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FIVE-WAVE CLASSICAL SCATTERING MATRIX AND INTEGRABLE EQUATIONS

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We study the five-wave classical scattering matrix for nonlinear and dispersive Hamiltonian equations with a nonlinearity of the type $u\,\partial u/\partial x$. Our aim is to find the most general nontrivial form of the dispersion relation $\omega(k)$ for which the five-wave interaction scattering matrix is identically zero on the resonance manifold. As could be expected, the matrix in one dimension is zero for the Korteweg–de Vries equation, the Benjamin–Ono equation, and the intermediate long-wave equation. In two dimensions, we find a new equation that satisfies our requirement.

Keywords: integrability, intermediate long-wave equation, Korteweg–de Vries equation, Benjamin–Ono equation, scattering matrix

Significant progress has been achieved in the field of nonlinear science during the last 40 years. The development of new mathematical tools led to singling out a class of integrable nonlinear partial differential equations. These nonlinear integrable equations play an important role in studying physical systems. By applying asymptotic procedures that use small parameters characterizing the physical regime of interest, we can reduce a very large class of nonlinear evolution equations to integrable equations (see [1]). Different approaches were developed for establishing the properties of these equations [2]. In this context, the present paper is based on the Zakharov–Schulman theorem (see [3]–[5]), which is related to the Poincaré analysis of the integrability of dynamical systems. The theorem is based on perturbation theory and on the introduction of the so-called classical scattering matrix, which relates two asymptotic states ($t \to \pm\infty$) for a classical Hamiltonian system. Loosely speaking, the theorem states that the existence of one additional integral of motion implies that the scattering matrix vanishes for each resonance process; the theorem also implies the existence of an infinite set of invariants. This, of course, is not sufficient for integrability (for which the completeness of the set of invariants must be proved). One of the consequences of the theorem is that the scattering matrix is not identically zero for nonintegrable systems. This result was recently used, for example, to prove the nonintegrability of the compact one-dimensional Zakharov equation [6].

Here, we discuss the integrability of some hydrodynamic wave equations. In particular, it is well known that the Korteweg–de Vries (KdV), the Benjamin–Ono (BO), and the intermediate long-wave (ILW) equations are examples of integrable systems with the same nonlinear operator and different linear dispersive terms. Using the Zakharov–Schulman theorem, we can investigate whether there are other integrable

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Hamiltonian wave equations characterized by the same nonlinear operator. A similar question was studied in [7], [8].

We consider the general wave equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \omega_k u = 0, \tag{1}$$

where u is a real function of x and ω_k is a convolution operator responsible for the wave dispersion. We assume that ω_k is an odd function of k, i.e., $\omega(k) = -\omega(-k)$, and hence $\omega(0) = 0$. Equation (1) can be written in Fourier space as

$$i\frac{\partial u_1}{\partial t} = \omega_1 u_1 + k_0 \int_{-\infty}^{\infty} u_2 u_3 \delta_{1-2-3}\, dk_{23}, \tag{2}$$

where $u_i = u(k_i)$, $dk_{23} = dk_2\, dk_3$, $\delta_{1-2-3} = \delta(k_1 - k_2 - k_3)$. We now introduce the normal variable $a(k)$ related to $u(k)$ as

$$u_k = \sqrt{k}(a_k\theta_k + a^*_{-k}\theta_{-k}),$$

where $\theta_k = \theta(k)$ is the Heaviside step function. In terms of the new variable, the equation has the Hamiltonian form

$$H = H_0 + H_{\text{int}} = \int_0^{\infty} \omega_1 |a_1|^2\, dk_1 + \int_0^{\infty} V_{1,2,3}(a_1 a_2^* a_3^* + a_1^* a_2 a_3)\delta_{1-2-3}\, dk_{123}, \tag{3}$$

where $V_{1,2,3} = \sqrt{k_1 k_2 k_3}\theta_1\theta_2\theta_3$.

