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# Describing dynamics of nonlinear axisymmetric waves in dispersive media with new equation

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ABSTRACT

A single nonlinear partial differential equation of the wave type for an axisymmetric case is obtained by the introduction of special auxiliary function. In contrast to cylindrical Korteweg–de Vries equation, new equation describes centrifugal and centripetal waves not only far from the center, but in its vicinity as well. With the use of this equation a number of specific problems on the evolution of the free surface disturbances are numerically solved for the cases of a horizontal bottom and a drowned concave. The research also demonstrates the difference between the results of calculations on the base of the complete equation and on the basis of the linearized equation.



## 1. Introduction

A study of gravitational waves of small but finite amplitude in shallow liquid layers is one of the classical problems of nonlinear physics. The well-known example is the Korteweg–de Vries equation, derived by the authors specifically for moderately long weakly nonlinear disturbances of a free surface (e.g., [1]). Only 75 years later it was generalized to the case of axisymmetric waves (see [2]). It should be emphasized that the so-called cylindrical Korteweg–de Vries equation was obtained and applied not only for the surface waves [3–8], but also for the waves of different physical nature (for example, ion-acoustic waves in plasma [9–12]).

The cylindrical Korteweg–de Vries equation allows investigating diverging waves. Moreover, it is valid only in the region that is quite far from the center of symmetry. These are fundamental limitations of the equation. Simulation of the evolution of waves moving in different directions until recently was possible only with the use of the systems of nonlinear equations containing the liquid velocity vector even in the linear terms of all equations (e.g., [13,14]).

A combined system of equations, which is more convenient for analysis, was proposed in [15]. The model consists of a basic nonlinear evolution equation for the disturbance of the free surface and two linear auxiliary equations, which are required to determine the horizontal liquid velocity vector involved only in the second-order terms of the main equation. This article covers only the results of two simple calculations for the case of the horizontal bottom. A slightly more compact version of the model and more detailed calculations of the centripetal and centrifugal waves, as well as of the collision of several disturbances (initially solitary in space) over the sloping bottom were published in [16].

Finally, in [17] a single non-standard wave equation was derived to model interaction of plane localized disturbances. In contrast to the Boussinesq equation, a new equation correctly describes the head-on collision of the moderate amplitude waves. It was shown analytically that in the first-order perturbation theory, the head-on collision of solitons is inelastic, and its nonlinear dynamics qualitatively differs from that of the Boussinesq equation.

The purpose of this article is derivation of a new model equation to describe the transformation of weakly nonlinear moderately long axisymmetric localized disturbances of the free surface of liquid layer, both divergent and convergent to the center of the tank with the bottom of the same symmetry. Below there are some

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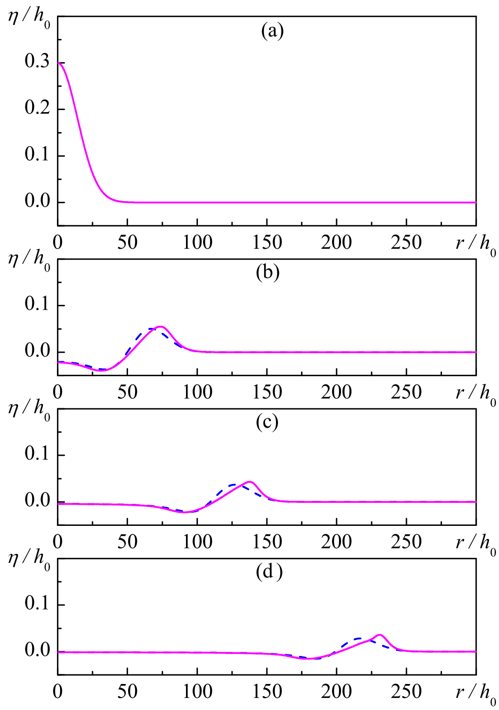

**Fig. 1.** Radial profiles of the initially quiescent bell-shape disturbance over the horizontal bottom in 4 points of time: $t = 0$ (a), $t^* = t\sqrt{g/h_0} = 60$ (b), $t^* = 120$ (c), and $t^* = 210$ (d); calculations based on Eq. (3) (solid curves) and on the linearized Eq. (3) (dashed curves).

