\tilde{\Omega}$, which is complementary to Ω, corresponds to possible soliton solutions. On the boundary $\partial\Omega$ between the cones the straight lines are tangent to the $\omega=\omega(k)$ dispersion curve, and at the points of tangency k_i the group and phase velocities coincide:

$$\left.\frac{\omega(k)}{k}\right|_{k_i}=\left.\frac{\partial\omega(k)}{\partial k}\right|_{k_i}. \tag{2.16}$$

For the dispersion law (2.15) this relation is written as

$$\left.\frac{d\varepsilon(\omega)}{d\omega}\right|_{\omega_i}=0. \tag{2.17}$$

It is natural to assume that the soliton amplitude vanishes at these critical points (since there should not be any stationary soliton solutions outside $\tilde{\Omega}$). As will be shown below, the behavior of a soliton solution near these critical points is universal. We demonstrate this fact in the case of the stationary equation (2.3). It is, however, fundamentally important that the result is general and can be used for other models. This fact was first investigated for capillary-gravitational solitons in deep water.[11–13] The spectrum of capillary-gravitational waves is known to have a minimum phase velocity for wave numbers lying in the intermediate region between the gravitational and capillary portions of the spectrum.

For simplicity, we assume that Eq. (2.17) has only one positive solution $\omega=\omega_0$ [because of the parity of $\varepsilon(\omega)$ there is one more root $\omega=-\omega_0$], and let the cone of angles $\tilde{\Omega}$ lie below the critical velocity:

$$v<v_{\mathrm{cr}}=\frac{c}{\sqrt{\varepsilon(\omega_0)}}.$$

Thus, the function $\varepsilon(\omega)$ has two identical maxima at symmetric points, and

$$\frac{d^2\varepsilon(\pm\omega_0)}{d\omega^2}<0.$$

In this case $\hat{L}$ is an invertible operator, and Eq. (2.4) can be written in the form

$$E_\omega=\frac{1}{L(\omega)}\alpha(E^3)_\omega. \tag{2.18}$$

Near the critical velocity $(v_{\mathrm{cr}}-v\ll v_{\mathrm{cr}})$ the plot of $L(\omega)$ as a function of ω is close to zero in small vicinities of the two points $\omega=\pm\omega_0$ because of its symmetry with respect to ω. Therefore, according to (2.18) the distribution of $E(\omega)$ is determined to a considerable extent by the function $1/L(\omega)$. Accordingly, in the t-representation the solution will be close to a monochromatic wave. It is important that the monochromaticity of the wave improves as v approaches v_{cr}. Therefore, $E(t')$ $(t'=t-x/v)$ will be sought in the form of an expansion in the harmonics $n\omega_0$:

$$E(t)=\sum_{n=0}^{\infty}[E_{2n+1}(\tau)e^{-i(2n+1)\omega_0 t'}+\text{c.c.}]. \tag{2.19}$$

Here we have formally introduced the small parameter

$$\epsilon=\sqrt{1-v/v_{\text{cr}}} \tag{2.20}$$

and the slow time $\tau=\epsilon t'$, so that the $E_{2n+1}(\tau)$ are the envelope amplitudes of each harmonic. The representation (2.19) means that the width of each harmonic along the frequency scale, $\delta\omega\sim\epsilon$, is small compared with the frequency ω_0, i.e., the Fourier spectrum (2.19) is a series of narrow peaks. The main peaks correspond to the first harmonic. Therefore, the action of $\hat{L}$ on (2.19) can be expanded into a series in powers of ϵ. Assuming that $E_{2n+1}\sim\epsilon^{2n+1}$ and substituting (2.19) into the stationary equation (2.4), with consideration of (2.17) in the first order we arrive at a stationary nonlinear Schrödinger equation:

$$\epsilon^2E_1-S\frac{\partial^2E_1}{\partial t^2}-\frac{3}{2}\alpha|E_1|^2E_1=0, \tag{2.21}$$

where

$$S=-\frac{v^2}{4c^2}\frac{d^2\epsilon(\omega_0)}{d\omega^2}>0. \tag{2.22}$$

Equation (2.21) has a soliton solution only if $\alpha>0$:

$$E_1(t')=\frac{2\epsilon}{\sqrt{3\alpha}}\,\text{sech}\left[\frac{\epsilon(t-x/v-t_0)}{\sqrt{S}}\right]. \tag{2.23}$$

This solution is unique to within a constant phase multiplier. It is the universal asymptote of the soliton solution. As v approaches v_{cr}, its amplitude vanishes according to a square-root law $\sim\sqrt{v_{\text{cr}}-v}$, and the soliton pulse width Δt increases in inverse proportion to this factor:

$$\Delta t=\sqrt{S}/\epsilon.$$

For times greater than Δt we must take into account the following expansion terms, particularly the third-order dispersion and the corrections to the cubic nonlinearity. In this time range the soliton behavior is no longer universal.

It is noteworthy that Eq. (2.21) does not have solitonlike solutions when $\epsilon^2=1-v/v_{\text{cr}}<0$.

When the tangent approaches the dispersion curve from above, S becomes negative. For this reason solitons exist only for defocusing media ($\alpha<0$).

The case where the point of tangency satisfies $\omega_0=0$ calls for a special treatment. Near the critical velocity the stationary equation (2.3) does not require the expansion (2.19). It is sufficient to expand $\epsilon(\omega)$ near $\omega=0$:

$$\varepsilon(\omega)=\varepsilon(0)+\frac{1}{2}\frac{d^2\varepsilon(0)}{d\omega^2}\omega^2.$$

According to this expansion, the stationary equation takes on the form

$$\epsilon^2E-S\partial_t^2E-\frac{1}{2}\alpha E^3=0, \tag{2.24}$$

where, as before, ϵ, S, and α are given, respectively, by Eqs. (2.20), (2.22), and (2.4) taken at $\omega=0$. The localized solution of Eq. (2.24) has the form of a soliton for the modified Korteweg-de Vries (MKdV) equation:

$$E(t-x/v)=\frac{2\epsilon}{\sqrt{\alpha}}\,\text{sech}\left[\frac{\epsilon(t-x/v-t_0)}{\sqrt{S}}\right].$$

3. QUASISOLITONS; HIGHER-ORDER DISPERSION

In this section we discuss the difference between solitons and quasisolitons in the case of a generalized nonlinear Schrödinger equation (GNLSE). The GNLSE has a more extensive class of soliton solutions than does the original Maxwell equation. Unlike the stationary solitons (2.23), these solutions are approximate and depend on four parameters. However, the mechanism for selecting the soliton solutions remains the same as for the stationary solitons considered in the preceding section.