The evolution equation is written as

$$i\frac{\partial a_k}{\partial t} = \frac{\delta H}{\delta a_k^*}. \tag{4}$$

For any Hamiltonian system of type (4), we can replace Hamiltonian (3) with the auxiliary Hamiltonian $H_\varepsilon = H_0 + H_{\text{int}}e^{-\varepsilon|t|}$, where $\varepsilon > 0$, and take the limit as $t \to \pm\infty$ in (4). For any initial condition $a(k,t)|_{t=0}$, the asymptotic fields $a(k,t)|_{t\to\pm\infty}$ are given by $c_\varepsilon(k)^{\pm}e^{-i\omega(k)t}$. These asymptotic limits are not independent and are related by $c_\varepsilon(k)^+ = S_\varepsilon[c_\varepsilon(k)^-]$, where S_ε is a nonlinear operator given by a series that converges for sufficiently large ε. The *classical scattering matrix* is defined (see [3] for the details) by taking the limit as $\varepsilon \to 0$. Hence,

$$S = \lim_{\varepsilon\to 0} \widehat{S}_\varepsilon, \qquad c_\varepsilon^{\pm}(k) \to c^{\pm}(k). \tag{5}$$

After the limit is taken, the series may diverge and become a formal series with $c^+(k)$ and $c^-(k)$ related as

$$c^+(k_1) = Sc^-(k_1) = c^-(k_1) + \sum_{n=3}^{\infty} \sum_{s_1,\dots,s_n} \int_0^{\infty} S_{1,2,\dots,n} \times \\ \times \delta(s_1k_1 + s_2k_2 + \dots + s_nk_n)c_1^{-s_2}\cdots c_n^{-s_n}\, dk_{2\dots n}. \tag{6}$$

Here, s_i can assume the value $+1$ or -1, and $c^{\pm s}$ is equal to $c^{\pm}$ for $s = 1$ and $(c^{\pm})^*$ for $s = -1$. The classical scattering matrix becomes

$$S_{1,2,\dots,n} = iT_{1,2,\dots,n}\delta(s_1\omega_1 + s_2\omega_2 + \dots + s_n\omega_n), \tag{7}$$

where $T_{1,2,\dots,n}$ is the scattering amplitude defined on the surface

$$\begin{gathered} s_1\omega_1 + s_2\omega_2 + \dots + s_n\omega_n = 0, \\ s_1k_1 + s_2k_2 + \dots + s_nk_n = 0. \end{gathered} \tag{8}$$

Depending on the s_i, the above conditions split into a set of relations that determines a *resonance manifold.* Zakharov and Schulman [3] proved an important theorem that states that if system (4) has an additional integral of motion I (in addition to the energy, momentum, and mass) of the form

$$I = \int f_k |a_k|^2 \, dk + \ldots,$$

where the dots imply terms of higher order in a_k, and if the dispersion relation is nondegenerate, then for each scattering process distinguished by a choice of the s_i in (8), the scattering amplitude vanishes, $T_{0,1,\ldots,n} = 0$, and the system has infinitely many constants of motion in involution. The theorem does not allow checking the integrability of an equation directly, but given a Hamiltonian system of form (4), it can be established that a nontrivial scattering, i.e., $T_{1,2,\ldots,n} \neq 0$ on some resonance manifold, is clear evidence of nonintegrability.

In the special case, Eq. (1) describes waves propagating in one direction. Therefore, the integrals in Hamiltonian (3) are defined for $k \geq 0$ in the formulation with normal variables. We are interested in finding the specific form of $\omega(k)$ for which the scattering amplitude vanishes on the resonance manifold. The three- and four-wave processes have only trivial solutions. The first interesting process is the five-wave process. In this case, the resonance manifold is given by the equations

$$\begin{aligned} &k_4 + k_5 = k_1 + k_2 + k_4, \\ &\omega(k_4) + \omega(k_5) = \omega(k_1) + \omega(k_2) + \omega(k_3). \end{aligned} \tag{9}$$

The scattering matrix is an enormous expression containing 80 terms (see [9], where a diagram technique for constructing these terms was developed). We assume that all positive wave numbers are ordered as

$$k_2 > k_4 > k_5 > k_3 > k_1.$$

Under this assumption, the five-wave amplitude scattering matrix reduces significantly and becomes

$$\begin{aligned} T_{123-4-5} = {} & F_{12}(F_{45} + G_{53} + G_{43}) + F_{13}(F_{45} + G_{25} + G_{24}) + G_{51}(F_{23} + G_{43} + G_{24}) + \\ & + G_{41}(F_{23} + G_{53} + G_{25}) + F_{45}F_{23} + G_{24}G_{53} + G_{25}G_{43}, \end{aligned} \tag{10}$$

where

$$F_{ij} = \frac{k_i + k_j}{\omega(k_i + k_j) - \omega(k_i) - \omega(k_j)}, \qquad G_{ij} = \frac{k_i - k_j}{\omega(k_i - k_j) - \omega(k_i) + \omega(k_j)} \tag{11}$$

with $i \neq j = 1, \ldots, 5$. The necessary condition for integrability is the cancellation of the five-wave amplitude on resonance manifold (9).

