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About shape of giant breather

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ABSTRACT

The pulse of freak waves on the surface of deep water can be a breather-type solution of the Euler equation. The shape of surface is periodic function of time in a moving frame. Only in the limit of very small steepness its shape is described by the Nonlinear Shredinger Equation (NLSE). For moderately small steepness we derived more exact envelope nonlocal equation similar to well-known Dysthe equation (DE), which is more convenient for analytical and numerical study. We have found approximate solution of this equation by the use of the variational approach.

1. Introduction

In our previous paper Dyachenko and Zakharov [1] we have shown numerically that the localized periodic in time high-amplitude breather, or giant breather could propagate along the surface of deep fluid without losing energy by radiation for a very long time (many thousands of periods). These breathers can be used for explanation of the freak wave phenomenon. Our numerical experiments show that the breathers can be identified with the NLSE-solitons only for very small values of steepness $\mu = \max|\eta_x| \leq 0.035$. Meanwhile breather do exist up $\mu \simeq 0.4$ at least. How to describe analytically the shape of breather in the "grey zone", $0.035 < \mu < 0.4$ that covers whole decade of steepness value? This is a challenging and very hard problem. The first step in this direction is derivation of more accurate (that NLSE) envelope equation which includes higher-order terms in expansion by power of μ, at least one extra term is necessary. First version of such equation was offered by K. Dysthe in 1979 Dysthe [2].

The Dysthe equation was actively solved numerically Akylas [3,4]; Clamond and Grue [5] and breather-type solution were observed. However, nobody tried to solve the Dysthe equation analytically and to determine the shape of giant breather. We do this in the present article.

There are several different versions of the Dysthe equation. Originally it was written for envelope of surface elevation. This choice of variable is very natural from the view-point of physics, but not optimal from mathematical view-point. Original Dysthe equations are not Hamiltonian, they do not admit any variational approach. In our next article we will show that this weak point can be fixed by a proper renormalization of variables. So far we derive a version of the Dysthe equation in the framework "Dyachenko equations" arising after conformal mapping of the domain filled with fluid to the lower half-plane. A new version of the Dysthe equation is non-Hamiltonian also, but it admits variational approach for construction of solitonic solution. We have implemented this approach using the most simple probe function – shape of NLSE soliton with indefinite amplitude. Even this primitive approach leads to fundamental conclusion – deviations from the NLSE are very important! They completely reshape the shape of breathers in the "grey zone" of steepness. Our analytical results are confirmed by comparison with the numerical data.

2. Conformal Dysthe equation

We assume that the fluid fills the area

$$-\infty < y < \eta(x,t)\,[-\infty < x < \infty].$$

The velocity field is potential, hence

$$V = \nabla\phi, \quad \nabla V = 0. \tag{2.1}$$

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Hydrodynamic potential $\phi(x,y,t)$ and shape of the surface $\eta(x,t)$ satisfy the equations

$$\begin{aligned}\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\phi_x^2+\phi_y^2\right)+g\eta &= -\frac{P}{\rho} \quad \text{at } y=\eta,\\ \frac{\partial\eta}{\partial t}+\eta_x\phi_x &= \phi_y \quad \text{at } y=\eta.\end{aligned} \tag{2.2}$$

We perform the conformal transformation to map the domain filled with fluid on Z-plane ($Z=x+iy$) to the lower half of the mathematical plane W, ($v<0$, $w=u+iv$).

The transformation is realized by function $Z(w,t)$. Potential ϕ is a harmonic function ($\Delta\phi=0$). Together with the stream function it is transformed to the complex velocity potential $\Phi(w,t)$.

Omitting the details equations (2.2) transform to the following ones:

$$Z_t = iUZ_u,\quad \Phi_t = iU\Phi_u - B + ig(Z-u). \tag{2.3}$$

Here U and B are

$$\begin{aligned}U &= \widehat{P}(V\overline{R}+\overline{V}R)\\ B &= \widehat{P}(V\overline{V}).\end{aligned} \tag{2.4}$$

Here $\overline{V}$ and $\overline{R}$ mean complex conjugate of V and R.

In (2.4) $\widehat{P}$ is the projector operator generating a function that is analytical in a lower half-plane

$$\widehat{P}(f) = \frac{1}{2}\left(1+i\widehat{H}\right)f,$$

where $\widehat{H}$ – is the Hilbert transformation.