The transparency window $\Delta\omega$ must be small compared with the mean value of the frequency ω_0: $\omega_0\gg\Delta\omega$. In this case an envelope can be introduced for the entire region. The most convenient and systematic approach for obtaining the equation for the envelopes is based on the Hamiltonian formalism.[14]

Let us consider Eq. (2.1), which we present in the form of a system of equations:

$$\frac{\partial\rho}{\partial x}+\frac{\partial^2\phi}{\partial t^2}=0,\quad \frac{\partial\phi}{\partial x}+\frac{1}{c^2}\left(\hat{\varepsilon}\rho+\frac{4\pi\chi}{c^2}\rho^3\right)=0. \tag{3.1}$$

The potential ϕ and the ''density'' ρ introduced here are related to the electric field E and the magnetic field H by the expressions

$$E=\frac{\sqrt{4\pi}}{c}\rho,\quad H=\sqrt{4\pi}\frac{\partial\phi}{\partial t}. \tag{3.2}$$

Equations (3.1) can be written in Hamiltonian form:

$$\frac{\partial\rho}{\partial x}=\frac{\delta\mathscr{H}}{\delta\phi},\quad \frac{\partial\phi}{\partial x}=-\frac{\delta\mathscr{H}}{\delta\rho}. \tag{3.3}$$

Here x plays the role of the time, and the Hamiltonian has the form of an integral with respect to time:

$$\mathscr{H}=\int\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2c^2}\rho\hat{\varepsilon}\rho+\frac{\pi\chi}{c^4}\rho^4\right]dt$$
$$\equiv\frac{1}{8\pi}\int\left[H^2+E\hat{\varepsilon}E+\frac{1}{2}\chi E^4\right]dt. \tag{3.4}$$

The quadratic part of $\mathscr{H}$ defines a linear dispersion law for $k=k(\omega)$, which coincides with (2.15). We can go over to the normal variables $a_\omega(x)$ using the replacements

$$\rho_\omega=\sqrt{\frac{\omega^2}{2k(\omega)}}\,(a_\omega^*+a_{-\omega}),$$
$$\phi_\omega=-i\sqrt{\frac{k(\omega)}{2\omega^2}}\,(a_\omega^*-a_{-\omega}), \tag{3.5}$$

where ρ_ω and ϕ_ω are the Fourier transforms of the density ρ and the potential ϕ, and $k(\omega)$ is understood in these formulas as a positive root of the dispersion relation (2.15). The substitution of these relations into Eq. (3.3) gives the equations of motion in the variables a_ω:

$$\frac{\partial a_\omega}{\partial x}=i\frac{\delta \mathscr{H}}{\delta a_\omega^*}, \tag{3.6}$$

where the Hamiltonian $\mathscr{H}$ takes the standard form (compare Ref. 14):

$$\mathscr{H}=\int k(\omega)|a_\omega|^2 d\omega+\frac{1}{2}\int T_{\omega_1\omega_2\omega_3\omega_4}a_{\omega_1}^* a_{\omega_2}^* a_{\omega_3}a_{\omega_4} \times\delta_{\omega_1+\omega_2-\omega_3-\omega_4}\Pi_i d\omega_i . \tag{3.7}$$

The matrix element T appearing therein is assigned by the formula

$$T_{\omega_1\omega_2\omega_3\omega_4}=\frac{3\chi}{4\pi c^4}\left[\frac{\omega_1^2\omega_2^2\omega_3^2\omega_4^2}{k(\omega_1)k(\omega_2)k(\omega_3)k(\omega_4)}\right]^{1/2}. \tag{3.8}$$

If the fourth-order susceptibility χ depends on the frequencies, the constant χ in the matrix element (3.8) is replaced by $\chi(\omega_1\omega_2\omega_3\omega_4)$ with the necessary symmetry properties (see Refs. 7 and 15), which ensure the following symmetry relations for T:

$$T_{\omega_1\omega_2\omega_3\omega_4}=T_{\omega_2\omega_1\omega_3\omega_4}=T_{\omega_1\omega_2\omega_4\omega_3}=T^*_{\omega_3\omega_4\omega_1\omega_2}. \tag{3.9}$$

In the Hamiltonian (3.7) we retained only the terms responsible for the scattering of waves, neglecting all the other processes, which make a contribution in the next (sixth) order with respect to the amplitude of the waves for narrow wave packets.

The Hamiltonian formulation of the equations of motion (3.6) guarantees "conservation" (absence of a dependence on x) of the Hamiltonian $\mathscr{H}$, as well as of the "momentum"

$$P=\int \omega|a_\omega|^2 d\omega, \tag{3.10}$$

which coincides exactly with the Poynting vector integrated over time:

$$P=\frac{c}{4\pi}\int_{-\infty}^{\infty}EH dt.$$

Let us now proceed to the derivation of the equation for the envelopes by introducing the packet envelope amplitude:

$$\psi(t,x)=\frac{1}{\sqrt{2\pi}}\int a_\omega e^{-i(\omega-\omega_0)t-ik_0(\omega_0)x}d\omega.$$

Here we assume that the spectrum of a_ω is concentrated in a narrow interval $\delta\omega$ near ω_0 and that $\omega_0\gg\delta\omega$. Accordingly, $\psi(t,x)$ is a slow function of the coordinates and the time.

Next, expanding $k(\omega)$ and $T_{\omega_1\omega_2\omega_3\omega_4}$ into a series in $\Omega=\omega-\omega_0$ at ω_0 we have

$$\kappa(\Omega)=k(\omega)-k(\omega_0)=\frac{1}{v_{\rm gr}}\Omega-k_0S\Omega^2-\gamma\Omega^3+\delta\Omega^4+\dots, \tag{3.11}$$

$$\begin{aligned}T_{\omega_1\omega_2\omega_3\omega_4}=T_0&+\frac{\partial T}{\partial\omega_1}(\Omega_1+\Omega_2+\Omega_3+\Omega_4)\\&+\frac{1}{2}\frac{\partial^2 T}{\partial\omega_1^2}(\Omega_1^2+\Omega_2^2+\Omega_3^2+\Omega_4^2)\\&+\frac{\partial^2 T}{\partial\omega_1\partial\omega_2}(\Omega_1\Omega_2+\Omega_3\Omega_4)\\&+\frac{\partial T}{\partial\omega_1\partial\omega_3}(\Omega_1\Omega_3+\Omega_1\Omega_4+\Omega_2\Omega_3\\&+\Omega_2\Omega_4)+\dots .\end{aligned} \tag{3.12}$$

