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Nonlinear Coherent Phenomena in Continuous Media

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ABSTRACT This review is devoted to description of coherent nonlinear phenomena in almost conservative media with applications to plasma physics, fluid dynamics and nonlinear optics. The main attention in the review is paid to consideration of solitons, collapses, and black holes. The latter is a quasi-stationary singular object which appear after the formation of a singularity in nonlinear wave systems. We discuss in details the qualitative reasons of the wave collapse and a difference between solitons and collapses, and apply to their analysis exact methods based on the integral estimates and the Hamiltonian formalism. These approaches are demonstrated mainly on the basic nonlinear models, i.e. on the nonlinear Schrödinger equation and the Kadomtsev-Petviashvili equation and their generalizations.

1 Introduction

All real continuous media, including vacuum, are nonlinear. Nonlinearity might be a cause of quite opposite physical effects. One of them is phase randomization leading to formation of a chaotic state - weak or strong wave turbulence. Wind-driven waves on the ocean surface is the classical example of that sort. Another group of effects is spontaneous generation of coherent structures. These structures may be localized in space or both in space and in time. Phases of Fourier harmonics, forming the structures, are strongly correlated.

Very often coherent structures coexist with wave turbulence. A simple example of the coherent structure is 'white caps' on the crest of gravity wave of high amplitude. Elementary visual observation shows that just before breaking, a wave crest takes the universal, wedge-type shape. Apparently, the harmonics composing this shape have correlated phases. The wave breaking is an important mechanism of energy and momentum dissipation on the ocean. A satisfactory theory of this basic effect is not yet developed.

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A more standard example of a coherent structure is a solitary wave on the surface of shallow water. These examples present two major types of coherent structures - collapses and solitons. Solitons are stationary, spatially localized wave packets, which are very common in nonlinear media. Collapses are almost as wide-spread phenomena as solitons. These are catastrophic processes of concentration of wave energy in localized space domains leading to absorption of at least part of this energy. Collapses are an important mechanism of the wave energy dissipation in almost conservative media, in particular, they play essential roles for many methods of fusion plasma heating.

Collapses and solitons are not all the coherent structures that can be found in nonlinear media. Rich families of coherent structures exist in active media, providing the balance between pumping and dissipation. Among them there are patterns described by the Ginsburg-Landau type equations and spiral waves in reaction-diffusion systems. Rolls and hexagons in the Benard convection are such examples. But even in almost conservative media one can find coherent structures different from solitons and collapses. One can mention, for instance, “black holes”, which are persistent localized regions of the wave energy dissipation arising in some cases after the act of wave collapse resulting in the formation of a singularity.

In this paper we shall discuss coherent structures in almost conservative media only. We concentrate our attention mostly on collapses and solitons, which are, in our opinion, closely related phenomena. In many important physical situations, collapse is a result of the soliton instability (for more details, see two reviews [1, 2] and references therein). We shall briefly discuss also the theory of black holes in the models describing by the nonlinear Schrödinger equation (NLSE). Using the Hamiltonian formalism gives us an opportunity to study the problem of coherent structures in its maximum generality (see also our recent review [3] devoted to this subject). Physical examples used in the paper are taken mostly from hydrodynamics, nonlinear optics, and plasma physics.

2 Phase randomization in nonlinear media

Let as consider wave propagation in a uniform boundless conservative medium. The wave field will be described by the complex normal variable $a_k(t)$, satisfying the equation of motion

$$\frac{\partial a_k}{\partial t} = -i\frac{\delta H}{\delta a_k^*}\,. \tag{1.1}$$

In the linear approximation

$$H = H_0 = \int \omega(k)|a_k|^2 dk, \tag{1.2}$$

where $\omega(k)$ is the dispersion law. In this case equation (1.1) is trivially integrated

$$\frac{\partial a_k(t)}{\partial t} + i\omega(k)a_k(t) = 0, \tag{1.3}$$

$$a_k(t) = C(k)e^{-i\omega(k)t}. \tag{1.4}$$

For a localized wave packet one should require $\int |C(k)|^2 dk < \infty$. For the system of monochromatic waves the distribution of a_k is a set of δ functions

$$C(k) = \sum C_n \delta(k - k_n). \tag{1.5}$$

In the linear approximation phases of each waves $\arg a_k = \phi_k$ grow linearly in time

$$\phi_k(t) = \phi_k(0) + \omega(k)t$$

and, respectively, the trajectory of the system winds on the infinitely-dimensional torus. The phase ϕ_k is defined modulo 2π. Therefore for two waves with incommensurable frequencies $\omega(k)$ and $\omega(k_1)$ difference (or sum) in phases $\phi_k(t) \mp \phi_{k_1}(t) = \phi_k\ (0) \mp \phi_{k_1}(0) + (\omega(k) \mp \omega(k_1))t$ with time becomes random function on the interval 2π. Thus, for continuous dependence $\omega = \omega(k)$ (except $\omega(k)$ =const), the linear dispersion leads to complete phase randomness for the wave distribution.

Now let us introduce into (1.1) a quadratic nonlinearity. It is enough to replace

$$H \to H_0 + H_1, \tag{1.6}$$

$$H_1 = -\frac{1}{2}\int V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\left(a_{\boldsymbol{k}}{}^* a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} + a_{\boldsymbol{k}} a_{\boldsymbol{k}_1}^* a_{\boldsymbol{k}_2}^*\right)\delta_{\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2} d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2\ . \tag{1.7}$$

Here $V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}$ are coupling coefficients for three-wave interaction. The equation of motion (1.1) takes now the form

$$\begin{aligned}\frac{\partial a_{\boldsymbol{k}}}{\partial t} + i\omega(\boldsymbol{k})a_{\boldsymbol{k}} + \frac{i}{2}\int \big\{ & V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2} a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2} \delta_{\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2} + \\ & 2V_{\boldsymbol{k}_1\boldsymbol{k}\boldsymbol{k}_2} a_{\boldsymbol{k}_1} a_{\boldsymbol{k}_2}^*\ \delta_{\boldsymbol{k}-\boldsymbol{k}_1+\boldsymbol{k}_2}\big\} d\boldsymbol{k}_1 d\boldsymbol{k}_2 = 0. \end{aligned} \tag{1.8}$$

The Equation (1.8) describes several nonlinear effects. Suppose that the equations

$$\boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3, \qquad \omega(\boldsymbol{k}_1) = \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) \tag{1.9}$$

have nontrivial real solutions, as for instance, if $\omega(0) = 0$ and $\omega'' > 0$. Suppose further that at $t = 0$

$$a_{\boldsymbol{k}} = C_1^{(0)}\delta(\boldsymbol{k} - \boldsymbol{k}_1) + C_2^{(0)}\delta(\boldsymbol{k} - \boldsymbol{k}_2) + C_3^{(0)}\delta(\boldsymbol{k} - \boldsymbol{k}_3), \tag{1.10}$$

where $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3$ satisfy the equations (1.9). Then at $t > 0$, in the limit of small enough intensities of waves, the complex amplitude $a_{\boldsymbol{k}}(t)$ can be sought in the form

$$a_{\boldsymbol{k}} = \sum_{i=1}^{3} C_i(t) e^{-i\omega(\boldsymbol{k}_i)t} \delta(\boldsymbol{k} - \boldsymbol{k}_i) \tag{1.11}$$

where $C_i(t)$ obey the system of ordinary differential equations, the so-called three-waves system [4]

$$\frac{\partial C_1}{\partial t} = iVC_2C_3, \quad \frac{\partial C_2}{\partial t} = iVC_1C_3^* \quad \frac{\partial C_3}{\partial t} = iVC_1C_2^*. \tag{1.12}$$

Here the coupling coefficient $V = V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}$.

Equation (1.12) can be easily solved in elliptic functions. The initial data

$$C_1 = 0, \qquad C_2 = C_2^{(0)}, \qquad C_3 = C_3^{(0)}$$

separate the solution describing growth of C_1. In particular, at small time

$$C_1 \simeq iVC_2^{(0)}C_3^{(0)}t.$$

This is the simplest nonlinear process - resonant "mixing" of two monochromatic waves.

The equations (1.12) describe also another very important nonlinear process, namely, the decay instability of the monochromatic waves. Let at $t = 0$

$$C_1 = Ae^{i\phi}, \qquad C_2 = q, \qquad C_3 = iq^* e^{i\phi}, \qquad |q| \ll A. \tag{1.13}$$

Now for small times

$$C_2 \simeq qe^{\gamma t} \tag{1.14}$$

where $\gamma = |v||A|$ is the growth rate of the so-called decay instability. This solution describes exponential growth of the waves C_2, C_3. Their phases $(C_2 = |C_2|e^{i\phi_2},\ C_3 = |C_3|e^{i\varphi_3})$ satisfy the condition

$$\phi_2 + \phi_3 = \phi + \pi/2. \tag{1.15}$$

Thus, the sum of phases ϕ_2 and ϕ_3 is fixed. But a phase of one of the waves in this pair (phase of q) is quite arbitrary. We found that in the most idealized case (when due to the decay instability only one pair of monochromatic waves is excited) this process yields the correlation for sum of phases of the excited waves and simultaneously introduces to the wave system an element of randomness, namely, the phase of q. In more realistic case the instability excites a whole ensemble of wave pairs satisfying the conditions (1.9) up to the accuracy of γ. Each exited pair adds one random phase. The exited waves are also unstable. Multiplication of the process of instability has to create in the system a lot of new waves with random phases and to cause finally complete turbulization of the wave field. We must stress that this scenario is just a very plausible conjecture. It would be very important to check it by a direct numerical experiment. The point of common belief is the following. As a result of multiple events of the wave mixing and decay instability, after some time phases become completely random. In this case the wave field can be described statistically by the correlation function

$$\langle a_{\boldsymbol{k}} a^*{}_{\boldsymbol{k}'} \rangle = n_{\boldsymbol{k}} \delta_{\boldsymbol{k}\boldsymbol{k}'}. \tag{1.16}$$

Here $n_{\boldsymbol{k}}$ is the quasi-particle density (or the wave action). This quantity for sufficiently small wave intensity satisfies the kinetic equation (for details see [5])

$$\frac{\partial n_{\boldsymbol{k}}}{\partial t} = St(n,n), \tag{1.17}$$

$$\begin{aligned} St(n,n) &= \int \big\{ A_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2} - A_{\boldsymbol{k}_1\boldsymbol{k}\boldsymbol{k}_2} - A_{\boldsymbol{k}_2\boldsymbol{k}\boldsymbol{k}_1} \big\} d\boldsymbol{k}_1 d\boldsymbol{k}_2, \\ A_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2} &= 4\pi |V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}|^2 \big(n_{\boldsymbol{k}_1} n_{\boldsymbol{k}_2} - n_{\boldsymbol{k}} n_{\boldsymbol{k}_1} - n_{\boldsymbol{k}} n_{\boldsymbol{k}_2} \big) \cdot \\ & \quad \delta_{\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2} \delta_{\omega(\boldsymbol{k})-\omega(\boldsymbol{k}_1)-\omega(\boldsymbol{k}_2)}. \end{aligned}$$

The kinetic equation accounts for the correlation in wave phases (1.15) in the first order with respect to the matrix element $V_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}$ that, in particular, provides a nonzero three wave correlation function $\langle a_{\boldsymbol{k}} a^*{}_{\boldsymbol{k}_1} a^*{}_{\boldsymbol{k}_2} \rangle = J_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2} \delta_{\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2}$.

The state of the wave field described by the kinetic equation (1.17) is called weak turbulence. Direct numerical examination of the theory of weak turbulence is one of the most interesting problem in computational physics at the time.