Then we introduce standard "Dyachenko variables" Dyachenko [6]

$$R = \frac{1}{Z_w},\quad \text{and } V = i\Phi_z = i\frac{\Phi_w}{Z_w} \tag{2.5}$$

In new variables the Euler equations read

$$R_t = i(UR_w - RU_w),\quad V_t = i(UV_w - RB_w) + g(R-1). \tag{2.6}$$

We must stress that

$$R\to 1,\quad V\to 0,\ \text{at } v\to -\infty.$$

R and V are periodic functions of u (or vanishing at $u\to\pm\infty$). All four functions R, V, U and B are analytic ones in the lower half-plane $v<0$. R has no zeros at $v<0$.

Consider evolution of weakly nonlinear wave train. We will use r instead of R

$$r = R-1.$$

Then equations (2.6) transform into

$$\begin{aligned}r_t + iV' &= i\left(-U' + Vr' - V'r + Ur' - rU'\right),\\ V_t - gr &= i\left(VV' - B' + UV' - rB'\right).\end{aligned} \tag{2.7}$$

(Here and below prime $'$ means derivative with respect to the conformal space variable u).

Now complex transport velocity U

$$U = \widehat{P}(V\overline{r}+\overline{V}r). \tag{2.8}$$

We will look for the breather solution. It is periodic in some reference frame moving with velocity c. In this reference frame equations (refr-V) read

$$\begin{aligned}r_t - cr' + iV' &= i\left(-U' + Vr' - V'r + Ur' - rU'\right) = F,\\ V_t - cV' - gr &= i\left(VV' - B' + UV' - rB'\right) = G.\end{aligned}$$

We look for the solution of these equations in the following form

$$r = \sum_{n=0}^{\infty} r_n(u,t)e^{in(\Omega t-ku)},\ k>0$$

$$V = \sum_{n=0}^{\infty} V_n(u,t)e^{in(\Omega t-ku)}. \tag{2.9}$$

Thereafter we will put $k=1$, $c=½\ \Omega=½$. The leading terms in expansion (2.9) are

$$r_1,\quad V_1\sim\epsilon\ll 1.$$

Then

$$r_n\sim V_n\sim\epsilon^n,\quad r_0\sim V_0\sim\epsilon^3. \tag{2.10}$$

The idea of expansion (2.9) is the following: r_n, V_n are "slow" functions of u. In other words

$$\frac{r_n'}{r_n}\sim\frac{V_n'}{V_n}\sim\epsilon\ll 1. \tag{2.11}$$

Slow dependence on u does not guarantee slow dependence on time. Strictly speaking, they should be presented as follow

$$r_n = \sum_{m=-\infty}^{\infty} r_n^m e^{im\Omega t},\ V_n = \sum_{m=-\infty}^{\infty} V_n^m e^{im\Omega t}.$$

However, one can show that the "fast" components in r_n and V_n are exponentially small $r_n^m\simeq e^{-m/\epsilon}\sim V_n^m$ and can be neglected. Only "slow" components ($m=0$) survive if $\epsilon\to 0$. For the slow components

$$\frac{\dot r_n}{r_n}\sim\frac{\dot V_n}{V_n}\sim\epsilon^2\ll 1. \tag{2.12}$$

Here $\dot r_n$ and $\dot V_n$ mean time-derivatives of r_n and V_n.

To proceed in derivation of envelope equation we have to learn how to calculate projective operator of functions like $a(u)e^{imu}$. Here $a(u)$ – any "slow" function of u. "Slowness" of $a(u)$ means that its singularities are posed on the distance of the order of $1/k\mu$, may be in both half-planes. Let us assume for the simplicity that $a(u)$ is rational. Projection on the lower half-plane means that we eliminate all poles of the function $a(u)e^{imu}$ in the lower half-plane, and keep untouched all poles in the upper half-plane. Let $m<0$. Now all residues in lower half-plane are exponentially small of the order of $e^{-m/\epsilon}$ and can be neglected. In this case projection is just a cosmetic operation. If $m>0$, it is much more serious surgery, but surviving poles in the upper half-plane have exponentially small residues, which can be neglected. We ended up with the following rule for calculation of projectors

$$\widehat{P}\left(e^{ikm}a(u)\right) = \begin{cases}0, & \text{if } m>0,\\ e^{ikm}a(u), & \text{if } m<0.\end{cases} \tag{2.13}$$

Only if $m=0$, projection is a nontrivial operation.