In the expression for $k(\omega)$ we have retained the terms up to fourth order in Ω, and in the matrix element T we have retained the terms that are quadratic in Ω. In expanding the matrix element, for simplicity, we considered it to be real and utilized its symmetry properties (3.9). Accordingly, the coefficients in (3.12) are

$$T_0=T_{\omega_0\omega_0\omega_0\omega_0},\quad \frac{\partial T}{\partial\omega_1}=\left.\frac{\partial T_{\omega_1\omega_2\omega_3\omega_4}}{\partial\omega_1}\right|_{\omega_k=\omega_0},$$

$$\frac{\partial^2 T}{\partial\omega_i\partial\omega_j}=\left.\frac{\partial^2 T_{\omega_1\omega_2\omega_3\omega_4}}{\partial\omega_i\partial\omega_j}\right|_{\omega_k=\omega_0}.$$

Next, performing the inverse Fourier transformation with respect to Ω, for ψ we obtain the generalized nonlinear Schrödinger equation

$$\begin{aligned}i\left(\frac{\partial\psi}{\partial x}+\frac{1}{v_{\rm gr}}\frac{\partial\psi}{\partial t}\right)&+K_0S\psi_{tt}+\beta_1|\psi|^2\psi\\&=-i\gamma\psi_{ttt}-4i\beta_2|\psi|^2\psi_t-\delta\psi_{tttt}+(\beta_3-\beta_4)[(\psi^2\psi_t^*)_t\\&-(\psi_t)^2\psi^*]+(\beta_3+\beta_5)\psi^*(\psi^2)_{tt}-\beta_6|\psi|^4\psi.\end{aligned} \tag{3.13}$$

The left-hand side of this equation corresponds to the classical nonlinear Schrödinger equation: the second term in it describes the propagation of a wave packet as a whole and, therefore, can be eliminated by going over to the local coordinate frame. The next term ($\sim S$) is responsible for quadratic dispersion. Now, for $d\epsilon(\omega_0)/d\omega_0=0$ the coefficient S coincides with the expression in (2.22). The last term on the left-hand side defines a nonlinear correction to the frequency of the monochromatic wave. The first two terms on the right-hand side are $\sim(\delta\omega/\omega_0)^3$. It is important that there are only two such terms. In this case the coefficient $\beta_2=2\pi\partial T/\partial\omega$ is nonzero even for a constant fourth-order susceptibility χ. When $\chi=$const holds, β_2 can vanish only if $k\sim\omega^2$. The remaining terms are $\sim(\delta\omega/\omega)^4$. Among them we took into account the terms $\sim|\psi|^4\psi$, which are of the same order of magnitude.

The coefficients β_i appearing in Eq. (3.13) take on a very simple form for the matrix element (3.8):

$$\beta_1=\frac{3}{2}k_0^2\chi\left(\frac{v_{\rm ph}}{c}\right)^4,\quad \beta_2=\frac{\beta_1}{\omega_0}\left(1-\frac{v_{\rm ph}}{2v_{\rm gr}}\right),$$

$$\beta_3=\beta_1\frac{k^{1/2}}{\omega_0}\frac{\partial^2}{\partial\omega_0^2}\left(\frac{\omega_0}{k^{1/2}}\right),\quad \beta_4=\beta_5=\frac{\beta_1}{\omega_0^2}\left(1-\frac{v_{\rm ph}}{2v_{\rm gr}}\right)^2. \tag{3.14}$$

According to its derivation, Eq. (3.13) should be classified as a Hamiltonian equation:

$$i\frac{\partial\psi}{\partial x}=-\frac{\delta\mathcal{H}}{\delta\psi^*}. \tag{3.15}$$

Here the Hamiltonian $\mathcal{H}$ can be represented in the form of a sum of Hamiltonians:

$$\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3+\mathcal{H}_4+\dots,$$

where

$$\mathcal{H}_1=\frac{i}{v_{\rm gr}}\int\psi^*\psi_t dt, \tag{3.16}$$

$$\mathcal{H}_2=-\int\left(k_0S|\psi_t|^2-\frac{\beta_1}{2}|\psi|^4\right)dt, \tag{3.17}$$

$$\mathcal{H}_3=\int\{i\gamma\psi^*\psi_{ttt}+i\beta_2(\psi^*\psi_t-\psi\psi_t^*)|\psi|^2\}dt, \tag{3.18}$$

$$\mathcal{H}_4=\int\left\{\delta|\psi_{tt}|^2-\frac{\beta_3}{2}|\psi|^2(\psi\psi_{tt}^*+\text{c.c.})-\frac{\beta_4}{2}(\psi_t^2\psi^{*2}+\text{c.c.})-\frac{\beta_5}{2}\psi^{*2}\partial_t^2\psi^2+\frac{\beta_6}{3}|\psi|^6\right\}dt. \tag{3.19}$$

Here $\mathcal{H}_2$ corresponds to the classical NLSE, and the next Hamiltonian corresponds to the complex MKdV equation. It is important that each of the successive Hamiltonians is smaller than the preceding one. However, this situation can change, if any of the expansion coefficients introduces additional smallness. As is seen from (2.23), the soliton width decreases as the quadratic dispersion coefficient S decreases. Therefore, when S is small (such a situation arises near the so-called zero-dispersion point), the cubic dispersion ($\sim\gamma$) must be taken into account with neglect of all the higher-order terms, as well as the term that is proportional to β_2. If β_1 is small, the nonlinear dispersion, which is proportional to β_2, must be taken into account with neglect of the cubic linear dispersion.

Let us now turn to an analysis of the solitonlike solutions for the generalized Schrödinger equation.

To illustrate how the mechanism (2.23) operates, we first consider the nonlinear Schrödinger equation with quadratic dispersion [which corresponds to the Hamiltonian (3.17)]:

$$i\frac{\partial E}{\partial x}+E_{tt}+2|E|^2E=0. \tag{3.20}$$

Here we have used dimensionless variables, and the nonlinearity is assumed to be focusing, $S\alpha>0$.