It might happen that equations (1.9) have no real solutions. In this case the first interacting term in the Hamiltonian has to be taken in the form

$$H_1 = \frac{1}{2} \int T_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} a_{\boldsymbol{k}}{}^* a_{\boldsymbol{k}_1}^* a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} \delta_{\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3} d\boldsymbol{k} d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3. \tag{1.18}$$

Equation (1.8) transforms now into the form

$$\begin{aligned} &\frac{\partial a_{\boldsymbol{k}}}{\partial t} + i\omega(\boldsymbol{k}) a_{\boldsymbol{k}} = \\ &-i \int T_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} a_{\boldsymbol{k}_1}^* a_{\boldsymbol{k}_2} a_{\boldsymbol{k}_3} \delta_{\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3} d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3. \end{aligned} \tag{1.19}$$

Equation (1.8) has the natural constants of motion, i.e., the Hamiltonian and the momentum

$$H = H_0 + H_1 \quad \text{and} \quad \boldsymbol{P} = \int \boldsymbol{k} a_{\boldsymbol{k}} a_{\boldsymbol{k}}{}^* d\boldsymbol{k}. \tag{1.20}$$

The Equation (1.19) has the additional invariant $N = \int a_{\boldsymbol{k}} a_{\boldsymbol{k}}{}^* d\boldsymbol{k}$. If $H_1 \ll H_0$ the equation (1.19) can be considered as linear and again has a solution (1.4), (1.5), but elementary process of the nonlinear wave interaction are now different.

If wave vectors $\boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3$ of three monochromatic waves satisfy the condition

$$-\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) = \omega(-\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3), \tag{1.21}$$

they pump a new wave with the wave vector

$$\boldsymbol{k} = -\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3. \tag{1.22}$$

This is a "resonant mixing" of wave triads. Another type of nonlinear interaction is an instability of monochromatic waves. As in the previous case they lead to excitation of wave pairs. In the case of instability of an individual wave with the wave vector $\boldsymbol{k}_0$ there excites a pair with the wave vectors $\boldsymbol{k}_2, \boldsymbol{k}_3$, satisfying the conditions

$$\boldsymbol{k}_2 + \boldsymbol{k}_3 = 2\boldsymbol{k}_0, \qquad \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) = 2\omega(\boldsymbol{k}_0). \tag{1.23}$$

Phases of new waves ϕ_2, ϕ_3 are connected with the phase of the initial wave ϕ_0 by the relation

$$\phi_2 + \phi_3 = 2\phi_0 + \pi/2. \tag{1.24}$$

Their difference $\phi_2 - \phi_3$ is again arbitrary. Hence this instability introduces an element of chaos to the system.

Another instability taking place in the system (1.19) is instability of wave pairs. If initially the wave field consists of two monochromatic waves with wave vectors $\boldsymbol{k}_0, \boldsymbol{k}_1$, two other waves grow exponentially, if their wave vectors $\boldsymbol{k}_2, \boldsymbol{k}_3$ satisfy the resonant conditions

$$\boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{k}_0 + \boldsymbol{k}_1, \quad \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) = \omega(\boldsymbol{k}_0) + \omega(\boldsymbol{k}_1). \tag{1.25}$$

Now

$$\phi_2 + \phi_3 = \phi_0 + \phi_1 + \pi/2.$$

The phase difference $\phi_2 - \phi_3$ is arbitrary again.

Combination of instability and wave mixing causes complete stochastization of phases. Weak turbulence in the framework of the model (1.19) is described by the kinetic equation

$$\frac{\partial n_{\boldsymbol{k}}}{\partial t} = St(n, n, n), \tag{1.26}$$

$$St(n,n,n) = 4\pi \int |T_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}|^2 \delta_{\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3} \delta_{\omega(\boldsymbol{k})+\omega(\boldsymbol{k}_1)-\omega(\boldsymbol{k}_2)-\omega(\boldsymbol{k}_3)}.$$

$$[n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}+n_{\boldsymbol{k}}n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}-n_{\boldsymbol{k}}n_{\boldsymbol{k}1}n_{\boldsymbol{k}_2}n_{\boldsymbol{k}}n_{\boldsymbol{k}_1}n_{\boldsymbol{k}_3}]d\boldsymbol{k}_1d\boldsymbol{k}_2d\boldsymbol{k}_3.$$

It should be noted that the equations (1.23) not necessarily have real solutions. In an isotropic medium $\omega=\omega(|\mathbf{k}|)$, the sufficient condition for their existence is $\omega'>0;\ \omega''<0$. If $\omega(0)=0$, $\omega'_k>0$, $\omega''>0$, the only solution of (1.22) is $\boldsymbol{k}_2=\boldsymbol{k}_3=\boldsymbol{k}_0$. In this case stochastization is less obvious and one has to expect formation of coherent structures. We study them in the next section.

3 Nonlinear Schrödinger equation

In some important physical situation, for instance, for waves on the surface of ideal fluid of finite depth $T_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}$ has indeterminacies at $\boldsymbol{k}_1=\boldsymbol{k}_2=\boldsymbol{k}_3=\boldsymbol{k}$. We will study only the simplest case when T is a continuous function on this submanifold. Denote $T(\boldsymbol{k})=T_{\boldsymbol{k}\boldsymbol{k}\boldsymbol{k}\boldsymbol{k}}$. Then the equation (1.19) has the exact solution

$$a_{\boldsymbol{k}}=Ae^{-i\tilde{\omega}(\boldsymbol{k}_0)t},\qquad \tilde{\omega}_{\boldsymbol{k}_0}=\omega_{\boldsymbol{k}_0}+T(\boldsymbol{k}_0)|A|^2. \tag{1.27}$$

Here, due to the obvious symmetry relation $T^*{}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}=T_{\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}\boldsymbol{k}_1}$, $T(\boldsymbol{k})$ is a real function.

Let us consider a solution of (1.19) that is close to the exact nonlinear monochromatic wave (1.27). Now

$$a_{\boldsymbol{k}}(t)=C(\boldsymbol{\kappa},t)e^{-i\omega(\boldsymbol{k}_0)t},\qquad \boldsymbol{\kappa}=\boldsymbol{k}-\boldsymbol{k}_0, \tag{1.28}$$

$$C(\boldsymbol{\kappa})=0\quad\text{if}\quad |\boldsymbol{\kappa}|\ll|k_0|.$$

Expanding $\omega(\boldsymbol{k})$ in the Taylor series

$$\omega(\boldsymbol{k})=\omega(\boldsymbol{k}_0+\boldsymbol{\kappa})=\omega(\boldsymbol{k}_0)+\kappa_p\frac{\partial\omega}{\partial k_p}+\frac{1}{2}\kappa_p\kappa_q\frac{\partial\omega}{\partial k_p\partial k_q}+...,$$

one can find that the Fourier transform from $C(\boldsymbol{\kappa},t)$,

$$\psi(\boldsymbol{r},t)=\int C(\boldsymbol{\kappa},t)e^{i\boldsymbol{\kappa}\boldsymbol{r}}d\boldsymbol{\kappa},$$

satisfies the nonlinear Schrödinger equation (NLSE)

$$\frac{\partial\psi}{\partial t}+(\boldsymbol{v}\nabla)\psi-i\omega_{\alpha\beta}\frac{\partial^2\psi}{\partial x_\alpha\partial x_\beta}=iT|\psi|^2\psi. \tag{1.29}$$

Here $\boldsymbol{v}=\partial\omega/\partial\boldsymbol{k}|_{\boldsymbol{k}=\boldsymbol{k}_0}$ is the group velocity,

$$\omega_{\alpha\beta}=\frac{1}{2}\frac{\partial\omega}{\partial k_\alpha\partial k_\beta}\Big|_{\boldsymbol{k}=\boldsymbol{k}_0},\qquad T=-T(\boldsymbol{k}_0).$$

Going to the frame of reference moving with the group velocity one can eliminate the first space derivative. We now obtain

$$\frac{\partial\psi}{\partial t} - i\omega_{\alpha\beta}\frac{\partial^2\psi}{\partial x_\alpha \partial x_\beta} = iT|\psi|^2\psi. \tag{1.30}$$

The monochromatic wave is described now by the solution of (1.28)

$$\psi = Ae^{iT|A|^2 t}. \tag{1.31}$$

One can study the stability of this solution, assuming

$$\psi = Ae^{iT|A|^2 t}(1 + \delta\psi e^{i(\Omega t - \boldsymbol{pr})}).$$

In the linear approximation $|\delta\psi| \ll 1$, one can obtain

$$\Omega^2 = (\omega_{\alpha\beta}p_\alpha p_\beta)^2 - 2T|A|^2\omega_{\alpha\beta}p_\alpha p_\beta. \tag{1.32}$$

If eigenvalues of the tensor $\omega_{\alpha\beta}$ have different signs, equation (1.31) yields instability at any sign of T. The domain of the instability in the p-space is concentrated along the cone

$$\omega_{\alpha\beta}p_\alpha p_\beta = 0. \tag{1.33}$$

If $p \simeq k_0$, this instability goes to the "second order decay instability" obeying the resonant conditions (1.23). If all eigenvalues of $\omega_{\alpha\beta}$ are of the same sign, instability takes place if

$$T\omega_{\alpha\beta}p_\alpha p_\beta > 0. \tag{1.34}$$

This instability is called the modulation instability (for details, see [1, 6, 7]). In this case the NLSE can be reduced to the form

$$i\psi_t + \Delta\psi + 2|\psi|^2\psi = 0. \tag{1.35}$$

We will call this equation the compact focusing NLSE. The domain of the instability of monochromatic wave is bounded now by the condition

$$|\omega_{\alpha\beta}p_\alpha p_\beta + \beta| < |T|A^2. \tag{1.36}$$

If $T\omega_{\alpha\beta}p_\alpha p_\beta < 0$, the monochromatic wave is stable and the NLSE can be simplified to the canonical form

$$i\psi_t + \Delta\psi - 2|\psi|^2\psi = 0. \tag{1.37}$$

This is the compact defocusing NLSE. Among non-compact NLSE the most interesting ones have the following canonical forms

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0, \tag{1.38}$$

$$i\psi_t + \Delta_\perp \psi - \psi_{xx} + 2|\psi|^2\psi = 0. \tag{1.39}$$

Here $\Delta_\perp = \partial^2/\partial y^2 + \partial^2/\partial z^2$. Equation (1.37) describes nonlinear modulations of gravity waves on a surface of deep water, while (1.38) is applicable to propagation of electromagnetic wave packets in media with negative (normal) dispersion. All species of the NLSE describe some coherent structures. Only for the compact cases (1.35) and (1.37) they are studied in a proper degree.

4 Solitons in the focusing NSLE

Development of instability of the monochromatic wave (condensate) in the framework of the compact focusing NLSE (1.35) does not lead to formation of weak-turbulent state directly. It leads first to formation of the coherent structures - solitons or collapses. In the quantum mechanical analogy, the NLSE (1.35) describes the motion of a particle in a self-consistent potential with attraction, where the attraction is the main cause of existence of the localized coherent structures. The nature of these structures depends essentially on the spatial dimension D. The most important coherent structure in (1.35) has maximal spatial symmetry. We will discuss only these structures.

Equation (1.35) can be rewritten as follows

$$i\psi_t + \psi_{rr} + \frac{D-1}{r}\psi_r + 2|\psi|^2\psi = 0, \qquad 0 < r < \infty. \tag{1.40}$$

This equation preserves two basic constant of motion: number of particles

$$N = 2^{D-1}\pi \int_0^\infty r^{D-1}|\psi|^2 dr \tag{1.41}$$

and the Hamiltonian

$$H = 2^{D-1}\pi \int_0^\infty r^{D-1}(|\psi_r|^2 - |\psi|^4)dr = X - Y, \tag{1.42}$$

where we denote

$$X = 2^{D-1}\pi \int_0^\infty r^{D-1}|\psi_r|^4 dr, \qquad Y = 2^{D-1}\pi \int r^{D-1}|\psi|^4 dr.$$

The Equation (1.40) has stationary solutions of the form

$$\psi = \varphi(r)e^{i\lambda^2 t} \tag{1.43}$$

where $\varphi(r)$ satisfies the equation

$$-\lambda^2\varphi + \Delta\varphi + 2\varphi^3 = 0 \tag{1.44}$$

Here $\Delta\varphi = \varphi_{rr} + (D-1)\varphi_r/r$. The solution (1.43) is a soliton if $\varphi(r) \to 0$ at $r \to \infty$ and integrals N, X, Y are finite. It is possible to show that the solutions of equation (1.44) for $D \leq 4$ decrease exponentially at infinity and this provides finiteness of the integrals N, X, Y.

