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Cascade generation of zonal flows by the drift wave turbulence

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ABSTRACT

The purpose of this research is to investigate the formation of zonal flows that can lead to the enhanced confinement of plasma in tokamaks. We show that zonal flows can be effectively formed by resonance triad interactions in the process of the inverse cascade. We discuss what energy sources are more effective for the formation of zonal flows.

1. Introduction

Zonal flows can serve as the internal transport barriers in tokamaks [1–3]. It has been suggested (see [4–12] and references cited therein) that zonal flows appear due to the interaction of drift waves. The nonlinear dynamics of these waves is often modeled by the Hasegawa–Mima equation [13]

$$\frac{\partial}{\partial t}(\Delta\psi - \psi) - \beta\frac{\partial\psi}{\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} = 0. \tag{1.1}$$

Here $\psi(x, y, t)$ is the electrostatic potential; x is the radial and y is the poloidal (or zonal) coordinates (the slab geometry is assumed); β is a parameter.[1]

The Hasegawa–Mima equation (1.1) describes the nonlinear dynamics of plasma drift waves with the dispersion law

$$\Omega(\mathbf{k}) = \frac{\beta p}{1+k^2} \quad \left[\mathbf{k} = (p, q),\ k^2 = p^2 + q^2\right], \tag{1.2}$$

p and q are zonal (poloidal) and radial wave numbers.[2] The formation of zonal flows corresponds in the Fourier space to the energy accumulation near the q-axis. The main interactions of the drift waves (1.2) are the triad resonances

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3, \qquad \Omega_{\mathbf{k}_1} = \Omega_{\mathbf{k}_2} + \Omega_{\mathbf{k}_3}. \tag{1.3}$$

However, the triad interaction coefficient vanishes if one of the waves in the triad (1.3) is a purely zonal mode (say, $p_3 = 0$). This circumstance lead researchers to account for slightly non-resonance interactions [12]. It was also suggested [4,7,8,10,15] that zonal flows appear as a result of the modulational instability. It corresponds to the 4-wave process, involving two almost resonance triad interactions connected through a virtual wave $(\mathbf{k}, \omega)$ whose frequency ω slightly differs from $\Omega(\mathbf{k})$. In a broader context zonal flows can be viewed as large scale convective cells [7,16].

The goal of the present Letter is to analytically describe the process of formation of zonal jets due to the triad interactions that are exactly in resonance. Herewith the fact that the triad interaction coefficient vanishes (when one of the waves in the triad is a purely zonal mode) is overcome in the following way. As energy is transferred towards the origin via the inverse cascade [5],

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[1] The parameter β characterizes various gradients of physical quantities, e.g. of density, temperature, or magnetic field. Various examples of plasma drift waves are listed in [14]. The units are chosen so that the effective ion giroradius ρ_s is 1.

[2] Note that p and q are the Fourier variables for y and x respectively.

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the maximum of the energy spectrum *increases*, and this increase can *compensate* the smallness of the triad interaction coefficient. By considering scaling solutions, we show that this compensation does happen.

We assume the magnitude of the field ψ to be sufficiently small, so that the nonlinear evolution can be viewed as the wave dynamics. The turbulence of the drift waves is described by the wave kinetic equation

$$\frac{\partial F_1}{\partial t} = 2\pi \int \frac{W_{23}(W_{23}F_2F_3 + W_{31}F_3F_1 + W_{12}F_1F_2)}{(1+k_1^2)(1+k_2^2)(1+k_3^2)} \times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\Omega_1+\Omega_2+\Omega_3)\,d\mathbf{k}_2\,d\mathbf{k}_3 \tag{1.4}$$

for the energy spectrum $F(\mathbf{k}, t)$ of the drift wave turbulence.[3] Here and throughout the Letter, when considering functions depending on different wave vectors $\mathbf{k}_j$ $(j = 1, 2, 3)$, we keep only their labels j. The functions W result from the triad interaction and are given by the formula

$$W_{12} = \left(k_1^2 - k_2^2\right)(p_2q_3 - p_3q_2); \tag{1.5}$$

similar expressions hold for W_{23} and W_{31}. The wave kinetic equation (1.4) was derived in [17–19]; a different (but equivalent) form was considered in [20] on the basis of the canonical Hamiltonian variables [21]; the latter form makes it evident that (1.4) is a particular case of the kinetic equation for bosons in the high concentration limit.

To have an analytically tractable problem, we assume the zonal wave number to be much smaller than the radial wave number (i.e. $|p| \ll |q|$); and the waves are either short ($k \gg 1$) or long ($k \ll 1$). Then we can consider similarity solutions

$$F(p, q, t) = t^m G\left(t^a p, t^b q\right) \tag{1.6}$$

of the wave kinetic equation (1.4). The similarity solution (1.6) is determined by the exponents a, b, m and by the "shape function" G, which is a function of two scalar variables.

The formation of zonal flow at $t \to \infty$ would mean that $a > b, m > 0$. And this is what we find.