It is noteworthy that, unlike the wave equation (2.1), a generalized NLSE, particularly the NLSE with quadratic dispersion, has an additional symmetry, viz., $E\to Ee^{i\phi}$, which appears as a result of the averaging of the equations of fast oscillations. Therefore, the envelope soliton solutions form a more extensive class of solutions than does the wave equation (2.1). According to our definition, they should be classified as quasisolitons. To find the corresponding solutions, we should set $E(x,t)=e^{i\lambda x}\psi(t+\beta x)$, where ψ obeys the equation[1)]

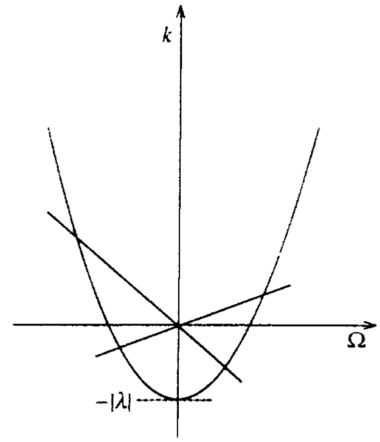

FIG. 1. Dispersion curve (3.23) for negative λ. Any straight line emerging from the origin of coordinates intersects the dispersion curve.

$$L(i\partial_t)\psi\equiv-i\beta\psi_t+\lambda\psi-\psi_{tt}=2|\psi|^2\psi. \tag{3.21}$$

In the case under consideration the conditions for Cherenkov radiation (2.14) are written in the following manner:

$$\beta\Omega=k(\Omega)\quad\text{or}\quad L(\Omega)=0, \tag{3.22}$$

where the dispersion relation for Eq. (3.21) takes the form

$$k(\Omega)=\lambda+\Omega^2. \tag{3.23}$$

Hence it is seen that for $\lambda<0$ the resonance condition (3.22) is satisfied for any value of the velocity (Fig. 1), and hence no solitons exist in this case. This is verified directly by solving Eq. (3.21): for $\lambda<0$ all the solutions are periodic or quasiperiodic. Soliton solutions are possible only for positive values of λ. Their velocities lie in the range $-2\sqrt{\lambda}\leqslant\beta\leqslant 2\sqrt{\lambda}$ (Fig. 2). At the points $\Omega=\pm\sqrt{\lambda}$ the straight line $k=\beta_{\rm cr}\Omega$ is tangent to the $k=k(\Omega)$ dispersion curve. According to the results of Sec. 2, the soliton solution should vanish at these points, as follows directly from the solution of Eq. (3.21):

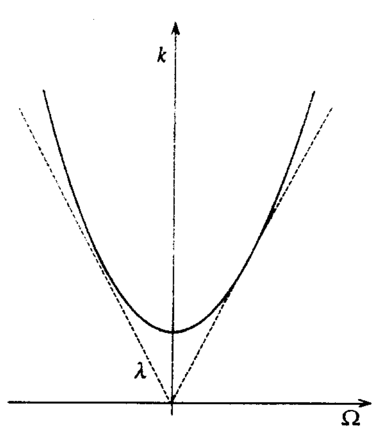

FIG. 2. Dispersion curve (3.23) for positive λ. The dashed lines which are tangent to the dispersion curve correspond to the critical velocities $\beta=\pm 2\sqrt{\lambda}$. These straight lines specify the boundary of the soliton cone of angles.

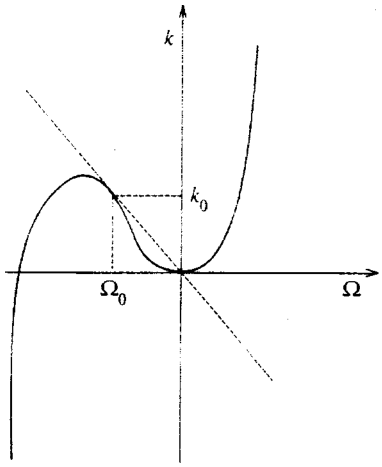

FIG. 3. Third-order dispersion $k=S\Omega^2+\gamma\Omega^3$. The dashed straight line is tangent to the dispersion curve at $\Omega=\Omega_0$, but intersects it at $\Omega=0$.

$$E=e^{i\lambda x}\frac{e^{i\beta t'}\Delta\Omega}{\cosh(\Delta\Omega t')},\quad \Delta\Omega=\sqrt{\lambda-\frac{\beta^2}{4}}. \tag{3.24}$$

Hence, the region for the existence of solitons is given by the inequality $\lambda>\beta^2/4$. The upper bound in this inequality specifies the critical velocity

$$\beta_{\rm cr}=\pm 2\sqrt{\lambda}.$$

It is important to note that the operator L in Eq. (3.21) is positive definite for $\lambda>\beta^2/4$.

Let us now turn to the third-order dispersion. We assume, as before, that the soliton solution contains an exponential multiplier

$$E(x,t)=e^{i\lambda x}\psi(t'),\quad t'=t+\beta x. \tag{3.25}$$

The corresponding operator $L(i\partial_t)$ has the form

$$L(\Omega)=-\beta\Omega+\lambda+S\Omega^2+\gamma\Omega^3. \tag{3.26}$$

This operator is sign-definite for any values of λ, β, S, and $\gamma\neq 0$. This means that the equation $L(\Omega)=0$ or the equivalent equation

$$\beta\Omega=\lambda+S\Omega^2+\gamma\Omega^3,$$

has at least one real solution: the dispersion curve for $k(\Omega)=\lambda+S\Omega^2+\gamma\Omega^3$ always intersects any straight line emerging from the origin of coordinates. For example, for $\lambda=0$ and $\beta\geqslant\beta_0=-S^2/(4\gamma)$ all the straight lines $k=\beta\Omega$ intersect the $k=k(\Omega)$ dispersion curve twice. For $\beta<\beta_0$ the straight lines have one point of intersection, and for $\beta=\beta_0$ tangency occurs (Fig. 3). However, one point of intersection is sufficient for the absence of solitons. On the other hand, the example of the KdV equation, which simultaneously has cubic dispersion and solitons, apparently contradicts the foregoing statement. Actually, there is no contradiction here. Everything is explained by the dependence of the matrix element on the wave vector, which provides for cancellation of the singularity in the equation of the form (2.18).

We can show in the example of the KdV equation

$$U_t+U_{xxx}+6UU_x=0, \tag{3.27}$$

how cancellation of a singularity occurs. For a soliton moving with the velocity v,

$$L(k)=ik(v+k^2).$$

For $v>0$ the equation $L(k)=0$ has one real root $k=0$. In this case the analog of Eq. (2.18) is

$$U_k=\frac{3ik(U^2)_k}{L(k)},$$

which clearly does not contain a singularity at $k=0$. The situation is similar for other equations of the KdV type (see, for example, Ref. 16).