The solution of equation (1.44) is a stationary point of the Hamiltonian for fixed number of particles N

$$\delta(H + \lambda^2 N) = 0. \tag{1.45}$$

The solution of (1.44) can be rescaled: $\varphi(r,\lambda) = \lambda\varphi_0(\lambda r)$, where $\varphi_0(\xi)$ satisfies the equation

$$-\varphi_0 + \Delta\varphi_0 + 2\varphi_0^3 = 0. \tag{1.46}$$

Hence $N = \lambda^{2-D}N_0$ with $N_0 = 2^{D-1}\pi \int\limits_0^\infty r^{D-1}\varphi_0^2(r)dr$. Let us perform the transform

$$\psi(r) \to a^{-D/2}\psi\left(\frac{r}{a}\right) \tag{1.47}$$

preserving the number of particles. As a result, the Hamiltonian takes a dependence on the parameter a

$$H(a) = \frac{X}{a^2} - \frac{Y}{a^D}. \tag{1.48}$$

According to (1.45) at the soliton solution $\partial H/\partial a|_{a=1} = 0$. Using (1.46) it is easy to get that at these solutions [1, 6]

$$X_s = \frac{D}{4-D}N_0^2/N_s, \quad Y_s = \frac{2}{4-D}N_0^2/N_s, \quad H_s = \frac{D-2}{4-D}N_0^2/N_s. \tag{1.49}$$

Here the index s denotes values of the integrals on the soliton solution. Since X, Y are positive, soliton solutions exist only if $D < 4$. Formulas (1.49), (1.58) make it possible to solve easily the question of soliton stability. If $D < 2$, $H_s < 0$, and the value $a = 1$ realizes the *minimum* of the Hamiltonian (1.48). Hence in this case one can assume that the soliton is stable. This result occurs to be true not only for scaling perturbations but also for the general ones that can be proved rigorously (see, for instance, [1],[8]).

This proof is based on the integral estimates of the Sobolev type. These inequalities arise as sequences of the general imbedding theorems between the spaces L_p and W_2^1 with the norms,

$$\|\psi\|_p = \left[\int |\psi|^p d\boldsymbol{r}\right]^{1/p}, \ (p > 0), \ \ \|\psi\|_{W_2^1} = \left[\int (|\psi|^2 + |\nabla\psi|^2) d\boldsymbol{r}\right]^{1/2}.$$

respectively. Namely, there exists such a constant $B > 0$ so that the following inequality between norms is valid (see, e.g., [9, 10]):

$$\left[\int |\psi|^p d\boldsymbol{r}\right]^{1/p} \leq B\left[\int (|\psi|^2 + |\nabla\psi|^2) d\boldsymbol{r}\right]^{1/2} \text{ if } D < \frac{2}{p}(p{+}4) \text{ and } \boldsymbol{r} \in \Re^D. \tag{1.50}$$

In this formula each integral is assumed to be convergent. Making in (1.50) the transform $\boldsymbol{r} \to \alpha\boldsymbol{r}$, it becomes

$$\int |\psi|^p d\boldsymbol{r} \leq B_1 \left[\alpha^q \int |\psi|^2 d\boldsymbol{r} + \alpha^{q-2} \int |\nabla\psi|^2 d\boldsymbol{r}\right]^{p/2} \quad \text{with } q = d\left(1 - \frac{2}{p}\right).$$

Calculation of the minimum of the r.h.s. of this inequality with respect to scaling parameter α gives the multiplicative variant of the Sobolev inequality [9, 10],

$$\int |\psi|^p d\boldsymbol{r} \leq C \left(\int |\psi|^2 d\boldsymbol{r}\right)^{(2-q)p/4} \left(\int |\nabla\psi|^2 d\boldsymbol{r}\right)^{qp/4}, \tag{1.51}$$

where C is a new constant.

In particular, for $p = 4$ we have (compare with [11])

$$\int |\psi|^4 d\boldsymbol{r} \leq C \left(\int |\psi|^2 d\boldsymbol{r}\right)^{(4-D)/2} \left(\int |\nabla\psi|^2 d\boldsymbol{r}\right)^{D/2} \tag{1.52}$$

This inequality can be improved by finding the best constant C in (1.52). For this aim consider the functional

$$J\{\psi\} = \frac{N^{(4-D)/2} X^{D/2}}{Y}, \tag{1.53}$$

so that

$$C^{-1} = \min J\{\psi\}. \tag{1.54}$$

To find C consider all extremals of the functional $J\{\psi\}$ and take among these the one which gives a minimal value for J. Note, this functional is invariant with respect to two independent dilatations: $\psi \to \alpha\psi$ and $\boldsymbol{r} \to \beta\boldsymbol{r}$. Therefore the corresponding Euler-Lagrange equation for the functional extremum leads to

$$-\psi + \nabla^2\psi + 2|\psi|^2\psi = 0,$$

coinciding with Eq. (1.46) for the soliton solutions. A minimal value of $J\{u\}$ is attained on radically-symmetric distribution without nodes and simultaneously satisfied by Eq.(1.46). This distribution is the ground state soliton for the stationary NLSE. Hence, with account of (1.49) the best constant is equal to

$$C = \frac{2}{N_{0d}(4-D)} \left(\frac{4-D}{D}\right)^{d/2}. \tag{1.55}$$

Here N_{0D} is the number of particles in the ground state soliton N_s depending on the dimension D. For example, in the 1D case $N_{01} = 2$, at $D = 2$, according to [14] $N_{02} = 5.84$ and for $D = 3$, $N_{03} = 9.47$ [2]. As a result, the inequality (1.52) reads (see, for instance, [1] and [12])

$$Y \leq CN^{(4-d)/2}X^{d/2}. \tag{1.56}$$

This inequality allows immediately to get a proof of 1D soliton stability. Substituting (1.56) at $D = 1$ into expression (1.42) for the Hamiltonian and taking into account relations (1.49) we arrive at the following estimate (see, for instance, [13])

$$H \geq X - CX^{1/2}N^{3/2} = H_s + (X^{1/2} - X_s^{1/2})^2. \tag{1.57}$$

Thus, a 1D soliton realizes the global minimum (in the given class!) of the Hamiltonian and therefore is stable not only with respect to small perturbations but also against finite ones[1].

If $D > 2$, the stationary point yields a positive value of H, so that, instead of being a minimum in the one-dimensional case, solitons realize the maximum of the Hamiltonian, which is now unbounded from below and can take (at $a \to 0$) arbitrary large negative values. On the other hand, transformations of the type

$$\psi(r) \to \psi(r)e^{isr^2} \tag{1.58}$$

increases the integral X, leaving integrals N and Y unchanged. Hence the soliton solution is a *saddle point*, leading to the conjecture about instability of the soliton for $2 < D < 4$. This fact can be proved rigorously too.

The case $D = 2$ is special. Now $N = N_0$ and $H \equiv 0$. This result should be discussed separately. The parameter λ characterizes an inverse spatial size (width) of the soliton. Independence of λ of the basic constants of motion for $D = 2$ means that the soliton is "soft" - it can be compressed or inflated without changing its energy and number of particles. In the linear approximation the soliton is marginally stable [15]. More detailed study shows that the soliton is unstable with respect to perturbations of finite amplitude.

The application of the procedure (1.57) at $D = 2$ gives

$$H \geq X\left(1 - \frac{N}{N_{02}}\right).$$

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[1]These inequalities were first used for the stability study of ion-acoustic solitons in magnetized plasma [73] based on the proof of the boundedness of the Hamiltonian. Later this approach was widely applied for the stability proof of the different kinds of solitons (see, for instance, [1]). The acknowledged best use of these inequalities for the collapse problem was presented by Weinstein [12]. Later more general results are reviewed in the paper [2].

From this estimate one can conclude that the Hamiltonian is bounded from below, taking non-negative values if the number of particles does not exceed the number of particles N_s at the ground state soliton solution. Its minimal value, equal to zero, is retained for distributions with vanishing mean square value of the wave number,

$$\langle k^2 \rangle = \frac{X}{N} \to 0.$$

Thus in this region of the phase space asymptotic states of any initial condition will be dispersively spreading distribution, i.e., asymptotically free fields.

In the three-dimensional case the analogous integral estimate for H [19],

$$H \geq X - CX^{3/2}N^{1/2}, \tag{1.59}$$

does not allow us to make any conclusion about soliton stability (note that maximum of the r.h.s. of (1.59) corresponds to a 3D soliton). Recall that the linear stability analysis predicts the instability of three-dimensional solitons [1, 15].

For $D = 1$ equation (1.40) is integrable [16] and has infinite number of extra constants of motion [16]. In this case a soliton can be found in the explicit form

$$\varphi_0(x) = \frac{1}{\cosh x}. \tag{1.60}$$

For $D = 4$ at $\lambda = 0$ the equation (1.46),

$$\varphi_0'' + \frac{3\varphi_0'}{r} + 2\varphi_0^3 = 0, \tag{1.61}$$

has the exact solution

$$\varphi_0 = \frac{2}{r^2 + 1}. \tag{1.62}$$

This is a limiting case for the soliton solutions. Now φ_0 vanishes powerfully at $r \to \infty$, that results in the logarithmical divergence of N_0 at $r \to \infty$.

5 Collapses in the NLSE

For $D \geq 2$ solitons are either unstable or do not exist. In this case the major coherent structure is a collapsing cavity (the region of higher wave intensity) leading to the formation of localized singularities of wave amplitude in a finite time.

One of the main reasons for the wave collapse existence is the Hamiltonian unboundedness. In such systems, like the NLSE, collapse can be represented as a process of falling down of some “particle” in a self-consistent unbounded potential. Indeed, the picture is more complicated than considered above. From the very beginning we have a spatially-distributed system with infinite number of degrees of freedom and therefore, rigorously speaking, it is hard to describe such a system by its reduction to a system of ODEs. The NLSE is a wave system and wave radiation plays a very essential role for blow-up.

Let Ω be an arbitrary region with a negative Hamiltonian H_Ω. Then using the mean value theorem for the integral Y_Ω,

$$\int_\Omega |\psi|^4 \, d\boldsymbol{r} \leq \max_{r\in\Omega} |\psi|^2 \int_\Omega |\psi|^2 d\boldsymbol{r},$$

one can get (compare with [6] and [72]) the following inequality

$$\max_{x\in\Omega} |\psi|^2 \geq \frac{|H_\Omega|}{N_\Omega}. \tag{1.63}$$

Here the expression in the r.h.s. of the inequality has the meaning of the mean energy per one quasi-particle. From this inequality, valid also when $\Omega = \Re^3$, it follows that $\max|\psi|^2$ as a function of t always is majorized by the conservative value. So, vanishing or yet some sufficient decreasing of the initially existed maximum of $|\psi|^2$ are impossible.

Let the Hamiltonian be negative initially in some separate region Ω, $H_\Omega < 0$, and the radiation emerge from this region. In the outer region, far from Ω, radiative waves will have small amplitudes. Consequently, their nonlinear interaction will be negligible with respect to their dispersion and they will have a positive Hamiltonian. Therefore due to the wave radiation, the Hamiltonian of the region H_Ω will become more and more negative increasing its absolute value, that is possible only due to the *unboundedness* of the Hamiltonian. Simultaneously, N_Ω as a positive value will decrease so that the ratio in the r.h.s. of the inequality (1.63) will increase. It automatically leads to the growth of the maximal value of $|\psi|^2$. Thus, radiation, as a dissipative process promotes the wave collapse.