Thereafter we put

$$V_1 = \epsilon\psi$$

and replace

$$\frac{\partial}{\partial u}\rightarrow \epsilon\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial t}\rightarrow \epsilon^2\frac{\partial}{\partial t}. \tag{2.14}$$

Using the rule (2.13) we find with accuracy up to ϵ^3

$$V_2 = \epsilon^2\Big(-i\psi^2+\frac{\epsilon}{2}\psi\psi'\Big), \quad r_2 = \epsilon^2\Big(\psi^2+i\psi\psi'\Big),$$
$$r_0 = i\epsilon^3\widehat{P}\Big(|\psi|^2\Big)', \quad V_0 = \epsilon^2\widehat{P}\Big(|\psi|^2\Big)' \tag{2.15}$$

r_1 and V_1 are related with relation

$$r_1 = V_1 - \frac{\epsilon}{2}V_1' \tag{2.16}$$

By the use of the relations (2.15) and (2.16) we end up with the following equation for ψ:

$$2i\dot{\psi}+\frac{1}{4}\psi''+|\psi|^2\psi = \epsilon\Big[\dot{\psi}'-\psi\widehat{H}\Big(|\psi|^2\Big)'-2i\Big(|\psi|^2\psi\Big)'\Big] \tag{2.17}$$

This is the Dysthe equation in our variables. In the limit of $\epsilon\rightarrow 0$ it gives standard NLSE.

3. Shape of breathers

$$\psi = \phi(u)e^{\frac{it}{2}}.$$

$\phi(u)$ – is complex function satisfying the equation

$$-\phi+\frac{1}{4}\phi''+|\phi|^2\phi = \epsilon\Big(\frac{i}{2}\psi'-2i\Big(|\phi|^2\phi\Big)'-\phi\widehat{K}\Big(|\phi|^2\Big)\Big). \tag{3.18}$$

Here $\widehat{K} = \widehat{H}(\partial/\partial u)$.

After separation of the amplitude and phase

$$\phi = Ae^{i\Phi},$$

we got two equations:

$$\frac{1}{4}\frac{\partial}{\partial u}A^2\Phi = \epsilon\Big(\frac{1}{4}\Big(A^2\Big)'-\frac{3}{2}\Big(A^4\Big)'\Big),$$

hence

$$\Phi' = \epsilon\Big(1-6A^2\Big). \tag{3.19}$$

Second equation (for amplitude) reads

$$-A+\frac{1}{4}A''+A^3-\frac{1}{4}A\Phi'^2 = -\epsilon\Big\{\Big(\frac{1}{2}+2A^2\Big)\Phi'+A\widehat{K}A^2\Big\}. \tag{3.20}$$

Keeping in (3.20) terms of the order of ϵ^2 is exceeding of accuracy. Thus it can be simplified up to the form

$$-A + \frac{1}{4}A'' + A^3 + \epsilon\, A\widehat{K}A^2 = 0. \tag{3.21}$$

Operator $\widehat{K}$ is pure negative. It acts on exponents as follow

$$\widehat{K}e^{iku} = -|k|e^{iku}.$$

It means that cubic terms in (3.21) have opposite effective signs and, as a result, this equation have solitonic solution only if $\epsilon<\epsilon_0$, ϵ_0 – is some critical value of ϵ.

We have reason to think that ϵ_0 is a relatively small number. Here is the explanation. $\widehat{K}$ is symmetric operator and equation (3.21) realize minimum of the functional

$$\frac{\partial H}{\partial A} = 0, \quad H = \int_{-\infty}^{\infty} -\frac{1}{2}A^2-\frac{1}{8}A'^2+\frac{1}{4}A^4+\frac{\epsilon}{4}A^2\widehat{K}A^2. \tag{3.22}$$