Solutions of the soliton type were recently obtained[17] for a generalized Schrödinger equation, which simultaneously takes into account the third-order dispersion and corresponds to its nonlinearity [in the present paper this corresponds to consideration of the Hamiltonians (3.17) and (3.18)]. If the relations between γ and β_2 are arbitrary, the soliton solution found in Ref. 17 has a spectrum concentrated at the frequencies $\Omega\sim 1/\gamma,\ 1/\beta_2$, i.e., at frequencies comparable to ω_0. In the unique case where the relation between the coefficients has the form

$$\frac{K_0S}{\beta_1}=\frac{3\gamma}{4\beta_2},$$

the soliton spectrum is displaced by a small amount. This case is special, i.e., Eq. (3.13) (written in dimensionless variables),

$$iE_x+E_{tt}+2|E|^2E=i\epsilon(E_{ttt}+6|E|^2E_t), \tag{3.28}$$

allows application of the inverse scattering problem technique (see, for example, Ref. 8). In this case the Hamiltonians (3.17) and (3.18) are conserved independently. They are both created by the same associated operator, viz., the Zakharov-Shabat operator.[2] The parameter ϵ in this equation is of order $\delta\omega/\omega$, and E takes values of order unity. Soliton solutions for this equation were first pointed out in Ref. 18. The simplest of them is the solution

$$E=e^{i\mu^2x}\frac{\mu}{Ch\mu(t-\epsilon\mu^2x)},$$

which transforms into a stationary soliton of the NLSE (3.24) when $\epsilon=0$.

One conclusion which can be drawn from the foregoing material is that the existence of soliton solutions for the third-order operators L is due to the presence of derivatives in the nonlinear term or, stated differently, the dependence of the matrix elements on the frequency. If there is no such dependence, or if it is insignificant, as is the case, for example, near the point of zero dispersion, there are no reasons for cancellation of the singularities in the equation of the form (2.18). Therefore, the results in Ref. 19 of the numerical observation of solitons for the NLSE with cubic dispersion should be revised (see also Ref. 20, which was devoted to this equation).

We shall henceforth confine ourselves to consideration of the case where there is no dispersion of the nonlinearity or it is insignificant. In such a situation third-order dispersion cannot provide for the existence of solitons, i.e., the next expansion terms must be taken into account.

For fourth-order dispersion the corresponding operator L has the form

$$L(\Omega)=-\beta\Omega+\lambda+S\Omega^2+\gamma\Omega^3+\delta\Omega^4. \tag{3.29}$$

The sign-definiteness of L is now determined by the sign of δ: the operator is positive definite for $\delta>0$ and negative definite in the opposite case.

The cubic term can always be eliminated from L by means of an appropriate frequency shift $\Omega\to\Omega+\nu$. Furthermore, using simple scaling and sign reversal, $L(\Omega)$ can be brought into the following two canonical forms:

$$L(\Omega)=-\beta\Omega+k(\Omega)=-\beta\Omega+\lambda+(\Omega^2-\gamma_0^2)^2, \tag{3.30}$$

$$L(\Omega)=-\beta\Omega+k(\Omega)=-\beta\Omega+\lambda+(\Omega^2+\nu_0^2)^2. \tag{3.31}$$

Then, applying the criterion (3.22) to the dispersion law (3.30) with $\lambda<0$, we can easily see that the resonance condition (3.22) is satisfied for all values of β and that the existence of solitons is, therefore, impossible in this region of parameters.

For positive $\lambda=\mu^4$ solitons are possible in the region $-\beta_{cr}\leqslant\beta\leqslant\beta_{cr}$, where

$$\beta_{cr}=4\Omega_0(\Omega_0^2-\nu_0^2) \quad \text{and} \quad \Omega_0^2=\frac{1}{6}(2\nu_0^2+\sqrt{16\nu_0^4+12\mu^4}). \tag{3.32}$$

Near the critical velocity (3.32) the dispersion is positive; therefore, localized solutions of the soliton type can exist only for focusing ($\delta\chi>0$) nonlinearity, while nonlinearity with respect to the quadratic dispersion would be defocusing. The form of the soliton in this case is determined from the equation

$$L(i\partial_t)\psi=2\sigma|\psi|^2\psi, \tag{3.33}$$

where $L(i\partial_t)$ is given by Eq. (3.30) or (3.31), $\sigma=\text{sgn}(\delta\chi)$ specifies the character of the nonlinear interaction: for $\sigma=1$ it is attractive, and for $\sigma=-1$ it is repulsive. Soliton solutions are possible only for a focusing medium.

The simplest solutions of (3.33) are stationary solitons. Their form is found by integrating the equation

$$\mu^4\psi+(\partial_t^2+\nu_0^2)^2\psi-2|\psi|^2\psi=0. \tag{3.34}$$

It is significant that a moving soliton for fourth-order dispersion has a profile which differs from a soliton for the NLSE with quadratic dispersion. It cannot be deformed into a stationary soliton by simple scaling and phase transformation.

To find the solution, Eq. (3.34) must be supplemented by the boundary conditions

$$\psi,\ \psi_t\to 0 \quad \text{as} \quad t\to\pm\infty.$$

The symmetry of Eq. (3.34) allows real symmetric (relative to t) solutions: $\psi(t)=\psi(-t)=\psi^*(t)$. At infinity ($t\to\pm\infty$) these solutions should decay exponentially: $\psi\sim e^{\nu t}\to 0$, where the exponent ν is determined from the equation

$$\nu^4+(\nu^2+\nu_0^2)^2+\mu^4=0.$$

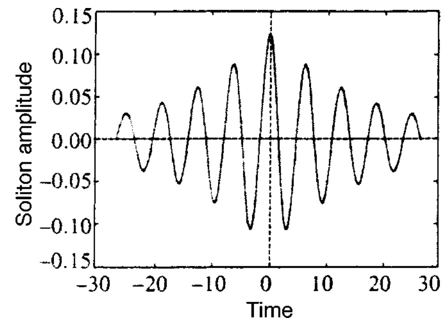

FIG. 4. Dependence of the soliton amplitude (in units of ν_0^2) on the time (in units of ν_0^{-1}) for $\mu/\nu_0=1/3$. The soliton envelope has the form of the function sech to good accuracy.