The occurrence of wave collapses can be proved by use of the virial theorem. From (1.40) one can derive the relation

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\boldsymbol{r} = 4[2H - (D-2)Y]. \tag{1.64}$$

At $D = 2$ this relation can be integrated twice

$$\langle r^2\rangle = \frac{\int r^2|\psi|^2 d\boldsymbol{r}}{\int |\psi|^2 d\boldsymbol{r}} = 4\frac{H}{N}t^2 + C_1 t + C_2, \tag{1.65}$$

where the constants C_1, C_2 are defined from the initial conditions. Hence it is seen that for $H < 0$, in spite of the values $C_{1,2}$ there always exists a finite time when the right hand side of (1.65) vanishes. Thus, $H < 0$ represent a sufficient condition for the collapse, which was found by Vlasov, Petrishchev and Talanov (the VPT criterion [17]). For $D = 3$ the equality (1.65) can be replaced by the inequality

$$\langle r^2 \rangle < 4\frac{H}{N}t^2 + C_1 t + C_2. \tag{1.66}$$

from which follows the same sufficient criterion $H < 0$ [6]. This estimate, however, is rather rough and can be improved. As was shown in a recent paper [19], the collapse threshold is defined by the unstable ground state soliton solution which in some sense plays the role of separatrix between collapsing and noncollapsing solution. It was proved in [19] that at $D = 3$ the equality (1.64) can be changed to the inequality

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 dr < 8(H - H_N). \tag{1.67}$$

Here $H_N = N_0^2/N$ is the value of the Hamiltonian of the ground state soliton (compare with (1.49)). Hence the equation (1.67) gives the sharper criterion for collapse [18, 19]

$$H \leq H_N. \tag{1.68}$$

What is the scenario of the collapse? For $D > 2$ NLSE (1.40) has the self-similar solution

$$\psi(r,t) = \frac{1}{(t_0 - t)^{1/2 + i\kappa(D)}} g(\xi), \qquad \xi = \frac{r}{\sqrt{t_0 - t}} \tag{1.69}$$

where $g(\xi)$ satisfies the equation

$$g_{\xi\xi} + \left(\frac{D-1}{\xi} + \frac{i\xi}{2}\right) g_\xi + \left(\frac{i}{2} - \kappa\right) g + 2|g|^2 g = 0, \tag{1.70}$$

$$g_\xi|_{\xi=0} = 0, \qquad g(\infty) = 0.$$

Here $\kappa = \kappa(D)$ is the eigenvalue of the nonlinear boundary problem (1.70). It is easy to show that as $\xi \to \infty$

$$g(\xi) \simeq \xi^{-(1+2i\kappa)}, \tag{1.71}$$

hence $|\psi|^2 \to 1/r^2 \quad \text{as} \quad t \to t_0$. For $D > 2$ the singularity (1.71) is integrable.

There is a plausible hypothesis: the self-similar solution (1.69) describes collapse in a general position. So far, the only way to check this conjecture is by a numerical experiment. Two series of experiments performed by two independent groups confirmed the hypothesis with a very high accuracy [20, 23]. This problem was also discussed in many other papers. We would like to draw attention to some of them - [74] and [30].

A generic case $D > 2$ can be called supercritical. The case $D = 2$ is critical. This case is especially interesting because it describes stationary self-focusing of electromagnetic waves in a nonlinear Kerr dielectric.

For $D \leq 2$ the singularity (1.71) is non-integrable, and the boundary problem (1.70) cannot have regular solutions. In the critical case $D = 2,\ N_s = N_0,\ H_s = 0$, and one can guess that the collapse is the compressing soliton [24]

$$|\psi|^2 = \frac{1}{f^2}\varphi_0^2\left(\frac{r}{f}\right) + ..., \qquad f = f(t_0 - t), \quad f(0) = 0. \tag{1.72}$$

In the strictly self-similar case $f(\xi) = \sqrt{\xi}$. As far the divergence at $D = 2$ is very weak (logarithmic) one can conjecture that now

$$f(\xi) = \sqrt{\frac{\xi}{b(\xi)}}.$$

Here $b(\xi)$ is a "slow" function and $b(0) = \infty$. It was shown [27, 28] that

$$b(\xi) \simeq \ln\ln\left(\frac{1}{\xi}\right).$$

This result is confirmed by numerical experiments with satisfactory accuracy [21, 34, 33, 22] and also analytically [29].

At the end of this section we want to discuss the possibility of collapse in the non-compact NLSE (1.37) and (1.38). First of all, it is easy to show that solitons are absent in this case. The explanation of this fact is very simple. In the transverse plane, equation (1.38) describes attraction between particles, but, in contrast, along the x-axis repulsion. Moreover, from the virial identities for mean transverse size and mean longitudinal size one can show that collapse of the wave packet as a whole is impossible at the stage of the compression of the wave packet in all directions ([35]). Numerical integration of these equations (as it was published in the first paper [36], devoted to this subject, as well as in the recent one [37]), demonstrates the fractal behavior of the system. The initial distribution with sufficiently large amplitude at the beginning demonstrates compression in the transverse plane; at the later stage the wave packet undergoes waving instability that results in splitting of the packet into two packets. At the next stage dynamics of each secondary packet repeats the fate of the original one.

6 Weak, strong and superstrong collapses

The central problem of the physical theory of collapse is the estimate of the efficiency of collapse as a nonlinear mechanism of wave energy dissipation. To achieve that we must include the nonlinear dissipative terms into equations describing the collapsing medium. The nonlinear Schrödinger equation could be modified as follows

$$i(\psi_t + \beta|\psi|^m\psi) + \psi_{rr} + \frac{D-1}{r}\psi_r + 2|\psi|^2\psi = 0. \tag{1.73}$$

Here for $\beta > 0$ the second term is responsible for the nonlinear dissipation. For a sufficiently large degree of nonlinearity m, the equation (1.73) has regular solutions for some small β. The amount of absorbed energy during the collapse is characterized by the integral

$$I \equiv \frac{dN}{dt} = \beta\pi \int dt \int\limits_0^\infty |\psi|^{m+2}(2r)^{D-1}dr. \tag{1.74}$$

Our aim now will be to estimate integral (1.74) at $\beta \to 0$. When approaching the collapse, there are two possibilities. In the critical case, $D = 2$, a strong collapse occurs when a finite amount of energy is accumulated at a collapse point and, as a result, the δ-type singularity is formed. The direct numerical solution of the equation (1.73) confirmed that idea. It was found that the part of energy absorbed during the collapse is about 15% to 25% from the value of N_{cr} [21, 22]. This part practically does not change with the decreasing of β, and slightly reduces when m is increased. In the supercritical case, the integrable singularity of wave energy density is formed in the collapse point. We have

$$|\psi|^2 \sim 1/r^2. \tag{1.75}$$

Let the characteristic size of the collapsing cavity be of the order of r_0 then the characteristic formation time of that scale is $\Delta t \sim r_0^2$. Substituting into (1.74), we obtain

$$I \sim \beta(\Delta t)^{(D-m)/2} \sim \beta r_0^{D-m}. \tag{1.76}$$

From (1.76) it is clear that the nonlinear damping is efficient if $m \geq D$. If we agree that all energy in the collapse zone, $\Delta N \sim r_0^{D-2}(D-2)$, is absorbed, we have

$$r_0 \sim [\beta(D-2)]^{\frac{1}{m-2}}, \quad I \sim (D-2)^{-\frac{m-D}{m-2}}\beta^{\frac{D-2}{m-2}}. \tag{1.77}$$

So, $I \to 0$ at $\beta \to 0$. Such a collapse can be called weak [8]. From (1.77) we can see that for $D \to 2$ the weak collapse becomes a strong one.

The previous considerations were based on the assumption that only the energy arrived at the collapse moment of time $t = t_0$ dissipates in the collapse point. That is not always true. In the point of collapse there could be formed a zone of energy dissipation that absorbs the energy from the surrounding area. In this case the life time of the collapse $\tau \gg \Delta t$, and one must solve the problem of the entire wave packet to estimate the absorbed energy. We suggest calling such a black hole regime a "superstrong collapse", because for a sufficiently large τ the full absorbed energy can exceed the absorption energy for the strong collapse regime. To describe the superstrong collapse it is necessary to obtain solutions of the stationary equation

$$i\beta|\psi|^m + \psi_{rr} + \frac{D-1}{r}\psi_r + 2|\psi|^2\psi = 2|\psi_0|^2\psi \tag{1.78}$$

with boundary conditions

$$\psi_r|_{r=0} = 0, \quad \psi \to \psi_0 \text{ as } r \to \infty.$$

The existence of the black hole also means that at the limit $\beta \to 0$, the equation

$$\psi_{rr} + \frac{D-1}{r}\psi_r + 2|\psi|^2\psi = 2|\psi_0|^2\psi \tag{1.79}$$

has a singular solution with a constant energy flux to the collapse point at $r = 0$

$$P = \lim_{r\to 0} \pi(2r)^{D-1}\ \mathrm{Im}(\psi\psi_r^*). \tag{1.80}$$

Let $D > 2$. Then the equation

$$\psi_{rr} + \frac{D-1}{r}\psi_r + 2|\psi|^2\psi = 0 \tag{1.81}$$

has the exact solution [23]

$$\psi = A_0/r, \qquad A_0 = \left(\frac{D-3}{2}\right)^{1/2}. \tag{1.82}$$

This solution can be used as a first step to construct a singular solution of the equation (1.81). Indeed, the solution near zero could be found as

$$|\psi| = \frac{A_0}{r}(1 + A_1 r^\mu + ...), \qquad \psi \to \psi_0, \qquad \mu = 2(4-D) > 0. \tag{1.83}$$

Here A_1 is an arbitrary constant, and $A_1 = qP^2$, where q is some multiplier. By the selection of P one can obtain the asymptotic solution of the equation (1.83) for $r \to \infty$. The numerical integration of (1.78) showed that in the interval $3 < D < 4$ the solution describing the superstrong collapse really can be constructed. Such solutions exist in a rather important physical case at $D = 3$, which corresponds to a $3-D$ nonstationary self-focusing. In this case the main asymptotic term at zero is a stationary solution [31, 32]

$$|\psi| = \frac{1}{2r|\ln\ r|^{1/2}}.$$

Superstrong collapse can exist also for power nonlinearity $|\psi|^{2n}\psi$ where $nD > 4$. Thus, at $D = 4$ the equation (1.81) has an exact singular solution

$$|\psi| = B/r,$$

whose amplitude is defined by the flux P from the equation

$$B^4(B^2-1)=P^2.$$

Finally, when $D>4$, equation (1.81) has quasi-classical stationary solutions with an asymptotic expansion at zero

$$\begin{aligned}|\psi| &= \frac{c}{r^\gamma}(1+c_1r^\nu+...), \quad c=P^2, \quad \gamma=\alpha(\alpha-1),\\ \nu &= \alpha(2D-8)>0, \quad \alpha=\frac{1}{3}, \quad c_1(D)>0.\end{aligned}$$

It is important that the quasi-classical criterion for this solution improves while approaching the singular point ($r=0$).

The existence of such solutions was also confirmed by the numerical integration of equation (1.78) at $\psi_0=0$ [23, 21].

7 Anisotropic black holes

As we saw in the previous section the black hole regime becomes quasi-classical starting from $D=4$. In this section we want to present an example showing how, due to the medium anisotropy, the "effective dimension D'' can be greater than 4 and, as a consequence, the black-hole regime can be realized.

We consider the upper-hybrid waves Langmuir waves in a plasma with sufficiently small magnetic field ($\omega_{pe}\gg\omega_{ce}$) when all changes in the dispersion law are expressed in the form of the additive term

$$\omega_k=\omega_{pe}\left(1+\frac{3}{2}k^2r_d^2+\frac{1}{2}\frac{\omega_{ce}^2}{\omega_{pe}^2}\frac{k_\perp^2}{k^2}\right), \tag{1.84}$$

where ω_{ce}, ω_{pe} are electron gyrofrequency and electron plasma frequency, respectively, $r_d=v_{Te}/\,\omega_{pe}$ is the Debye radius, $\boldsymbol{k}_\perp$ is the component transverse to the external magnetic field $\boldsymbol{B}_0$, directed along the z axis. In the dispersion law (1.84), the first term describes the potential electron plasma oscillations with a plasma frequency. Other terms are due to slower processes. In isotropic case ($\omega_{ce}=0$) (1.84) transforms into the dispersion law for the Langmuir waves.