Let us implement the variational approach and will find solution in the form

$$A = \frac{a}{\cosh 2u}. \tag{3.23}$$

a – is still unknown value. We plug (3.23) into (3.22). The last term can be easily calculated in terms of Fourier Transform by the following formulae:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-ikx}}{\cosh^2 2x}\mathrm{d}x = \frac{\sqrt{2\pi}}{8}\frac{k}{\sinh\frac{k\pi}{4}},$$

and

$$\int_0^{\infty}\frac{k^3\mathrm{d}k}{\sinh^2 2k} = \frac{3}{32}\zeta(3).$$

As a result

$$H = -\frac{2}{3}a^2+\Big(\frac{1}{6}-0.22\epsilon\Big)a^4.$$

Condition $\partial H/\partial A = 0$ gives

$$a = \sqrt{\frac{2}{1-1.32\epsilon}}. \tag{3.24}$$

In the limit $\epsilon\rightarrow 0$ we get the NLSE result, $a = \sqrt{2}$. One can see that relatively small ϵ leads to the strong deviation from the NLSE limit.

4. Comparison with numerical experiment

To check the theory presented in this article we compared breather-type solution described in our paper Dyachenko and Zakharov [1] with the soliton shape (3.23). In the Fig. 1 envelope is the following:

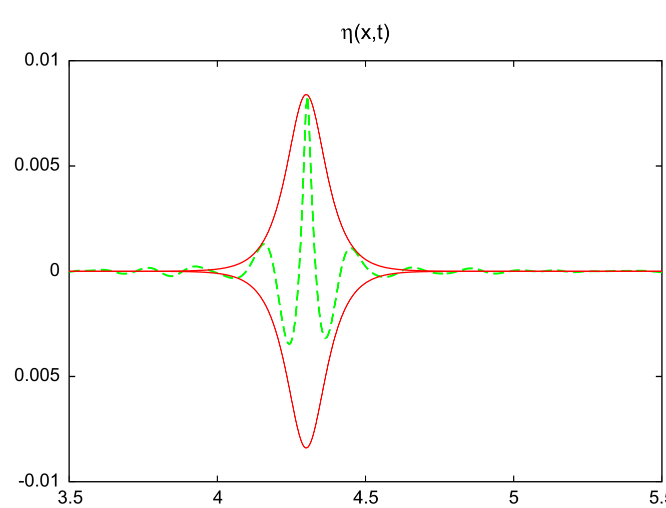

Fig. 1. Envelope from Dysthe equation fits breather from the exact equations.

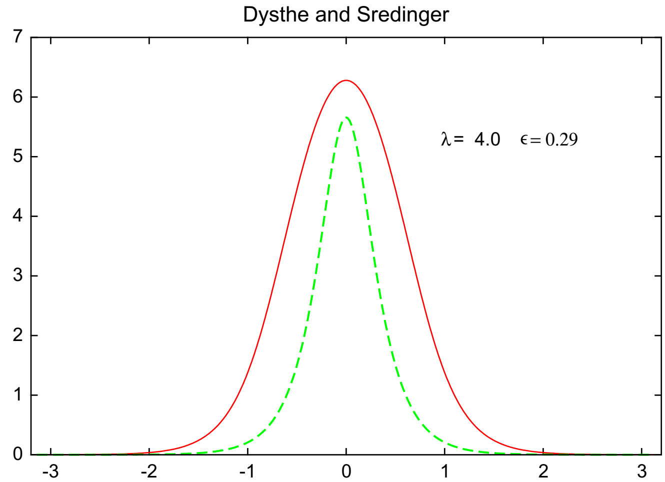

Fig. 2. Solitons for NLSE and Dysthe equation. $\lambda = 4.0$, $\epsilon = 0.290$.

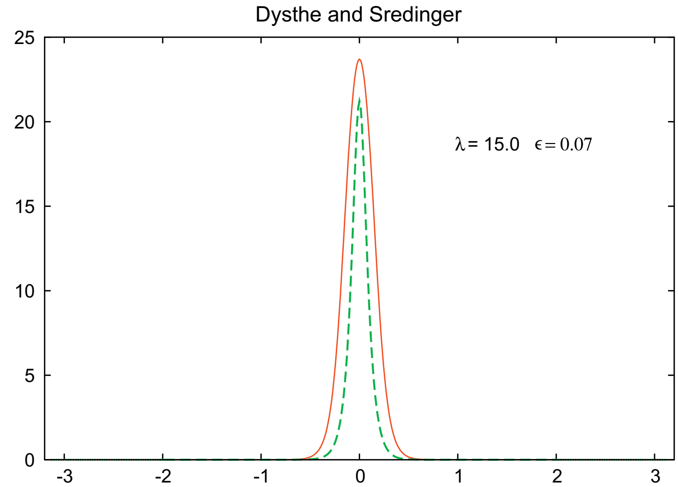

Fig. 4. Solitons for NLSE and Dysthe equation. $\lambda = 15.0$, $\epsilon = 0.070$.