In what follows, we show that the KdV, BO, and ILW equations have this property.

The KdV equation. Let $\omega(k) = -k^3$. Then

$$F_{ij} = \frac{1}{3k_i k_j}, \qquad G_{ij} = -\frac{1}{3k_i k_j}. \tag{12}$$

A simple calculation reduces the five-wave amplitude matrix to

$$T_{123-4-5} = \frac{1}{9k_1k_2k_3k_4k_5}(k_4 + k_5 - k_1 - k_2 - k_3) = 0. \tag{13}$$

We note that $T_{123-4-5} = 0$ even outside the resonance manifold (only momentum conservation is needed), i.e., the condition on the frequencies in (9) is not used.

The BO equation. The BO equation models the evolution of long one-dimensional internal gravity waves in a stratified fluid in the limit of deep water. For such an equation, $\omega(k) = -|k|k$, but we can set $\omega(k) = -k^2$ for positive wave numbers. Then

$$F_{ij} = \frac{k_i + k_j}{2k_i k_j}, \qquad G_{ij} = -\frac{1}{2k_j}, \tag{14}$$

and we can elegantly show that the five-wave scattering amplitude is given by

$$T_{123-4-5} = L_{12345}(k_4^2 + k_5^2 - k_1^2 - k_2^2 - k_3^2) + M_{12345}(k_1 + k_2 + k_3 - k_4 - k_5), \tag{15}$$

where L_{12345} and M_{12345} are two positive functions of the wave numbers. Cancellation of the scattering matrix is hence obvious from Eq. (9).

The ILW equation. The ILW equation describes long internal gravity waves in a stratified fluid of finite depth. The dispersion relation is given by

$$\omega(k) = ak^2 \coth bk - ck \tag{16}$$

and reduces to the KdV equation in the limit as $b \to 0$ with $a = 1$ and $c = 0$ and to the BO equation in the limit as $b \to \infty$ with $a = 3/b$ and $c = 3/b^2$. To verify the cancellation for such a dispersion relation, we first note that $T_{123-4-5}$ is invariant under the transformation $\omega(k) \to \omega(\alpha k) + \beta k$, where $\alpha \neq 0$ and β are arbitrary constants. Moreover, we can introduce an extra independent variable p and set

$$\omega(k) \to \omega(k,p) = k^2 \frac{1 + e^p}{1 - e^p}. \tag{17}$$

The resonance surface must satisfy the equations

$$\begin{gathered} p_4 + p_5 = p_1 + p_2 + p_3, \qquad k_4 + k_5 = k_1 + k_2 + k_3, \\ \omega_{k_4,p_4} + \omega_{k_5,p_5} = \omega_{k_1,p_1} + \omega_{k_2,p_2} + \omega_{k_3,p_3}. \end{gathered} \tag{18}$$

The five-wave amplitude matrix now depends on 10 variables $T_{12345} = T(k_1, \ldots, k_5, p_1, \ldots, p_5)$, and the cancellation of this matrix on the resonance manifold was successfully verified using symbolic computation with `Maple`.

We now turn our attention to the most general case and consider

$$T(k_1, \ldots, k_5) = 0 \tag{19}$$

as a *functional equation* for the unknown $\omega(k)$ with $k_1, \ldots, k_5$ satisfying resonance condition (9). We recall that if $\omega(k)$ is a solution of the functional equation, then

$$\tilde{\omega}(k) = a\omega(bk) + ck \tag{20}$$

is also a solution for arbitrary $a, b \neq 0$ and c.

We formulate the following proposition.

Proposition 1. *Any solution of the functional equation that is analytic near zero and is such that* $\omega(0) = 0$ *is equal to one of the functions* $\omega_1(k) = k^2$, $\omega_2(k) = k^3$, $\omega_3(k) = -k^2(e^{2k}+1)/(e^{2k}-1) = k^2 \coth k$. *If* $a, b \neq 0$ *are complex numbers, then there exists a fourth solution* $\omega_4(k) = ik^2(e^{i2k}+1)/(e^{i2k}-1) = -k^2 \cot k$.