The nonlinear effect, in a small-amplitude region ($E^2/8\pi nT\ll m/M$), m and M being the electron and ion masses, respectively, corresponds to the nonlinear frequency shift due to the interaction with slow adiabatic plasma flows induced by high-frequency plasma oscillations. In this limit, the equation for the envelope of high-frequency oscillations in dimensionless variables can be written as follows [38]

$$\Delta(i\psi_t+\Delta\psi)-\sigma\Delta_\perp\psi+\nabla(|\nabla\psi|^2\nabla\psi)=0, \tag{1.85}$$

where ψ is the envelope of high-frequency waves and $\sigma = \omega_{ce}^2/2\omega_{pe}^2$. Respectively, the low-frequency plasma fluctuations follow adiabatically the pondermotive pressure of high-frequency waves,

$$n = -|\nabla\psi|^2.$$

The equation (1.85) at zero magnetic field transforms into the Zakharov equation describing collapse of Langmuir waves [6] in the so-called static approximation.

The equation (1.85) can be further reduced under additional assumptions. It is known that due to weak turbulent processes, such as induced scattering of ions or four-wave interaction, the energy transfer by cascade to the region $\omega_k \to \omega_{pe}$. If one studies these processes in more details, it is possible to find that they lead, in the first stage, for waves with $(kr_d)^2 < \sigma$, to a rapid decrease of $k_\perp$, and only subsequently, to a reduction of k_z up to a zero value. This means that the wave condensate will have characteristic longitudinal scales smaller than the transverse ones. Under this assumption Eq. (1.85) reads as follows

$$\frac{\partial^2}{\partial z^2}\left(i\psi_t + \frac{\partial^2}{\partial z^2}\psi\right) - \Delta_\perp\psi + \frac{\partial}{\partial z}\left(\left|\frac{\partial\psi}{\partial z}\right|^2 \frac{\partial\psi}{\partial z}\right) = 0. \tag{1.86}$$

where we put, without any restriction, the constant $\sigma = 1$, that corresponds to a simple rescaling.

The equation (1.86) can also be written in the Hamiltonian form

$$i\frac{\partial^2}{\partial z^2}\psi_t = \frac{\delta H}{\delta\psi^*}, \tag{1.87}$$

where the Hamiltonian

$$H = \int (|\psi_{zz}|^2 + |\nabla_\perp\psi|^2 - \frac{1}{2}|\psi_z|^4) d\boldsymbol{r} \equiv I_1 + I_2 - I_3. \tag{1.88}$$

The possible stationary solutions of this equation should correspond to the soliton-like solution

$$\psi = \psi_0 \exp(i\lambda^2 t),$$

where ψ_0 satisfies the equation

$$\frac{\partial^2}{\partial z^2}\left(-\lambda^2\psi_0 + \psi_{0zz}\right) - \Delta_\perp\psi_0 + \left(|\psi_{0z}|^2\,\psi_{0z}\right) = 0. \tag{1.89}$$

Localized solutions of this equation simultaneously represent stationary points of the Hamiltonian for a fixed number of particles $N = \int |\psi_z|^2 d\boldsymbol{r}$ (coinciding up to a constant multiplier with the energy of high-frequency waves),

$$\delta(H + \lambda^2 N) = 0. \tag{1.90}$$

Performing now the scaling transformation retaining N,

$$\psi \to \frac{a^{1/2}}{b}\psi\left(\frac{z}{a}, \frac{r_\perp}{b}\right),$$

instead of (1.47) for the NLSE, H (1.88) becomes a function of two scaling parameters,

$$H(a,b) = \frac{I_1}{a^2} + \frac{I_2}{b^2}a^2 - \frac{I_3}{ab^2}. \tag{1.91}$$

The variational problem (1.90) now yields two relations between integrals I_l $(l = 1,2,3)$ on the solution ψ_0

$$-2I_1 + 2I_2 + I_3 = 0, \; -I_2 + I_3 = 0. \tag{1.92}$$

Another relation follows after multiplication of (1.89) by ψ_0 and integration

$$\lambda^2 N + I_1 + I_2 - 2I_3 = 0. \tag{1.93}$$

After a simple algebra based on (1.92) and (1.93), one can show that

$$I_1 = -2\lambda^2 N < 0,$$

contradicting the sign of I_1 which is positive definite. This contradiction implies that for (1.86) stationary soliton solutions do not exist [39]. This is possible to understand considering the NLSE (1.40) as an example. According to (1.49), soliton solutions in the NLSE exist for $D \leq 4$ and are absent for $D > 4$ that corresponds to the well-known general fact, i.e., to increase of the role of nonlinear effects with growth of dimension D. Consider now the parabolic family $b = \gamma a^2$, where γ is a constant. For this kind of curve, the first two terms in $H(a,b)$ (1.91) have the same (self-similar) behavior (dependence)

$$H(a,\gamma) = \frac{1}{a^2}\left(I_1 + \frac{I_2}{\gamma^2}\right) - \frac{1}{a^5}\left(\frac{I_3}{\gamma^2}\right). \tag{1.94}$$

Hence, firstly, one can see that the Hamiltonian is unbounded from below as $a \to 0$ that is one of the necessary conditions for the existence of collapse. Secondly, the comparison of (1.94) with (1.48) shows that the equation (1.86) is equivalent to the NLSE with $D = 5$. According to our classification presented in the previous sections, the dimension $D = 5$ corresponds to superstrong collapse providing existence of quasi-stationary black-hole regime. Moreover, this regime can be described in terms of semi-classical approach. The latter assumes solutions of the equation (1.86) in the form $\psi = Ae^{i\Phi}$ where we impose the following (semi-classical) restrictions on the phase Φ and the amplitude A

$$|\Phi_t|T \gg 1, \quad |\Phi_z|L_z \gg 1, \quad |\nabla_\perp \Phi|L_\perp \gg 1. \tag{1.95}$$

Here T is the characteristic time of the amplitude variation, L_z and $L_\perp$ are characteristic longitudinal and transverse scales of the amplitude, respectively.

Under these assumptions, in the leading order we have the Hamilton-Jacobi equation for the eikonal Φ

$$\Phi_t + \Omega(\nabla\Phi) - n = 0 \tag{1.96}$$

where $\Omega(\boldsymbol{k}) = k_z^2 + k_\perp^2/k_z^2$ is the dispersion relation for small-amplitude waves describing by the linearized equation (1.86), $\boldsymbol{k} = \nabla\Phi$ is the wave vector and $n = |\psi_z|^2 \simeq A^2\Phi_z^2$ is the wave intensity. At the next order we arrive at the continuity equation for n

$$n_t + \mathrm{div}(n\boldsymbol{V}) = 0. \tag{1.97}$$

Here $\boldsymbol{V} = \partial\Omega/\partial\boldsymbol{k}$ is the group velocity. Eqs. (1.96), (1.97) retain the Hamiltonian structure

$$\begin{aligned} n_t &= \frac{\delta H}{\delta\Phi}, \quad \Phi_t = -\frac{\delta H}{\delta n}, \\ H &= \int \left[\Omega(\nabla\Phi)n - \frac{n^2}{2}\right] d\boldsymbol{r}. \end{aligned}$$

It is possible to show that Eqs. (1.96) and (1.97) have the whole family of collapsing solutions starting from semi-classical strong collapse up to the weakest collapse corresponding to self-similar solution of the equation (1.86) (for details, see [40, 39]). All semi-classical collapsing regimes occur to be unstable. Therefore at the initial stage of a collapse we have the formation of a weak singularity which later on serves as the origin for the appearance of a black hole. To find a structure of a black hole, it is enough to take semi-classical equations (1.96), (1.97) and to seek for solutions in the form of an anisotropic funnel

$$\Phi = \frac{1}{z}\phi(\eta), \quad n = \frac{1}{z^4}g(\eta). \tag{1.98}$$

Here $\eta = r_\perp/z^3$ is a new self-similar variable and the function $g(\eta)$, as it is easy to show, obeys the ordinary differential equation

$$gg' + 3\eta(g + 3\eta g')^4 = 0. \tag{1.99}$$

Solutions of this equation only depend on the constant

$$P = \int_0^\infty (1 - 3\eta g' g^{-1}) g' d\eta,$$

which is the energy flux into the singularity. Numerical calculations of (1.99) showed the existence of monotonically vanishing solutions with the asymptotics $g \sim \eta^{-1/3}$ at $\eta \to \infty$. It should be added that the semi-classical criterion (1.95) for the solution (1.98) improves as $\boldsymbol{r}$ approaches the singular point.

The similar situation arises for lower-hybrid waves near the lower-hybrid resonance ω_{LH}. In the case, when $\omega_{ce} \gg \omega_{pe}$, the dispersion law of waves is

$$\omega_k = \omega_{LH}\left(1 + k_\perp^2 R^2 + \frac{1}{2}\frac{m}{M}\frac{k_z^2}{k_\perp^2}\right),$$

where $R = [3/4 + (3T_i/T_e)]r_{ce}$, r_{ce} is the electron gyro-radius, while m and M are the electron and ion mass, respectively. For low wave intensity, as for UH waves, the low-frequency plasma-density variation is related to the high-frequency pondermotive force through [41, 42]

$$n = i[\nabla\psi \times \nabla\psi^*]_z\ . \tag{1.100}$$

Here, as in a previous case, we write (1.100) in dimensionless variables, ψ stands for the envelope of the high-frequency electric potential of LH waves.

The evolution equation for the envelope is obtained by usual average over the high-frequency ω_{LH} [41, 42]

$$\Delta_\perp(i\psi_t + \Delta_\perp\psi) - \alpha\partial_z^2\psi - \nabla_\perp([\nabla_\perp\psi \times \nabla_\perp\psi^*]_z[\boldsymbol{n} \times \nabla_\perp\psi]) = 0. \tag{1.101}$$

Here $\boldsymbol{n} = \boldsymbol{B}_0|\boldsymbol{B}_0|$ and α is a constant. This equation can be written in the Hamiltonian form

$$\Delta_\perp i\psi_t = \frac{\delta H}{\delta\psi^*}, \tag{1.102}$$

$$H = \int\left(|\Delta_\perp\psi|^2 + |\psi_z|^2 + \frac{1}{2}[\nabla_\perp\psi \times \nabla_\perp\psi^*]^2\right)d\boldsymbol{r} \equiv I_1 + I_2 - I_3.$$

The same analysis as it was done for UH waves demonstrates that solitons are absent for the model (1.101). The Hamiltonian under scaling transformations, $\psi(z, r_\perp) \to a^{-1/2}\psi(z/a, r_\perp/b)$, regaining the wave energy $N = \int|\nabla_\perp\psi|^2 d\boldsymbol{r}$, behaves as follows [39]

$$H(b,\gamma) = \frac{1}{b^2}\left(I_1 + \gamma^2 I_2\right) - \frac{1}{b^4}\left(\frac{I_3}{\gamma}\right),$$

where $H(b,\gamma)$ is taken along the parabolas $a = \gamma b^2$. Thus, for the effective dimension $D = 4$ weak collapse forms initially a singularity, which eventually transforms into a black hole. It is interesting to note that the effective dimension D for the black-hole regime corresponds to a lower boundary of the semiclassical black holes [45].

At the end of this Section we would like to note the recent experimental observations of quasi-stationary localized structures in the auroral ionosphere (at altitudes near 800 km) [43]. The wavelet analysis of these measurements by the plasma wave interferometer aboard the AMICIST rocket demonstrated in the region of lower-hybrid frequency the existence of long-life-time solitary structures possessing rotating eigenmodes [44]. These observations are consistent with the results of three-dimensional numerical experiments which showed the presence of a cavity density. In our opinion, these objects are the first candidates for black holes.

8 Structure in media with weak dispersion

Let us consider another situation where phase stochastization plays a less important role than coherent structures. This is propagation of waves in a media with weak dispersion. Suppose first that dispersion is absent entirely. In an isotropic medium

$$\omega(k) = c|k| \tag{1.103}$$

where c has a meaning of sound speed.