$$A = \frac{a}{\cosh \lambda x}.$$

with $a = 0.0084$, and $\lambda = 17$. If it were NLSE envelope with the same $\lambda = 17$, than a would be 0.0048. So, one can see increasing of the breather amplitude according to the formulae (3.24).

5. Stationary solutions of the Dysthe equation

The main goal of this section is to compare soliton solutions of the Dysthe and the Shredinger equations. Instead of equation of (2.17) it is more convenient to deal with standard NLSE equation with additional nonlocal term, namely

$$-\lambda^2 A + A'' + A^3 + \epsilon A \widehat{K} A^2 = 0. \quad (5.25)$$

We will look for the soliton solution of equation (5.25) with various values of ϵ. It is usual to apply iterative numerical algorithm to solve this nonlinear equation. The simplest algorithm might be the following (written for Fourier harmonics):

$$\left(A_k^{s+1} = \frac{M}{\left(\lambda^2 + k^2\right)} \left[A^s (A^{s2} + \epsilon \widehat{K} A^{s2}) \right]_k \right) \quad (5.26)$$

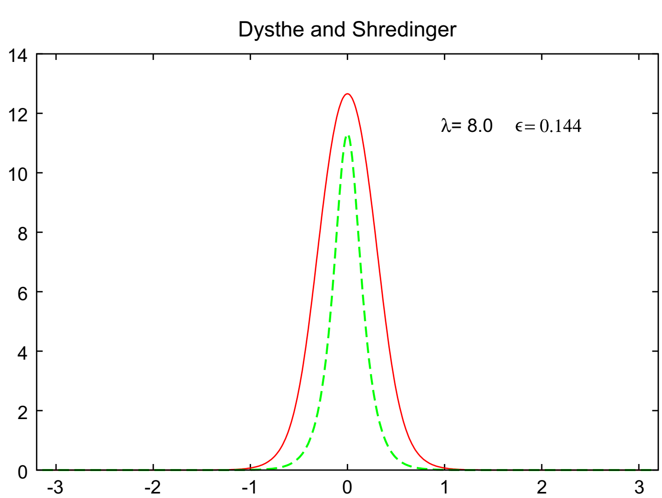

Fig. 3. Solitons for NLSE and Dysthe equation. $\lambda = 8.0$, $\epsilon = 0.144$.

Here M is the factor to provide convergence of the iteration, in particular

$$M = \frac{\int A^{s2} \mathrm{d}u}{\int \frac{A_k^s}{\left(\lambda^2 + k^2\right)} \left[A^s (A^{s2} + \epsilon \widehat{K} A^{s2}) \right]_k \mathrm{d}k}$$

We look for solution which are symmetric with respect to y-axis, so it can be expanded as cosine Fourier series with real Fourier coefficients.

Parameter ϵ, which is equal to

$$\epsilon = \frac{1}{(k_0 L)^2}$$

corresponds to the inverse squared number of carrier waves per dimensionless length. The above algorithm provides absolute accuracy of the order of 10^{-14} (close to round-off errors) with couple of thousands of iterations. It takes a few seconds on usual PC.

In all Figs. 2–4 NLSE-solitons are lower and more narrow with respect to solitons for Dysthe equation. This is in qualitative agreement with formula (3.24).

6. Conclusion

According to estimation (3.24) higher nonlinear approximation, the Dysthe equation, leads to wider and higher soliton. One can see from (3.24) that it might exist critical value of ε for which soliton solution breaks. It corresponds to limiting breather that we observed in fully nonlinear simulation, see Dyachenko and Zakharov [1], and Fig. 1. Local steepness for such breather is close the steepness of limiting Stokes wave, $\mu \simeq 1/\sqrt{3}$.

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