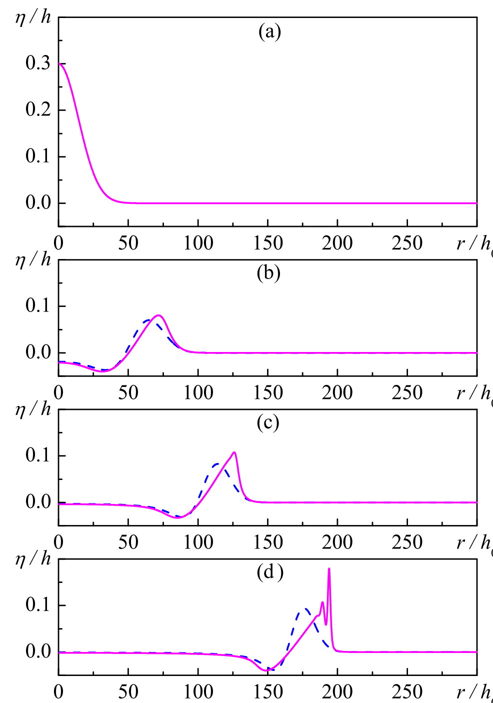

**Fig. 2.** Radial profiles of the initially quiescent bell-shape disturbance over the bottom with concave in 4 points of time: $t = 0$ (a), $t^* = 60$ (b), 120 (c), and 210 (d); calculations based on Eq. (3) (solid curves) and on the linearized Eq. (3) (dashed curves).

applications of the equation that serve to find solutions to several typical problems.

## 2. Assumptions and the model equation

Assume that, firstly, the characteristic horizontal scale of the disturbances $L_w$ is substantially greater, and the amplitude $\eta_a$ is considerably smaller than the equilibrium layer depth of the layer $h$ ($h/L_w \sim \varepsilon^{1/2}, \eta_a/h \sim \varepsilon$, where $\varepsilon$ is a small parameter). Secondly, the slope of the fixed solid bottom is insignificant (the derivative of the layer depth along any horizontal direction $dh/dl \sim \varepsilon^{3/2}$). Thirdly, the capillary effects are relatively small ($\sigma/(\rho g h^2) < 1$, here $\sigma$ is the surface tension, $\rho$ is the liquid density, and $g$ is the free fall acceleration). Fourthly, the stationary components of the liquid flow are equal to zero, and, at last, dissipation may be neglected. In addition, as the propagation velocity of gravitational waves is much lower than the sound speed in the fluid we can use the incompressible liquid approximation. Within the frames of these assumptions the following system of equations was derived in [15,16]:

$$\frac{\partial^2\eta}{\partial t^2} - gh\nabla^2\eta - g\,\nabla h\cdot\nabla\eta - \frac{g}{2}\,\nabla^2(\eta^2) - h\,\nabla^2\left(\mathbf{u}^2\right) - \\ - \left(\frac{h^2}{3} - \frac{\sigma}{\rho g}\right)\nabla^2\frac{\partial^2\eta}{\partial t^2} = 0 \qquad (1)$$

$$\mathbf{u} = \nabla\varphi, \qquad \nabla^2\varphi = -\frac{1}{h}\frac{\partial\eta}{\partial t}. \qquad (2)$$

Here $t$ is the time, and $\nabla = (\partial/\partial x,\ \partial/\partial y)$ is the gradient operator in the horizontal plane. Eq. (1) was obtained from the system of Euler's equations and the continuity equation. Eqs. (2) are the potential flow condition and the linearized law of conservation of mass for the layer. They are sufficient to determine the averaged horizontal fluid velocity $\mathbf{u}$, which is contained only in the second-order terms of the main equation (1).