Now resonant conditions (1.9) can be satisfied only if all three vectors $\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3$ are parallel. In particular, they are satisfied, if $\boldsymbol{k}_2 = \boldsymbol{k}_3 = \boldsymbol{k}, \boldsymbol{k}_1 = 2\boldsymbol{k}$. It means that the monochromatic wave cannot exist for a long time; it produces second harmonic, then zero and higher harmonics. Phases of all harmonics are correlated. This creates favorable conditions for the formation of coherent structures. This family is especially rich, if the dispersion relation is not exactly linear

$$\omega(|\boldsymbol{k}|) = c(|\boldsymbol{k}| + L(\boldsymbol{k})), \quad |L(\boldsymbol{k})| \ll \boldsymbol{k}, \quad L(0) = 0. \tag{1.104}$$

If $L''(\boldsymbol{k}) > 0$, the resonant conditions (1.9) are satisfied when all three vectors $\boldsymbol{k}_i$ are almost parallel. Then it is possible to consider the situation when the support of the function $a(\boldsymbol{k})$ is concentrated on an almost one dimensional set. In other words, one can present the wave vector in the form

$$\boldsymbol{k} = (p, \boldsymbol{q})$$

and consider the complex amplitude $a(p, \boldsymbol{q}) \neq 0$ only if $|\boldsymbol{q}| \ll p,\ p > 0$. Here $p, \boldsymbol{q}$ are components along and across the direction of the wave propagation. Now

$$|\boldsymbol{k}| = \sqrt{p^2 + \boldsymbol{q}^2} \simeq p + \frac{1}{2}\frac{\boldsymbol{q}^2}{p}, \tag{1.105}$$

and one can put approximately

$$\omega(p, \boldsymbol{q}) \simeq c\left(p + \frac{1}{2}\frac{\boldsymbol{q}^2}{p} + L(p)\right), \qquad L(-p) = -L(p). \tag{1.106}$$

The most natural model of acoustic waves is compressible ideal hydrodynamics with dependence of internal energy of both density ρ and its gradient. In particular, ion-acoustic waves in isotropic plasma relate to this kind of waves and can be described in terms of ideal hydrodynamics with dispersion. We can use this model for calculation of the coupling coefficient for three-wave interaction. Skipping the details (see [3], e. g.), we present the result of these calculations

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) \simeq V(p, p_1, p_2) = \mu (p p_1 p_2)^{1/2}; \qquad p_i > 0 \tag{1.107}$$

where μ is a constant expressing through the characteristics of the media: mean density, sound speed and internal energy.

Let D be the space dimension. One can introduce a new unknown function

$$u(x, r, t) = -\frac{1}{(2\pi)^D} \int\limits_{p>0} \sqrt{p}(a_{p,q} + a^*_{-p,-q}) e^{ip(x-t)+i\boldsymbol{q}\boldsymbol{r}}\, dp d\boldsymbol{q}. \tag{1.108}$$

After simple transformation one can find that in this case (1.8) takes the form

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \hat{L}\, u - \nu \frac{\partial^2 u}{\partial x^2}\right) = -\Delta_\perp u. \tag{1.109}$$

Here

$$\hat{L}\left(\frac{\partial}{\partial x}\right) = i\, \hat{L}\left(-i\frac{\partial}{\partial x}\right) = -\,\hat{L}\left(-\frac{\partial}{\partial x}\right) \tag{1.110}$$

is the operator responsible for dispersion and $\Delta_\perp$ is the Laplacian with respect to $\boldsymbol{r}$. In the two dimensional space $\Delta_\perp u = u_{yy}$, in the three-dimensional case $\boldsymbol{r} = (y, z)$ and $\Delta_\perp u = u_{yy} + u_{zz}$. In equation (1.109) we introduced the dissipative (viscous) term νu_{xx}.

If $\omega^2 = \omega^2(k^2)$ is an analytic function of k^2 $L(p)$ is an odd function: $L(p) = -L(-p)$. In the simplest case

$$L(p) = \pm p^3, \quad \hat{L}\left(\frac{\partial}{\partial x}\right) = \mp \frac{\partial^3}{\partial x^3}. \tag{1.111}$$

In the 2D case for $\nu = 0$ we obtain now the Kadomtsev-Petviashvili equations [46, 47]:

the KPI equation -

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}\right) = -\frac{\partial^2 u}{\partial y^2} \tag{1.112}$$

and the KPII equation -

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = -\frac{\partial^2 u}{\partial y^2}. \tag{1.113}$$

The KPI equation describes acoustic-type waves with positive dispersion. These are magneto-acoustic waves in strongly magnetized plasma with propagation angles not close to transverse and parallel directions of an external magnetic field, phonons under some conditions [48], gravity-capillary waves for the shallow water limit.

The KPII equation corresponds to the negative dispersion. Gravity waves for shallow water, ion-acoustic waves in isotropic plasma, magneto-acoustic waves for perpendicular to magnetic field propagation belong to such type of waves.

The general equation (1.109) can be called the generalized KP equation. We note that this equation is written in a frame moving with the sound speed c, which coincides with the group velocity of the waves at $k = 0$. All other terms describe slow process with respect to this propagation: the second term represents the nonlinear renormalization of the sound speed, the third one is responsible for weak dispersion and, finally, the term in the r.h.s. of (1.109) refers to transverse diffraction of acoustic waves.

In many interesting cases, $L(p)$ is not an analytic function in the vicinity of $p = 0$. In this case $L\,(\partial/\partial x)$ is a pseudo-differential operator. For instance, among the operators

$$L(p) = \pm p|p|^{2s}, \quad s > 0, \quad \hat{L} = \mp\frac{\partial}{\partial x}\left|\frac{\partial}{\partial x}\right|^{2s} \tag{1.114}$$

the choice $s = 1/2$, $\Delta_\perp u = 0$, $\nu = 0$ corresponds to the well-known Benjamin-Ono equation applicable for description of internal waves. The generalized KP equation (GKP) describes the wide spectrum of coherent structures including solitons, collapses and black holes. Let us consider the simplest examples of such structures.

Neglecting by dispersion, dissipation and diffraction, one arrives at the Hopf equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{1.115}$$

The general solution of this equation is given in the implicit form

$$x = ut + F(u) \tag{1.116}$$

where $F(u)$ is arbitrary function. Let $F(u) = -ut_0 + u^3$. The corresponding solution has a self-similar form

$$u = (t_0 - t)^{1/2} g\left(\frac{x}{(t_0 - t)^{3/2}}\right). \tag{1.117}$$

Here $g(\xi)$ is the solution of the cubic equation

$$g^3 = g + \xi. \tag{1.118}$$

The solution (1.117) describes self-similar wave collapse. As a result, the first derivative of u becomes infinite in a finite time. Indeed, according to (1.117)

$$\frac{\partial u}{\partial x}|_{x=0} \simeq \frac{1}{t_0 - t}.$$

At the moment of collapse ($t = t_0$) $u = x^{1/3}$. This example of the wave collapse is known as the wave breaking.

Let us consider the influence of the neglected factors to the process of collapse. Taking into account the dependence of the diffraction term we have the dispersionless KP equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial^2 u}{\partial y^2}. \tag{1.119}$$

In 3D media one has to replace $\partial^2 u/\partial y^2$ by $\Delta_\perp u$. Weak dependence on the perpendicular coordinate in the solution (1.116) might be taken into account by replacing in the solution (1.116) $t_0 \to t_0 + \epsilon y^2$. In this case, comparing competing terms in (1.119) ,

$$\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) \simeq \frac{1}{(t_0 - t)^2}; \quad \frac{\partial^2 u}{\partial y^2} \simeq \frac{\epsilon}{(t_0 - t)^{3/2}}, \tag{1.120}$$

one can see that weak dependence on the perpendicular coordinates does not arrest the wave breaking.

On the contrary, both dissipation and dispersion arrest the collapse. In the presence of dissipation the equation (1.109) transforms (in the one-dimensional case) into the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}. \tag{1.121}$$

As soon as for the wave breaking $uu_x \simeq 1/(t_0 - t)^{1/2}$ and the dissipation term can be estimated as $u_{xx} \simeq 1/(t_0 - t)^{5/2}$, collapse is seen to be arrested even by an infinitesimally small viscosity ν. To estimate the dissipation efficiency one can exploit the identity

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u^2 dx = -\nu\int_{-\infty}^{\infty} u_x^2 dx. \tag{1.122}$$

As soon as $u_x^2 \simeq 1/x^{4/3}$, the integral in the right hand side of (1.122) converges as t approaches t_0

$$\nu\int_{-\infty}^{\infty} u_x^2 dx \simeq \frac{\nu}{(t_0 - t)^{1/2}}.$$

Similarly, the total amount of absorbed energy is

$$\Delta E = \int_0^{t_0} dt \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u^2 dx \simeq 2\nu {t_0}^{1/2}.$$

Note that $\Delta E \to 0$ as $\nu \to 0$. By definition this is *weak collapse.*

What is the remote results of the collapse in this case? The Burgers equation (1.121) has a solution in the form of a stationary propagating shock wave

$$u = \frac{2s}{1+\exp[s(x+st)/\nu]}. \tag{1.123}$$

Calculating now the rate of dissipation for this solution, one can get

$$\nu\left(\frac{\partial u}{\partial x}\right)^2 = \frac{s^4}{4\nu\cosh^4[s(x+st)/\nu]}.$$

Thus, dissipation is concentrated in a very small domain near $x = -st$. The shock wave is a moving sink of energy, i.e. a black hole of codimension 1. It can be compared with the black hole of dimension zero (codimension three) which can arise after the formation of weak singularity in the 3D NLSE (see Section 6).

Another fundamental effect, arresting the collapse of gradients, is the wave dispersion. Suppose that

$$L\left(\frac{\partial}{\partial x}\right)u = -\frac{\partial}{\partial x}\left|\frac{\partial}{\partial x}\right|^{2s}u, \tag{1.124}$$

where $|\partial/\partial x|^{2s}$ is the operator with symbol $|k|^{2s}$. Then the equation (1.109) in 1D case takes the form of the generalized Korteweg-de-Vries equation (GKDV).

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left|\frac{\partial}{\partial x}\right|^{2s}u = 0 \tag{1.125}$$

For $s = 1$ this equation transforms into the classical KDV equation. At $s = 1/2$ (1.125) coincides with the Benjamin-Ono equation.

Comparison of linear and nonlinear terms in (1.125) shows that the collapse of gradients (wave breaking) is impossible for any $s > 0$. The equation (1.125) can be presented as follows

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\frac{\delta H}{\delta u} \tag{1.126}$$

where

$$H = T - U, \quad T = \frac{1}{2}\int_{-\infty}^{\infty}\left(\left|\frac{\partial}{\partial x}\right|^{s}u\right)^2 dx, \quad U = \frac{1}{6}\int_{-\infty}^{\infty}u^3 dx. \tag{1.127}$$

The equation (1.125) preserves the Hamiltonian H and the momentum $P = \frac{1}{2}\int_{-\infty}^{\infty}u^2 dx$.

Let us look for soliton solutions of (1.125) in the form of a stationary propagating wave $u = u(x - Vt)$. After one integration we obtain

$$-Vu + \frac{1}{2}u^2 - \left|\frac{\partial}{\partial x}\right|^{2s}u = 0. \tag{1.128}$$

This equation follows from the variational problem

$$\delta(H + VP) = 0, \tag{1.129}$$

demonstrating that soliton solutions are stationary points of the Hamiltonian for fixed momentum P. As far as the operator $|\partial/\partial x|^{2s}$, it is positive definite. Therefore solitons (as localized objects) can exist only for $V > 0$. Only in this case the linear operator $\hat{R} = V + |\partial/\partial x|^{2s}$ in the stationary equation (1.128) is positive definite and reversible. Otherwise the solution of (1.128) has oscillating asymptotics at infinity.