Proof. The idea behind the proof is to reduce functional equation (19) to an ordinary differential equation and find solutions. We set $k_2 = k_4 + u$ and $k_5 = k_3 + v$. The conservation of momentum in (9) then leads to $k_1 = v - u$. Considering the constraint on frequencies (conservation of energy in (9)) and expanding in a Taylor series near $u = v = 0$, we now obtain

$$v = \frac{\omega'(k_4)}{\omega'(k_3)} u + o(u). \tag{21}$$

Moreover, expanding functional equation (19) near $u = v = 0$ and using Eq. (21), we obtain an ordinary differential equation that in the leading order contains a large number of terms. Without loss of generality, assuming that $\omega(k) = a_2k^2 + a_3k^3 + \dots$ and expanding the ordinary differential equation near $k_3 = k_4 = 0$, we obtain the condition $a_2a_3 = 0$, which leads to the following cases.

Case 1. Let $a_2 \neq 0$. Then $a_3 = 0$, which implies $\omega(k) = k^2 + a_4k^4 + a_5k^5 + \dots$. We expand the ordinary differential equation near $k_3 = 0$. In the leading nontrivial order, we obtain the third-order ordinary differential equation

$$-5\omega'(k_4)^2 + 8k_4\omega''(k_4)\omega'(k_4) + \omega'(k_4)k_4^2\omega'''(k_4) - 3k_4^2\omega''(k_4)^2 = 0, \tag{22}$$

whose only solution of the form $\omega(k) = k^2 + a_4k^4 + a_5k^5 + \dots$ is $\omega(k) = k^2$.

Case 2. Let $a_3 \neq 0$. Then $a_2 = 0$. Without loss of generality, we assume that $f(k) = k^3 + a_4k^4 + a_5k^5 + \dots$. We again expand the ordinary differential equation near $k_3 = 0$. In the leading nontrivial order, we obtain the fourth-order ordinary differential equation

$$\begin{aligned}-4\omega'(k_4)^2\omega'''(k_4) + 12\omega'(k_4)^2 - k_4\omega'(k_4)^2\omega''''(k_4) - 2\omega'(k_4)k_4^2\omega'''(k_4) +{}\\ + 4\omega'(k_4)k_4\omega''(k_4)\omega'''(k_4) - 18k_4\omega''(k_4)\omega'(k_4) +{}\\ + 6\omega'(k_4)\omega''(k_4)^2 - 3\omega''(k_4)^3k_4 + 6k_4^2\omega''(k_4)^2 = 0.\end{aligned} \tag{23}$$

Any solution of the form $\omega(k) = k^3 + a_4k^4 + a_5k^5 + \dots$ is equivalent to either $\omega(k) = k^3$ or $\omega(k) = k^2 \coth k$ (or $\omega(k) = -k^2 \cot k$ for complex constants a and b in (20)). We note that there are no nontrivial solutions in the case where $a_2 = a_3 = 0$.

Our results indicate that the five-wave amplitude scattering matrix for Eq. (2) also vanishes for the dispersion relation

$$\omega(k) = ak^2 \cot bk - ck, \tag{24}$$

where a, b, and c are constants. The resulting equation hardly has any physical sense because $\omega(k) = \infty$ at $bk = \pi n$. But in the limit as $b \to 0$ with $a = 3/b$ and $c = 3/b^2$, the dispersion relation reduces to $\omega(k) = k^3$, i.e., coincides with dispersion relation for the KdV model. Hence, there exists one more equation that is possibly quite interesting. Let $u = u(x, y, t)$ be a function of two spatial coordinates (x, y). Then $\omega = \omega(k, p)$ is given by

$$\omega(k, p) = ak^2 \coth ap. \tag{25}$$

The five-wave amplitude matrix vanishes in this case. Expanding in a Taylor series for small a, we obtain the equation

$$u_t + uu_x + \partial_x^2 \partial_y^{-1} u = 0. \tag{26}$$

The same equation can be obtained starting from $\omega(k,p) = ak^2 \cot ap$ and expanding in a Taylor series for small a. This equation resembles, but is not equivalent to, the Khokhlov–Zabolotskaya equation (see, e.g., [10])

$$u_t + uu_x + \partial_y^2 \partial_x^{-1} u = 0, \tag{27}$$

which describes the propagation of a confined beam in a slightly nonlinear medium without dispersion or absorption (see [11]).

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