Consider dynamics of nonlinear axisymmetric solitary disturbances of the free surface ($\eta \to 0$ when $r \to \infty$, where $r = \sqrt{x^2 + y^2}$ is the polar radius), propagating simultaneously in the direction of increasing $r$ coordinate and to the center of the tank. Now introduce a new auxiliary function $\psi$ using the equations $\partial\psi/\partial r = r\eta$ and $\partial\psi/\partial t = -r(h+\eta)u_r$ (here $u_r$ is the radial component of fluid velocity). Then the mass conservation law is satisfied identically in the elementary volume of the layer (there is an equality of the mixed derivatives $\partial^2\psi/\partial r\,\partial t$ and $\partial^2\psi/\partial t\,\partial r$). In our case it becomes possible to replace $u_r$ by $-(\partial\psi/\partial t)/(rh)$ in the second-order terms. Further, integrate the equation over the $r$ coordinate from $r$ to $\infty$. And, as a result, the model system of Eqs. (1) and (2) is reduced to the equation

$$\frac{\partial^2\psi}{\partial t^2} - gh\,r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \frac{g}{2}\,r\frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(\frac{\partial\psi}{\partial r}\right)^2\right] - \\ - \frac{1}{h}\,r\frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(\frac{\partial\psi}{\partial t}\right)^2\right] - \beta\,r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^3\psi}{\partial t^2\partial r}\right) = 0, \qquad (3)$$

where $\beta = h^2/3 - \sigma/\rho g$. At that the initial-boundary problem is set simply as follows:

$$\psi(0,r) = p(r), \qquad \frac{\partial\psi}{\partial t}(0,r) = q(r),$$

$$\psi(t,0) = p(0), \qquad \frac{\partial\psi}{\partial r}(t,0) = 0.$$

Here $p(r)$ and $q(r)$ are some limited differentiable functions, and the last two conditions result from the limited nature of values $u_r$ and $\eta$. Remind that the bottom is certainly axisymmetric.

Search for the analytical solutions to the nonlinear partial differential equation (3) is a difficult task that is why only the numerical results are presented below.

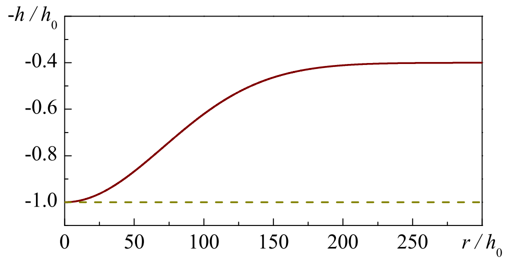


**Fig. 3.** Two considered radial profiles of the tank bottom.

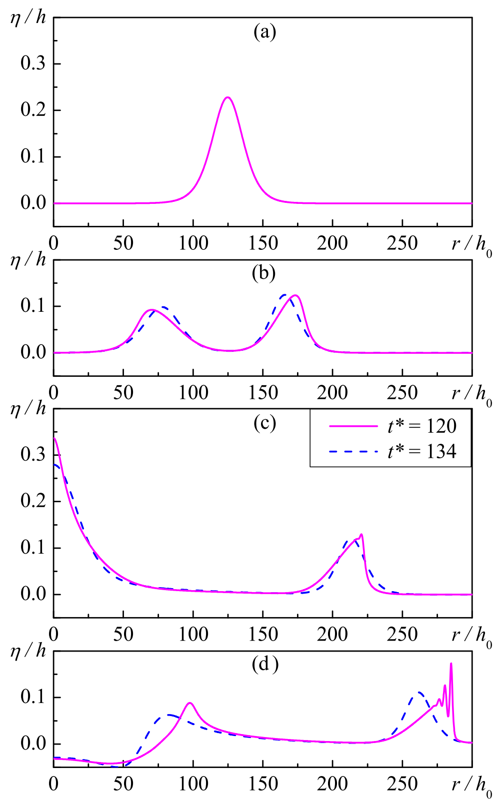


**Fig. 4.** Evolution of radial profiles of the initially quiescent annular disturbance over the drowned concave in 4 points of time: $t = 0$ (a), $t^* = 60$ (b), 120 or 134 (c), and $t^* = 210$ (d); calculations based on Eq. (3) (solid curves) and on the linearized Eq. (3) (dashed curves).