2. Short waves

When $|p| \ll |q|$ and $k \gg 1$, the wave kinetic equation (1.4) takes the form

$$\frac{\partial F_1}{\partial t} = 2\pi \int \frac{W_{23}(W_{23}F_2F_3 + W_{31}F_3F_1 + W_{12}F_1F_2)}{q_1^2q_2^2q_3^2} \times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\Omega_1+\Omega_2+\Omega_3)\,d\mathbf{k}_2\,d\mathbf{k}_3, \tag{2.1}$$

$$\text{with } \Omega = \frac{\beta p}{q^2}, \quad W_{12} = \left(q_1^2 - q_2^2\right)(p_2q_3 - p_3q_2). \tag{2.2}$$

This equation admits scaling in both p and q, and we look for the similarity solution (1.6). The function (1.6) indeed satisfies (2.1) if

$$m - 2a - 3b = -1, \tag{2.3}$$

while the "shape" function $G(p, q)$ should satisfy the following integral-differential equation

$$mG_1 + a\,p_1\frac{\partial G_1}{\partial p_1} + bq_1\frac{\partial G_1}{\partial q_1} = 2\pi \int \frac{W_{23}}{q_1^2q_2^2q_3^2}(W_{23}G_2G_3 + W_{31}G_3G_1 + W_{12}G_1G_2) \times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\Omega_1+\Omega_2+\Omega_3)\,d\mathbf{k}_2\,d\mathbf{k}_3. \tag{2.4}$$

(In accordance with our notations, $G_j = G(p_j, q_j)$, $j = 1, 2, 3$.) The "shape" equation (2.4) by itself does not determine the similarity solution: We need to specify the exponents a, b, m.

One equation comes from the energy conservation

$$\int F(p, q, t)\,dp\,dq = \text{Const} \quad \Rightarrow \quad m - a - b = 0. \tag{2.5}$$

We cannot use the enstrophy conservation to find one more equation for the exponents a, b, m, since the enstrophy follows the direct cascade and is dissipated at small scales.

However, the kinetic equation additionally conserves the integral [22]

$$I_{\text{short}} = \int \frac{p^2}{q^6} F(p, q, t)\,dp\,dq. \tag{2.6}$$

The conservation of this quantity gives

$$m - 3a + 5b = 0. \tag{2.7}$$

From (2.3), (2.5), and (2.7), we find

$$a = \frac{3}{5}, b = \frac{1}{5}, m = \frac{4}{5} \quad \Rightarrow \quad F(p, q, t) = t^{\frac{4}{5}} G\left(t^{\frac{3}{5}} p, t^{\frac{1}{5}} q\right). \tag{2.8}$$

It is also possible to consider a case when there is the energy source of constant intensity and conservation of the extra invariant (2.6):

$$\int F(p, q, t)\,dp\,dq \propto t \quad \Rightarrow \quad m - a - b = 1, \tag{2.9}$$

$$\int \frac{p^2}{q^6} F(p, q, t)\,dp\,dq = \text{Const} \quad \Rightarrow \quad m - 3a + 5b = 0. \tag{2.10}$$

Then

$$a = \frac{7}{5}, b = \frac{3}{10}, m = \frac{27}{10} \quad \Rightarrow \quad F(p, q, t) = t^{\frac{27}{10}} G\left(t^{\frac{7}{5}} p, t^{\frac{3}{10}} q\right). \tag{2.11}$$

In a general scaling case, when there are sources of the energy and the extra invariant (2.6) that pump these conserved quantities with arbitrary power-law intensities

$$\text{the energy source} \quad \int F(p, q, t)\,dp\,dq \propto t^{\mu}, \tag{2.12}$$

$$\text{the extra invariant source} \quad \int \frac{p^2}{q^6} F(p, q, t)\,dp\,dq \propto t^{\nu}, \tag{2.13}$$

we have

$$m - a - b = \mu, \qquad m - 3a + 5b = \nu. \tag{2.14}$$

From (2.3) and (2.14)

$$a = \frac{3 + 4\mu - \nu}{5}, \qquad b = \frac{2 + \mu + \nu}{10}, \qquad m = \frac{8 + 19\mu - \nu}{10}. \tag{2.15}$$

The case (2.8) with the conservation of the energy and the extra invariant corresponds to $\mu = \nu = 0$.

The case (2.11) with the energy source of constant intensity and the conservation of the extra invariant corresponds to $\mu = 1, \nu = 0$.

In both cases (2.8) and (2.11) — since $a > b$ — the inverse cascade leads to the formation of zonal flow. But in the second case, the zonal flow formation is more pronounced. From the consideration of similarity solutions for the sea wave turbulence [23], we have learned that the exponents μ, ν can take real values in some intervals; in particular, μ can even exceed 1.

[3] We assume that the energy spectrum is broad enough so that the drift waves can be viewed as incoherent, and the kinetic equation (1.4) is applicable.

We believe that similar consideration would work for the general kinetic equation (1.4) with arbitrary p,q (not necessarily, $|p| \ll |q|$). In general, Eq. (1.4) additionally conserves the integral [24]

$$I = \frac{1}{2}\int \frac{\eta_{\mathbf{k}}}{\Omega_{\mathbf{k}}} F_{\mathbf{k}}(t)\,d\mathbf{k}, \quad \text{where}$$

$$\eta_{\mathbf{k}} = \arctan\frac{q+p\sqrt{3}}{p^2+q^2} - \arctan\frac{q-p\sqrt{3}}{p^2+q^2} - 2\sqrt{3}\Omega(\mathbf{k}). \tag{2.16}$$

However, in the general situation, instead of considering the similarity solutions, we would need to resort to numerical simulations.