The roots of this equation are assigned by the expressions

$$\nu=\pm\left[\frac{1}{2}(\sqrt{\mu^4+\nu_0^4}-\nu_0^2)\right]^{1/2}\pm i\left[\frac{1}{2}(\sqrt{\mu^4+\nu_0^4}-\nu_0^2)\right]^{1/2}. \tag{3.35}$$

They are all complex. This means, in particular, that all stationary solitons should have an oscillating structure. If $\mu\sim\nu_0$ holds, the real and imaginary parts of the exponent ν are of the same order. Critical tangency occurs when $\mu=0$. Near this point the real part of ν' is small for a finite value of the imaginary part:

$$\nu=\pm\mu^2/\nu_0\pm i\nu_0. \tag{3.36}$$

Envelope solitons of the universal form (2.23) appear in just this limit.

For large μ ($\mu\gg\nu_0$) the roots have the asymptote

$$\nu=\mu\frac{\pm 1\pm i}{\sqrt{2}}.$$

Figures 4–6 show the solitons for different values of μ and ν_0. In the limit $\mu\to 0$ (Fig. 4) the soliton has a clearly expressed envelope soliton form, and at large μ ($\mu\gg\nu_0$) the soliton has only one oscillation on its scale (Fig. 6). At large distances (large times) the solitons for all the values of μ and ν_0 have exponentially decaying, oscillating tails. As the ratio μ/ν_0 increases, the amplitude of the soliton increases, and its width decreases. The solitons obtained here, like the real

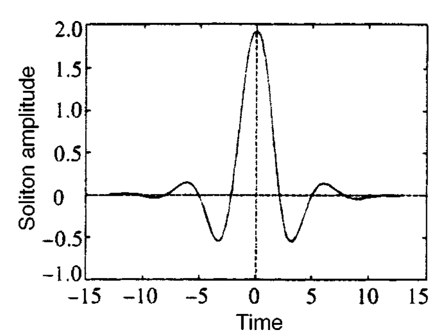

FIG. 5. Form of a soliton when $\mu/\nu_0=1$. The amplitude of the soliton (in units of ν_0^2) increases, and its width (in units of ν_0^{-1}) decreases. Oscillations are still observed on the scale of the soliton.

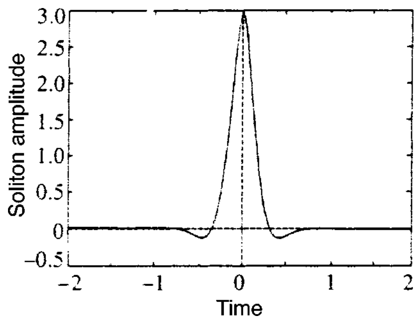

FIG. 6. Form of a soliton when $\mu/\nu_0=10$. The oscillating tail is scarcely visible.

solutions of Eq. (3.34), are simultaneously solutions in the form of stationary solitons for Eq. (2.1) with the dielectric constant

$$\varepsilon(\omega)=\varepsilon_0-a(\omega^2-\omega_0^2)^2 \quad \text{and} \quad \chi a>0.$$

As for the dispersion (3.31), here the situation is similar to what occurs for the NLSE with quadratic dispersion (3.20). Solitons are possible for $\lambda>-\nu_0^4$. The only difference from quadratic dispersion is the change in the value of the critical velocity. Near these points the structure of the solitons has the universal form (2.23).

4. STABILITY OF SOLITONS

Let us examine the stability of the solitons obtained in the preceding section. We first show how stability can be proved for the NLSE with quadratic dispersion (3.20). The Hamiltonian for it has the form

$$H=\int(|\psi_t|^2-|\psi|^4)dt\equiv I_1-I_2, \tag{4.1}$$

and the soliton solution (3.24) has the form of the stationary point of the Hamiltonian for a fixed momentum

$$P=-i\int\psi\psi_t^*dt$$

and a fixed number of particles (energy) $N=\int|\psi|^2dt$:

$$\delta(H+\beta P+\lambda N)=0.$$

Following Ref. 21, we shall prove stability in the sense of Lyapunov, i.e., we shall show that the soliton has a minimum for H at fixed P and N. For this purpose, it is convenient to represent λ in the form of a sum of $\beta^2/4$ and the positive quantity μ^2. We next consider the functional $F=H+\beta P+(\beta^2/4)N$, which, as can easily be seen, is the same Hamiltonian in a moving coordinate frame: the replacement of the wave function $\psi\to\psi e^{it\beta/2}$ transforms F into H (4.1). Thus, for stability it is sufficient to establish that H has a minimum in the stationary soliton.

Let us consider the integral $I_2=\int|\psi|^4dt$. It is easy to prove that the following chain of inequalities holds (see also Refs. 21 and 22):

$$\begin{aligned}\int_{-\infty}^{\infty}|\psi|^4dt&\le\max_t|\psi|^2\int_{-\infty}^{\infty}|\psi|^2dt\\&=\int_{-\infty}^{t_{\max}}\frac{d|\psi|^2}{dt}dt\int_{-\infty}^{\infty}|\psi|^2dt\\&\le 2N\int_{-\infty}^{t_{\max}}|\psi||\psi_t|dt\le 2N\int_{-\infty}^{\infty}|\psi||\psi_t|dt\\&\le 2N^{3/2}\left[\int_{-\infty}^{\infty}|\psi_t|^2dt\right]^{1/2}.\end{aligned} \tag{4.2}$$

This inequality can be enhanced by finding the best constant [instead of 2 in (4.2)]. The maximum value of the functional

$$G[\psi]=\frac{I_2}{N^{3/2}I_1^{1/2}}$$

clearly solves this problem. To find the maximum of $G[\psi]$ it is sufficient to consider all the stationary points of this functional and then to select the one which has the maximum value of G. All the stationary points of $G[\psi]$ are determined from the following equation, which coincides with the equation for a stationary soliton:

$$-\mu^2\psi+\psi_{tt}+2|\psi|^2\psi=0,$$

where $\lambda=\mu^2>0$. Hence it can easily be seen that the maximum of $G[\psi]$ is achieved in a real soliton solution, which is unique (to within a constant phase multiplier):

$$\psi_s=\frac{\mu}{\cosh(\mu t)}.$$

After this, all the integrals in $G[\psi]$ are easily calculated:

$$N=2\mu,\quad I_{1s}=\frac{2}{3}\mu^3,\quad I_{2s}=\frac{4}{3}\mu^3,$$

and the inequality (4.2) ultimately takes the form

$$\int_{-\infty}^{\infty}|\psi|^4dt\le\frac{1}{\sqrt{3}}N^{3/2}\left[\int_{-\infty}^{\infty}|\psi_t|^2dt\right]^{1/2}. \tag{4.3}$$

The substitution of this inequality into (4.1) gives the following estimate:

$$H\ge H_s+(\sqrt{I_1}-\sqrt{I_{1s}})^2,$$

where $H_s=-2\mu^3/3<0$ is the value of the Hamiltonian in the soliton solution. This estimate becomes exact in the soliton solution, proving the stability of the solitons with quadratic dispersion in the sense of Lyapunov. We stress that this proof provides for the stability of solitons not only with respect to small perturbations, but also with respect to finite perturbations.