By multiplying (1.128) by $u/2$ and integrating with respect to x one can get the relation

$$\frac{3}{2}U - T = VP. \tag{1.130}$$

Let $u(x)$ be a solution of (1.128). Consider the functions $u(x, a) = a^{-1/2} u\,(x/a)$ depending on the scaling parameter a. Then

$$H = \frac{T}{a^{2s}} - \frac{U}{a^{1/2}} \tag{1.131}$$

(here U, T are calculated on the solution $u(x)$). For this kind of deformation the condition (1.127) now reads as $\partial H/\partial a|_{a=1} = 0$ or

$$\frac{1}{2}U - 2sT = 0, U = 4sT,$$

which in combination with (1.130) gives

$$T = \frac{VP}{6s-1}, \quad U = \frac{4sVP}{6s-1}, \quad H = \frac{1-4s}{6s-1}VP. \tag{1.132}$$

Here both functionals U and T are positive definite for $s > 1/6$. For the model (1.125) this defines the region of the soliton existence. For $s > 1/4$ the Hamiltonian on the soliton solutions is negative, $H_s < 0$. In this case it is possible to show that the stationary point $u(x)$ is not only the local but the global minimum of the functional H that, in accordance with the Lyapunov theorem, provides the soliton stability.

For $s < 1/4$ $H_s > 0$. Letting $a \to 0$ in (1.112) one can see that in this case H can be made arbitrary negatively large. Solitons in this region of the parameter V represent saddle points and one may expect that they are unstable.

For any s the equation (1.124) allows the self-similar substitution

$$u = (t_0 - t)^{1+1/2s} F\,(\xi)\,, \quad \xi = \frac{x}{(t_0 - t)^{1/(1+2s)}}. \tag{1.133}$$

For this family of self-similar solutions we have

$$P \simeq (t_0 - t)^{\frac{1-4s}{1+2s}}. \tag{1.134}$$

Hence one can see that a localized solution can exist only for $s > 1/4$, coinciding with the interval for soliton stability. The solution (1.133) at $0 < s < 1/4$ describes weak collapse leading to the formation of an integrable singularity: $u \approx 1/|x|^{2s}$. At $s = 1/4$ we have the regime of strong collapse corresponding to the critical NLSE considered in Section 6.

The results of this Section can be easily extended to a more general equation

$$\frac{\partial u}{\partial t} + u^p \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left| \frac{\partial}{\partial x} \right|^{2s} u = 0. \tag{1.135}$$

We now consider reference [50] where for the critical KDV equation at $s = 1$ and $p = 2$ the corresponding theory for strong collapse was developed. In particular, it was shown that the asymptotic form of the collapsing distribution approaches the soliton form at the collapse time, and the absorbed energy into singularity corresponds to the soliton energy. For $p > 2$ $(s = 1)$ collapse becomes weak. Stable solitons appear for $p < 2$ [49].

Very interesting coherent structures are described by the dissipative generalized KDV equation

$$\frac{\partial u}{\partial t} + u^p \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left| \frac{\partial}{\partial x} \right|^{2s} u = \nu \frac{\partial^2 u}{\partial x^2}. \tag{1.136}$$

For $s > 1/4$ this equation describes the so-called collisionless shock waves discovered by R.Z. Sagdeev [51]. For the unstable case $s \leq 1/4$ the coherent structures have not been studied yet.

9 Singularities on a fluid surface

For sea surface waves, the wave breaking leads to an infinite second derivative of the surface profile (so that angles or cones appear on the surface). In this field, the first important results date back to the middle of the last century and belong to the famous Stokes [52]. Using the apparatus of complex analysis, Stokes discovered that the critical angle for the surface slope of stationary gravity waves for the deep water case was equal to 120°. For larger angles stationary gravity waves were assumed to be absent. Checking analyticity violation is the most sensitive tool for studying that set of collapses. Loss of analyticity of vortex sheets at the nonlinear stage of the Kelvin-Helmholz instability [53] is such an example. Various aspects of the singularity formation for vortex sheet motion have so far been studied in a number of papers, both numerically and analytically [53]-[58]. The paper [56] should be mentioned in particular, which provides a considerable numerical evidence of arising of the infinite surface curvature in a finite time. The root (in space) character of the arising singularity has also been checked in [56].

Below we present some recent results [59, 60] devoted to the free surface hydrodynamics of an ideal fluid. Adopting only the small slope approximation, in absence of both gravity and capillarity, this system was effectively examined. In particular, it was shown that for two-dimensional flows the velocity component v, tangent to the free surface, obeys the equation, formally coinciding with the Hopf equation (1.115),

$$\frac{\partial v^{\pm}}{\partial t} + v^{\pm}\frac{\partial v^{\pm}}{\partial x} = 0. \tag{1.137}$$

Here $v^{\pm}$ is analytical continuation of $v(x,t)$ to the upper $^{(+)}$ and lower $^{(-)}$ half-planes of the variable x. On the real axis $v = \frac{1}{2}(v^{+} + v^{-})$ and functions $v^{(\pm)}$ are complex conjugate. The free surface elevation $\eta(x,t)$ in this approximation ($|\nabla\eta| \ll 1$) is defined from integration of the equation

$$\frac{\partial \eta}{\partial t} = -\hat{H}\, v. \tag{1.138}$$

Here

$$(\hat{H}f)(x) = \frac{1}{\pi} V.P. \int_{-\infty}^{+\infty} \frac{f(x')}{x' - x}\, dx'.$$

is the Hilbert transform. Both equations for v and η are integrable. The integrability of these equations originates from the solution of the Laplace equation in the fluid bulk.

Autonomy of the equation for the tangent velocity component from elevation η is one of the main features of this system[2]. It admits, as for (1.115), the standard method of characteristics, but the analyticity requirement for functions $v^{\pm}$ leads in comparison with solution (1.116) to some changes in the form of general solution. Omitting all details of the general solution analysis (see [60, 59]), we present here only the main results.

The formation of singularities on the free surface for small angle approximation can be considered as the process of the wave breaking in the complex plane to which the solution can be extended. This results in the motion of both branch points of the analytical continuation of the velocity potential and singular points of the analytical extension of the surface elevation. When for the first time the most "rapid" singular point reaches the

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[2]Equation (1.137), after separation of imaginary and real parts, transforms into a system of the gas dynamic type with negative pressure,

$$u_t + (uv)_x = 0,$$

$$v_t + vv_x = \frac{1}{2}(u^2)_x$$

where u is normal component of the velocity. It is interesting that this system also follows from the quasi-classical limit of the fifth NLSE.

real axis it just indicates the appearance of the singularity. Respectively three kinds of singularities are possible. For the first kind at the touching moment, the tangent velocity on the surface has an infinite first derivative and simultaneously the second space derivative of the surface coordinate $z = \eta(x,t)$, i.e. η_{xx}, also tends to infinity. These are weak singularities of the root character ($\eta_{xx} \sim\mid x \mid^{-1/2}$). This kind of singularities turns out to be consistent with an assumption about small surface angles. It is shown that the interaction of two movable branch points of the tangent velocity can lead under some definite conditions to the formation of the second type of singularities - wedges on the surface shape. Close to the collapse time the self-similar solution for such singularities is compatible with the complete system of equations describing arbitrary angle values. The third type is caused by the initial analytical properties of $\eta_0(x)$, resulting in the formation of strong singular surface profile.

As was shown in [61], the equation of motion for free surface hydrodynamics with finite depth in the absence of capillarity can also be integrated effectively in the small angle approximation. Of course, the root singularities, as well as all others have the same asymptotic behavior as for the deep water case. Another interesting effect is connected with the possibility to integrate the free surface hydrodynamics in the limit of large surface gradients. In this case, as it was shown in [62], the equation can be reduced to the so-called Laplacian growth equation (LGE)[3] which allows application of the pole decomposition. The latter means that a system of equations has an exact solution in the form of finite sum of poles, residues of which are constants and pole positions (in complex plane) obey a closed dynamical system of ordinary differential equations. In the case of the LGE this dynamical system allows complete integration. Similarly, the solution can be written in an implicit form (for more details see [65, 66, 62, 61]).

10 Solitons and collapses in the generalized KP equation

Let us take into account the diffraction term, that corresponds to consideration of the dependence on perpendicular coordinates in the GKDV equation. Assuming maximum symmetry in the perpendicular plane we

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[3] At first this equation was derived in 1945 by Polubarinova-Kochina and Galin [63, 64] for boundary flows in porous media. Later it became clear that this equation is applicable for description of the boundary motion for phase transition of the first kind.

will study the following version of the equation (1.109)

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\partial}{\partial x}\left|\frac{\partial}{\partial x}\right|^{2s}u\right)=\frac{\alpha}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}u. \qquad (1.139)$$

Here $\alpha=\pm1$, d is the dimension of the perpendicular plane, r is the radius in this plane.

The equation (1.139) can be written in the Hamiltonian form (1.126)

$$H=T-U+W \qquad (1.140)$$

where

$$T = \frac{1}{2}\int\left(\left|\frac{\partial}{\partial x}\right|^{s}u\right)^2 dxdr,\quad U=\frac{1}{6}\int u^3dxdr, \qquad (1.141)$$

$$2W = \frac{\alpha}{2}\int(\nabla_\perp w)^2dxdr \quad w=\int_{-\infty}^{x}udx. \qquad (1.142)$$

This equation conserves the Hamiltonian and the momentum

$$P=1/2\int u^2dxdr. \qquad (1.143)$$

Soliton solutions of the equation (1.139) have the form

$$u=u_s(x-Vt,r) \qquad (1.144)$$

with the boundary condition $u_s\to0$ in all directions at infinity, $\sqrt{x^2+r^2}\to\infty$, and requiring finite momentum $P<\infty$.

Solitons are solutions of the stationary KP equation

$$\hat{R}w=\left[\left(V+\left|\frac{\partial}{\partial x}\right|^{2s}\right)\frac{\partial^2}{\partial x^2}+\frac{\alpha}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}\right]w=\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right), \qquad (1.145)$$

which can be presented as the variational problem (1.129) where H and P are given by (1.141) and (1.143), respectively. Soliton solutions exist only for $\alpha=+1$ and positive V. For all other cases the operator $\hat{R}$ is not sign definite and it cannot provide a vanishing soliton solution at $\sqrt{x^2+r^2}\to\infty$ (for more details see [67, 72, 68, 13]).

Multiplying now (1.145) by $w/2$ and integrating with respect to x and r, one obtain

$$-VP-T-W+\frac{3}{2}U=0. \qquad (1.146)$$

Let us take a trial function for the variational problem (1.129) with H and P given by (1.141) and (1.143) in the form, retaining the total momentum P,

$$u(x,r,a,b)=a^{-1/2}b^{-d/2}u_s\left(x/a,r/b\right). \qquad (1.147)$$

As a result, the Hamiltonian becomes the function of two scaling parameters a and b

$$H = \frac{T}{a^{2s}} + \frac{a^2}{b^2}W - \frac{U}{a^{1/2}b^{d/2}}. \tag{1.148}$$

By inserting (1.147) into (1.129) we get

$$\left.\frac{\partial H}{\partial a}\right|_{a=b=1} = 0, \quad \left.\frac{\partial H}{\partial b}\right|_{a=b=1} = 0. \tag{1.149}$$

This yields

$$-2sT + 2W + \frac{1}{2}U = 0, \quad -2W + \frac{d}{2}U = 0. \tag{1.150}$$

Solving now the linear system (1.146) and (1.150) one can get

$$\begin{aligned} T &= \frac{d+1}{6s-1-d(1+s)}VP_s,\ U = \frac{4s}{6s-1-d(1+s)}VP_s \\ W &= \frac{sd}{6s-1-d(1+s)}VP_s,\ \ H_s = \frac{d(1+s)+1-4s}{6s-1-d(1+s)}VP_s. \end{aligned} \tag{1.151}$$

These formulas become identical to (1.132) at $d = 0$. From (1.151) one can see that on the soliton solutions T and W must have the same (positive) sign. Hence it follows that multidimensional solitons exist only if $\alpha > 0$. In other words, multidimensional solitons exist only for the KPI equation and its generalization. This conclusion ($\alpha > 0, V > 0$) corresponds completely to the requirement of sign-definiteness of the operator R in the equation (1.145). In the following we shall assume $\alpha = 1$.