3. Long waves

In the long wave limit, when $|p| \ll |q| \ll 1$,

$$\Omega(\mathbf{k}) = \beta p\big(1-q^2\big), \qquad \eta_{\mathbf{k}} = \frac{p^3}{q^2}, \tag{3.1}$$

and the wave kinetic equation (1.4) takes the form

$$\frac{\partial F_1}{\partial t} = 2\pi\int W_{23}(W_{23}F_2F_3 + W_{31}F_3F_1 + W_{12}F_1F_2) \times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\Omega_1+\Omega_2+\Omega_3)\,d\mathbf{k}_2\,d\mathbf{k}_3,$$
$$\text{with } W_{12} = \big(q_1^2-q_2^2\big)(p_2q_3-p_3q_2). \tag{3.2}$$

When we substitute dispersion law $\Omega(\mathbf{k})$ from (3.1) into (3.2), the linear part drops out, and the frequency delta function takes the form

$$\delta(\Omega_1+\Omega_2+\Omega_3) = \delta\big(\beta p_1q_1^2 + \beta p_2q_2^2 + \beta p_3q_3^2\big).$$

Eq. (3.2) has similarity solution (1.6) if

$$m - 2a - 5b = -1, \tag{3.3}$$

while the "shape" function $G(p,q)$ should satisfy the integral-differential equation

$$mG_1 + ap_1\frac{\partial G_1}{\partial p_1} + bq_1\frac{\partial G_1}{\partial q_1} = 2\pi\int W_{23}(W_{23}G_2G_3 + W_{31}G_3G_1 + W_{12}G_1G_2) \times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\Omega_1+\Omega_2+\Omega_3)\,d\mathbf{k}_2\,d\mathbf{k}_3. \tag{3.4}$$

Assuming again the presence of pumping

$$\text{the energy source} \quad \int F(p,q,t)\,dp\,dq \propto t^{\mu}, \tag{3.5}$$

$$\text{the extra invariant source} \quad \int \frac{p^2}{q^2}F(p,q,t)\,dp\,dq \propto t^{\nu} \tag{3.6}$$

(cf. (2.12)–(2.13)), we have

$$m - a - b = \mu, \qquad m - 3a + b = \nu. \tag{3.7}$$

From (3.3) and (3.7)

$$a = \frac{1+3\mu-2\nu}{5}, \qquad b = \frac{2+\mu+\nu}{10}, \qquad m = \frac{4+17\mu-3\nu}{10}. \tag{3.8}$$

Again, the following two particular cases are of special interest.

1. No sources, i.e. conservation of the energy and the extra invariant: $\mu=\nu=0$ (cf. (2.8))

$$a = b = \frac{1}{5}, m = \frac{2}{5} \quad\Rightarrow\quad F(p,q,t) = t^{\frac{2}{5}}G\big(t^{\frac{1}{5}}p, t^{\frac{1}{5}}q\big). \tag{3.9}$$

In this case — since $a=b$ — the similarity solution does not show any tendency towards zonal flow. From the extra conservation of the integral (2.16) we know [25] that the energy concentrates in the 30° angle around zonal flow: $|p/q| < 1/\sqrt{3}$. In other words, the polar angle $\theta = \arctan(q/p)$ (in absolute value) exceeds 60°.[4]

2. The energy source of constant intensity and conservation of the extra invariant: $\mu=1, \nu=0$ (cf. (2.11))

$$a = \frac{4}{5}, b = \frac{3}{10}, m = \frac{21}{10} \quad\Rightarrow\quad F(p,q,t) = t^{\frac{21}{10}}G\big(t^{\frac{4}{5}}p, t^{\frac{3}{10}}q\big). \tag{3.10}$$

In this case, we do have generation of zonal flows — since $a > b$. This zonal flow generation is not implied by the balance argument [25], but it follows from the consideration of the similarity solution because the latter takes into account the *nonlinear coefficient* of the triad interaction.

4. Conclusion

We have seen that the inverse cascade can lead to the formation of zonal jets (at least in the model cases considered in this Letter). To see how this cascade generation works in general (without similarity), it is necessary to use numerical simulations.

It is hard to compare the presented cascade generation with the modulational instability: It is not clear what regime this instability will saturate to. Furthermore, which mechanism is the most effective in generation of zonal flows can depend on the *initial* field of small scale drift waves [15,27].

We have seen that the zonal jets can be generated more effectively if there is the energy source, and moreover, it becomes stronger over time ($\mu > 1$ in the similarity situations). Herewith, there should be no source of the extra invariant (2.16). It is possible to achieve such situation if the growth rate is positive in some domain of the $\mathbf{k}$-space and negative in some other domain (in other words, we should have instability in one place and decrement in another place of the $\mathbf{k}$-plane).

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