Now let us turn to fourth-order dispersion. We represent the corresponding functional $F=H+\beta P+\lambda N$ in the form of a sum of the mean value of the operator $L(i\partial_t)$ (3.29) and the nonlinear term:

$$F=\int\psi^*L(i\partial_t)\psi dt-\int|\psi|^4dt. \tag{4.4}$$

To prove the stability of solitons, we must find the analog of the inequality (4.3) for the mean $\langle L(i\partial_t)\rangle$.

Let $L(\Omega)$ be the positive definite polynomial $\Omega\in(-\infty,\infty)$ of degree $N=2l$. Then $L(\Omega)$ can be expanded as

$$L_{2l}(\Omega)=\sum_{p=0}^{l} L_{2l-2p}(\Omega_p)\prod_{i=1}^{p-1}(\Omega-\Omega_i)^2, \tag{4.5}$$

where Ω_i and the polynomials $L_{2l-2p}(\Omega)$ are constructed from L_{2l} according to the following rule. Let $\Omega=\Omega_0$ be the minimum point of $L_{2l}(\Omega)$: $\min L_{2l}(\Omega)=L_{2l}(\Omega_0)$. The latter allows us to write $L_{2l}(\Omega)$ in the form

$$L_{2l}(\Omega)=L_{2l}(\Omega_0)+(\Omega-\Omega_0)^2L_{2l-2}(\Omega),$$

where $L_{2l-2}(\Omega)$ is a nonnegative polynomial of degree $2l-2$. The expansion of the polynomial $L_{2l-2}(\Omega)$ gives a new nonnegative polynomial of degree $2l-4$. Further recursion leads us to formula (4.5). It is important that all the coefficients in this expansion are nonnegative: $L_{2l-2p}(\Omega_p)\geqslant 0$. It is also clear that $L_0(\Omega_l)=C_{2l}$.

Expansion (4.5) generates the corresponding expansion for the mean value of $L_{2l}(i\partial_t)$:

$$\langle L_{2l}(i\partial_t)\rangle\equiv\int \psi^* L_{2l}(i\partial_t)\psi dt$$
$$=L_{2l}(\Omega_0)N_0+L_{2l-2}(\Omega_1)N_1+\ldots+L_0(\Omega_l)N_l, \tag{4.6}$$

where

$$N_p=\int|\psi_p|^2dt;\quad \psi_p=\prod_{q=0}^{p-1}(i\partial_t+\Omega_q)\psi,\quad p\geqslant 1;$$

$$\psi_0\equiv\psi.$$

This representation shows how the square of the norm of the positive definite polynomial operator expands in the norms N_p with the nonnegative coefficients $L_{2l-2p}(\Omega_p)$.

For the positive definite fourth-order dispersion (3.29)

$$L(\Omega)=\lambda-\beta\Omega+D\Omega^2+\gamma\Omega^3+\Omega^4$$

the expansion (4.5) reads as

$$L(\Omega)=\mu^4+\eta^2(\Omega-\Omega_0)^2+(\Omega-\Omega_0)^2(\Omega-\Omega_1)^2, \tag{4.7}$$

where μ^4 replaces $L_4(\Omega_0)$, and η^2 replaces $L_2(\Omega_1)$. With no loss of generality, we can set $\Omega_0=-\Omega_1=\nu_0$ in Eq. (4.7) (this corresponds to the replacement $\psi\rightarrow\psi\exp\{-i(\Omega_0+\Omega_1)t/2\}$), so that Eq. (4.7) takes the form

$$L(\Omega)=\mu^4+\eta^2(\Omega-\nu_0)^2+(\Omega^2-\nu_0^2)^2. \tag{4.8}$$

The difference between the dispersions (3.30) and (3.31) stems from the fact that the quantity $2\nu_0^2-\eta^2$ can be positive or negative. For (3.30) $2\nu_0^2>\eta^2$, and for (3.30) $2\nu_0^2<\eta^2$. In accordance with (4.8), the integral expansion of the norm of the operator L is written as

$$\langle L(i\partial_t)\rangle=\mu^4N+\eta^2\int|(i\partial_t+\nu_0)\psi|^2dt$$
$$+\int|(\partial_t^2+\nu_0^2)\psi|^2dt. \tag{4.9}$$

This representation means that a moving soliton can be regarded as a stationary point of the new Hamiltonian

$$H'=\eta^2\int|(i\partial_t+\nu_0)\psi|^2dt+\int|(\partial_t^2+\nu_0^2)\psi|^2dt-\int|\psi|^4dt \tag{4.10}$$

when the number of particles N is fixed:

$$\delta(H'+\mu^4N)=0. \tag{4.11}$$

If the Hamiltonian H' is bounded from below for a fixed value of N, and its lower bound corresponds to a soliton, the soliton will be stable.

In terms of the new Hamiltonian the soliton solution obeys the equation

$$\mu^4\psi_s+\eta^2(i\partial_t+\nu_0)^2\psi_s+(\partial_t^2+\nu_0^2)^2\psi_s-2|\psi_s|^2\psi_s=0. \tag{4.12}$$

Next, multiplying this equation by ψ_s^* and integrating over t, we arrive at the following relation between the integrals appearing in H':

$$\mu^4N_s+\eta^2\int|(i\partial_t+\nu_0)\psi_s|^2dt+\int|(\partial_t^2+\nu_0^2)\psi_s|^2dt$$
$$-2\int|\psi_s|^4dt\equiv H_s'+\mu^4N_s-\int|\psi_s|^4dt=0.$$

Another relation follows after the multiplication of (4.12) by $t\partial_t\psi_s^*$ and integration:

$$(\mu^4+\eta^2\nu_0^2+\nu_0^4)N_s+(2\nu_0^2-\eta^2)\int|\partial_t\psi_s|^2dt$$
$$-3\int|\partial_t^2\psi_s|^2dt-\int|\psi_s|^4dt=0.$$

Combining these two relations, we obtain

$$H_s'=(\eta^2\nu_0^2+\nu_0^4)N_s+(2\nu_0^2-\eta^2)\int|\partial_t\psi_s|^2dt$$
$$-3\int|\partial_t^2\psi_s|^2dt.$$