From relations (1.151), we get the necessary conditions for existence of solitons

$$s > \frac{1+d}{6-d}. \tag{1.152}$$

The sufficient condition for soliton stability is again $H_s < 0$, implying

$$s > \frac{d+1}{4-d}. \tag{1.153}$$

In the interval

$$\frac{1+d}{6-d} < s \leq \frac{d+1}{4-d}, \tag{1.154}$$

solitons are unstable.

Let us consider now the most important physical examples of equation (1.139). For $d = 1$ and $s = 1$ (1.139) is nothing more than the classical (2D) KPI equation (1.112) (where it is necessary to change $u \to -u$ and $t \to -t$). We see that the condition (1.153) is satisfied now, and the soliton is stable [69]. The soliton in this case has the form of a lump and can be found analytically [70]. Another example arises if $s = 1$ and $d = 2$. In this case (1.139) is the KPI equation for a three-dimensional media. Now the criterion (1.154) is satisfied and the soliton exists but is unstable. This fact was established in the paper [69]. Development of the soliton instability results into collapse of acoustic waves confirmed by numerical experiments [71, 72].

Consider the GKPI equation in the three-dimensional case with power nonlinearity

$$\frac{\partial}{\partial x}[u_t + \gamma(\gamma-1)u^{\gamma-2}u_x] = \Delta_\perp u \tag{1.155}$$

for which the Hamiltonian is of the form

$$H = \frac{1}{2}\int (u_x)^2 d\boldsymbol{r} + \frac{1}{2}\int (\nabla_\perp w)^2 d\boldsymbol{r} - \int u^\gamma d\boldsymbol{r}. \tag{1.156}$$

This equation generalizes the KDV equation (1.135) with power nonlinearity to many dimensions. In particular, the classical KPI equation corresponds to $\gamma = 3$. The case $\gamma = 4$ is possible if for some physical reason the three-wave matrix element vanishes. For instance, such a situation takes place for special angles of propagation of acoustic-type waves in a ferromagnet [75]. In this case, as it was shown at first in this paper it is possible to write down the analog of the virial theorem.

Consider the quantity $I = \int r_\perp^2 u^2 d\boldsymbol{r}$ which, because of conservation of the x component of the momentum, $P_x = \frac{1}{2}\int u^2 d\boldsymbol{r}$, has the meaning of mean transverse size of the wave distribution. Let us find the first derivative of I with respect to time. By means of (1.155) we have

$$I_t = -4\int u(\boldsymbol{r}_\perp \nabla_\perp) w d\boldsymbol{r}.$$

Calculating now the second derivative of I one can get

$$I_{tt} = 4\left[2\int (\nabla_\perp w)^2 d\boldsymbol{r} - d(\gamma-2)\int u^\gamma d\boldsymbol{r}\right].$$

By use of (1.156) the r. h. s. of this equation can be rewritten

$$I_{tt} = 4\left[4H - 2\int (u_x)^2 d\boldsymbol{r} + \beta\int u^\gamma d\boldsymbol{r}\right], \tag{1.157}$$

where $\beta = 4 - d(\gamma-2)$. At $d = 2$ (the 3D case) and $\gamma = 4$ the coefficient $\beta = 0$. In this case from the equation (1.157) one can get the following inequality [75]

$$I_{tt} < 16H. \tag{1.158}$$

Hence we have the same sufficient condition $H < 0$, as for the NLSE. For the classical 3D KPI equation ($d = 2$, $\gamma = 3$) the coefficient $\beta = 4 - 2(\gamma-2) > 0$ and in the virial identity (1.157) the two last terms have different signs and therefore, even for $H < 0$, it is difficult to get a certain answer about the sign of the r. h. s. of (1.157) and that is so, despite the unboundedness of the Hamiltonian from below. But if the Hamiltonian of some region Ω is negative, then, following to the arguments analogous to section 3, it is possible to show that radiation of waves from this area promotes collapse. Radiation reduces the Hamiltonian of the cavity Ω so that H_Ω becomes more negative. Simultaneously, due to the unboundedness of the Hamiltonian, the maximal value of the wave amplitude into the cavity will increase, and this process continues up to the singularity formation [72]. At the moment there are no analytical arguments whether the collapse time (for $d = 2$ and $\gamma = 3$) is finite or infinite. Meanwhile, the numerical experiments performed in [71, 72] indicate that this time is finite.

One more physical example is $s = 1/2$, $d = 1$. We have now the following equation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial^2}{\partial x^2}\hat{H}u\right) = \frac{\partial^2 u}{\partial y^2}. \tag{1.159}$$

This equation describes two-dimensional Tolman-Schlichting waves in the laminar boundary layer. In this case again the condition (1.154) is fulfilled and 2D solitons are unstable.

In the case of soliton instability

$$s < \frac{d+1}{4-d} \tag{1.160}$$

the equation (1.139) describes weak collapse. The corresponding self-similar solution is

$$u = (t_0 - t)^{-2s/(2s+1)} F\left(\frac{x}{(t_0 - t)^{1/(2s+1)}}, \frac{r}{(t_0 - t)^{(s+1)/(2s+1)}}\right). \tag{1.161}$$

Collapse leads to the formation of an integrable singularity for $t \to t_0$

$$u = \frac{1}{x^{2s}}\varphi\left(\frac{r}{x^{s+1}}\right). \tag{1.162}$$

More detailed structures of collapses as well as the role of dissipation in arresting collapse and the formation of black holes have not properly been studied so far.

11 Self-focusing in the boundary layer

Now we demonstrate how the tools considered above work for the two-dimensional model,

$$u_t = \frac{\partial}{\partial x}\hat{k}u - 6uu_x = \frac{\partial}{\partial x}\frac{\delta H}{\delta u} \tag{1.163}$$

where the Hamiltonian becomes

$$2H = \int \left(\frac{1}{2} u\hat{k}u - u^3 \right) d\boldsymbol{r} \equiv \frac{1}{2} I_1 - I_2.$$

Here $\hat{k}$ is the integral operator, its Fourier transform is the modulus $|\boldsymbol{k}| = (k_x^2 + k_y^2)^{1/2}$. This equation describes low-frequency oscillations of the boundary layer within the high Reynolds number, $Re \gg 1$, with the mean velocity profile $\boldsymbol{v} = \hat{x}\ U(z)$ $(0 \leq z < \infty)$. The function $U(z)$ is assumed to be a monotonically growing function with a constant value at the infinity. The dimensionless amplitude u is connected with the velocity fluctuations along mean flow by means of the relation

$$\delta v_x \approx -6huU'(z), \tag{1.164}$$

where $h = U(0)/U'(0)$ is a thickness of the boundary layer.

Equation (1.163) was derived first by V.I. Shrira [77]. It represents the two-dimensional generalization of the well-known Benjamin-Ono (BO) equation describing long waves in stratified liquids. One should note that for this problem this equation was also derived in the 1D case first in the papers [81], taking into account the small viscosity.

The simplest soliton in this model are of the form $u = u_s(x - Vt, y)$. Their shape is defined from the equation

$$-Vu_s - \hat{k}u_s + 3u_s^2 = 0. \tag{1.165}$$

For the 1D case the solution of this equation can be found explicitly

$$u_s = \frac{2V}{3(x^2V^2 + 1)} \quad (V > 0). \tag{1.166}$$

In the 2D case the model has a ground state soliton that is a cylindrically symmetric solution without nodes. Such a solution was found in [82] numerically.

It is very important to note that the velocities of the 2D ground state solitons as well as their amplitudes are positive quantities. In physical variables both the 1D and 2D solitons, upon applying the relation (1.164), move in the upstream direction and have negative amplitudes. The latter means that in the real hydrodynamic system solitons look like holes in the mean velocity profile, therefore they move slower than the main flow. When the soliton amplitude grows, the soliton velocity decreases and *vice versa.* This physical reasoning suggests a possibility for the appearance of the wave collapse in this system and the instability of 1D solitons with respect to two-dimensional perturbations as well (for details, see [78, 79]). This instability is analogous to the Kadomtsev-Petviashvili instability [46, 47]. In the two-dimensional case, as was shown in [80], it is possible to develop a quasi-classical nonlinear theory of this instability taking a solution in the form of the 1D soliton (1.166) with slowly varying parameters depending on y and t.

The soliton (1.163) in this model, as many others, represents a stationary point of H for fixed x-projection of the momentum $P = 1/2 \int u^2 d\boldsymbol{r}$

$$\frac{\delta}{\delta u}(H + VP_x) = 0.$$

A minimum of H (for fixed P) is found in the one-dimensional case. It follows from the estimates analogous to (1.56)

$$\int u^3 d\boldsymbol{r} \le C \left(\int u\hat{k}u d\boldsymbol{r} \right)^{D/2} \left(\int u^2 d\boldsymbol{r} \right)^{(3-D)/2},$$

with the best constant C attaining its value at the ground state soliton

$$C = I_{2s} I_{1s}^{-D/2} \left(2P_s\right)^{(D-3)/2}.$$

Hence it is easy to get the estimate

$$H \ge H_s + 1/2(I_1^{1/2} - I_{1s}^{1/2})^2, \tag{1.167}$$

which becomes precise on the 1D soliton. Thus, in the one-dimensional case the soliton is proved to be stable with respect to 1D perturbations, but optionally against small ones [83, 78, 79].

In 2D case this system demonstrates the critical behavior like the 2D cubic NLSE. In particular, the Hamiltonian is bounded from below by zero,

$$H \ge \frac{1}{2}\left[1 - \left(\frac{P}{P_s}\right)^{1/2}\right] \int u\hat{k}u d\boldsymbol{r},$$

if the total perturbation power does not exceed the critical value equal to P_s. If initially the Hamiltonian is negative, $H < 0$, then it is unbounded from below. The latter follows from the scaling transformation, retaining P,

$$u_s(\boldsymbol{r}) \to \frac{1}{a^{d/2}} u_s(\boldsymbol{r}/a). \tag{1.168}$$

Under these transformations H becomes a function of the scaling parameter a,

$$H(a) = \frac{I_{1s}}{2a} - \frac{I_{2s}}{a^{d/2}}, \tag{1.169}$$

and is unbounded from below as $a \to 0$, starting from $d \ge 2$. It is enough to state that in this case the formation of a singularity is possible due to small amplitude waves radiated from the region with negative Hamiltonian. In this case the inequality corresponding to (1.63) is of the form

$$\max_{x\in\Omega} u \leq \frac{|H_\Omega|}{2P_\Omega}. \tag{1.170}$$

Numerical integration of the equation (1.163) confirmed the main theoretical predictions.

For all initial conditions with $P > P_s$ and $H < 0$, the significant growth of amplitude was observed at the peak moving with increasing acceleration. The temporal behavior of the peak velocity and of the peak amplitude are familiar, indicating that the collapse is of self-similar nature. Upon approaching the singularity the peak anisotropy vanishes, and the peak distribution becomes nearly symmetric.

For the initial conditions with $P < P_s$ $(H > 0)$ a slow evolution took place: the distribution of u slowly decayed near the maximum. The spectrum evolution for $P_x < P_{x,cr}$ demonstrated the energy transfer to the long-wave region, which on a qualitative level is in agreement with the estimate (1.170).

In conclusion of this section, we would like to point out several interesting experiments [76], summarizing the results of many years of experimental studies of the onset of the coherent structures in the boundary layer of the blowing plate by the mechanical vibrating system near the edge of the plate (see, also [84], [85]). According to these experimental data, one-dimensional solitons are exited at the initial stage, later (for larger distances from the plate edge) one-dimensional solitons demonstrate their instability which results "in the formation of thorns", i.e., the localized three-dimensional coherent structures. Self-focusing of the above structures is observed at longer distances. A later stage of the development of the thorns-solitons leads to the formation of vortices and to their eventual separation.

The above theory and numerical experiments as well explain all these experimental observations, but not the formation of vortices, for which equation (1.163) becomes inapplicable. The threshold character of the wave collapse in the boundary layer described by Eq. (1.163) also explains why in many other experimental studies in the boundary layer such bright phenomena as self-focusing of solitons and collapse have not been observed or have not been distinguished on the background of the turbulent noise. The collapse is possible to observe starting from the finite energy of the pulse as it was in experiments [76]. If the pulse amplitude is small enough then this phenomena is absent.

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