For both dispersions the Hamiltonian H_s' is bounded from above in the soliton solution by the number of particles multiplied by a certain positive factor: for (3.30)

$$H_s'\leqslant\left[\frac{1}{12}(2\nu_0^2-\eta^2)^2+\eta^2\nu_0^2+\nu_0^4\right]N_s,$$

and for (3.31)

$$H_s'\leqslant(\eta^2\nu_0^2+\nu_0^4)N_s.$$

We now prove that H' has a lower bound for a fixed value of N. For this purpose we first evaluate the two integrals

$$J_1=\int|(i\partial_t+\nu_0)\psi|^2dt\quad\text{and}\quad J_2=\int|(\partial_t^2+\nu_0^2)\psi|^2dt$$

in terms of two other integrals: N and $I_2=\int|\psi|^4dt$. It is easy to see that the estimate (4.3) is valid for the first integral J_1:

$$\int_{-\infty}^{\infty}|\psi|^4dt\leqslant\frac{1}{\sqrt{3}}N^{3/2}\left[\int_{-\infty}^{\infty}|(i\partial_t+\nu_0)\psi|^2dt\right]^{1/2}. \tag{4.13}$$

Using the inequality (4.3) again, we can obtain the estimate sought for J_2, if we first perform integration by parts in $\int|\psi_t|^2dt$ using the Cauchy–Bulyakovskiĭ inequality,

$$\int |\psi_t|^2dt=-\int \psi^*(\psi_{tt}+\nu_0^2\psi)dt+\int \nu_0^2|\psi|^2dt$$
$$\leqslant N^{1/2}\left[\int |(\partial_t^2+\nu_0^2)\psi|^2dt\right]^{1/2}+\nu_0^2N,$$

and then substitute the result obtained into (4.3):

$$J_2\geqslant\frac{1}{N}\left(\frac{3I_2^2}{N^3}-\nu_0^2N\right)^2. \tag{4.14}$$

Using the inequalities (4.13) and (4.14) we obtain an estimate of H' in terms of N and I_2:

$$H'\geqslant f(I_2)=\frac{3I_2^2}{N^3}+\frac{1}{N}\left(\frac{3I_2^2}{N^3}-\nu_0^2N\right)^2-I_2. \tag{4.15}$$

Continuing this inequality, we obtain

$$f(I_2)\geqslant 2\,\frac{\sqrt{3}I_2}{N^2}\left(\frac{3I_2^2}{N^3}-\nu_0^2N\right)-I_2.$$

Finally, from this we arrive at the desired inequality, i.e., the boundedness of the Hamiltonian:

$$H'\geqslant-\frac{4\sqrt{3}N}{9}\left[1+\frac{\sqrt{3}N}{6\nu_0^2}\right]^{3/2}. \tag{4.16}$$

According to Lyapunov's theorem, this proves the stability of the stationary point of the Hamiltonian corresponding to its minimum. This minimum point is a certain soliton solution of Eq. (4.12). It need not be unique. It is noteworthy that, according to the estimate (4.16), the Hamiltonian can take negative values. If initially we have $H'<0$, the maximum value of $|\psi|^2$ will be bounded from below by the conserved quantity (compare Ref. 21):

$$\max_t|E|^2\geqslant|H'|/N.$$

Thus, an initially existing intensity maximum cannot vanish as the pulse propagates (as x increases). On the other hand, small-amplitude radiation should ensure relaxation of the initial distribution toward a certain soliton state, which is possible owing to the lower bound on the Hamiltonian.

To conclude this section we wish to say a few words about the stability of the stationary solitons (2.23). Near the critical velocity this question can be treated within the parabolic NLSE (3.20), for which the answer is already known. As for the stability of solitons with velocities far from the critical value, the terms for dispersion of the next order must be taken into account. As we saw in this section, the fourth-order terms, which ensure that the corresponding operator L is positive, also provide for the stability of solitons. We assume that the positive definite four-order polynomial operators should ensure the stability of one-dimensional solitons. It is possible that the solitons will be unstable only for operators which increase at infinity ($|\Omega|\to\infty$) in proportion to $\sqrt{|\Omega|}$.

5. CONCLUDING REMARKS

In conclusion, we would like to note that the selection rules for solitons based on the criteria (2.8) and (2.9) are valid for arbitrary dimensionality. It is significant that the conditions for the existence of solitons remain unchanged: the corresponding operator L must be sign-definite. In addition, the fourth-order dispersion for all physical dimensionalities D ensures the existence of stable solitons for the GNLSE with cubic nonlinearity (with neglect of its dispersion). This follows from the estimate of the dispersion term of the Hamiltonian in terms of I_2 and N. In this case the inequality (4.3) has the form

$$\int |\psi|^4d^Dx\leqslant C\left[\int |\Delta\psi|^2d^Dx\right]^{D/4}\left[\int |\psi|^2d^Dx\right]^{2-D/4}. \tag{5.1}$$

Substituting this estimate into the Hamiltonian

$$H=\int |\Delta\psi|^2d^Dx-\int |\psi|^4d^Dx$$

gives its lower bound:

$$H\geqslant\int |\Delta\psi|^2d^Dx-C\left[\int |\Delta\psi|^2d^Dx\right]^{D/4}\left[\int |\psi|^2d^Dx\right]^{2-D/4}$$
$$\geqslant-\left(\frac{4}{D}-1\right)\left(\frac{4}{CD}\right)^{4/(D-4)}N^{(8-D)/(4-D)}.$$

Apart from soliton stability, for media with Kerr nonlinearity this also proves that wave collapse ceases because of fourth-order dispersion for the physical dimensionalities $D=2,3$.

One last remark: in the present work we confined ourselves to consideration of equations with only cubic nonlinearity, although in the general expansion of the electric displacement D (2.2) the term which is quadratic with respect to the amplitude must be taken into account. If tangency occurs at a nonzero frequency, the quadratic anharmonic terms are not resonant near the critical velocity and can be eliminated by a canonical transformation (for further details regarding this, see the review in Ref. 14). These terms lead to renormalization of the four-wave matrix element (3.8). Thus, the universality of the behavior of solitons near the critical velocity remains in force.

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1)In dimensional variables the parameter β introduced here, which has the meaning of the reciprocal of the velocity, is equal to the difference between the soliton velocity and the group velocity divided by $v_{\rm gr}^2$. In this case β is assumed to be small compared with $1/v_{\rm gr}$.

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