## 3. Examples of disturbances transformation

Although the nonlinear terms in Eq. (3) are different from the respective ones in two-dimensional Boussinesq equation, the numerical scheme from [18] was only slightly modified. We also note that in all problems considered below, a liquid was initially at rest, that is $q(r) = 0$.

Fig. 1 shows the transformation of the initially quiescent bell-shape disturbance ($\eta(0, r) = \eta_a \exp[-(r^*/20)^2]$, where $r^* = r/h_0$) over the horizontal bottom. Firstly, the level of liquid lowers behind the head front, and then returns to its equilibrium position. The leading disturbance is neither a linear wave nor a soliton, because our solitary disturbance consists of a leading wave and a "tail". Therefore the radial decay of leading wave is neither proportional to $r^{-1/2}$ or $r^{-2/3}$. In [5] the authors noted that in this case, both the leading wave and the "tail" have a radial decay $\sim r^{-1/3}$. In our calculations, the following close dependence $\sim r^{-0.37}$ for $25 < r^* < 250$ was found.

Fig. 2 displays the evolution of the initially quiescent bell-shape disturbance over the bottom with drowned concave: $h(r) = h_0[1 - 0.6(1 - \exp[-(r^*/100)^2])]$ (see Fig. 3). It is clearly seen that in the region of a smaller layer depth the propagation velocity of the cylindrical wave is noticeably lower and its amplitude is slightly larger than above the horizontal bottom. In addition over the uneven bottom the disturbance profile is transformed, firstly, into a triangular shape with a steep head and a sloping back fronts, and then oscillations arise at the head. The effect of nonlinearity is stronger here. Emphasize that this figure demonstrates the value $\eta(r)/h(r)$.

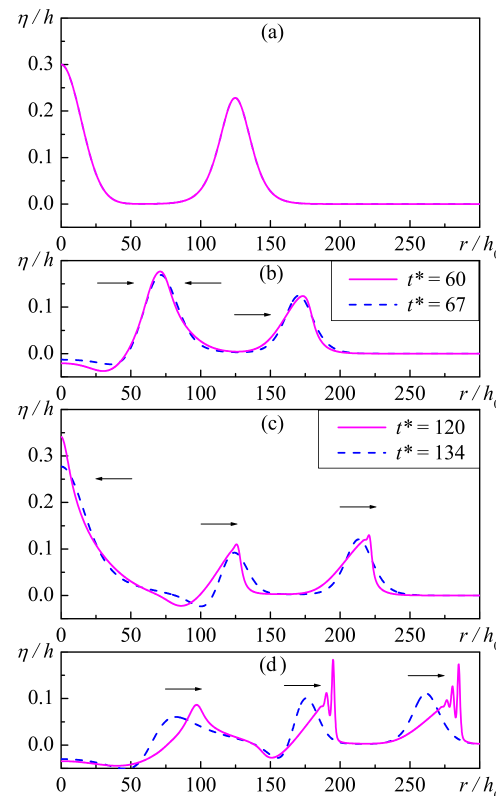


**Fig. 5.** Transformation of radial profiles of the initially quiescent bell-shaped and ring-shaped disturbances over the bottom with concave in 4 points of time: $t^* = 0$ (a), 60 or 67 (b), 120 or 134 (c), and $t^* = 210$ (d); calculations based on Eq. (3) (solid curves) and on the linearized Eq. (3) (dashed curves).

Similar solutions are shown in Fig. 4 for initially quiescent annular disturbance $\eta(0, r) = \eta_a \operatorname{sech}^2[(r^* - 125)/(4L/h_0)]$, which is four times wider than the plane soliton, because $L = 2\sqrt{\beta(1/3 + h_0/\eta_a)}$ is the width of such soliton (see, e.g., [17]). Firstly, as it was expected, both converging and diverging cylindrical waves are formed. Then the first of them is evolving similar to the converging wave in Fig. 2, and the second one leads to a splash "fountain" in the center of the tank, and after that another centrifugal wave is formed. In Fig. 4(c) it is seen that the maximum splash calculated on the basis of the linearized Eq. (3) takes place later and is 17% lower than the one calculated by the complete Eq. (3).

Fig. 5 demonstrates interaction of ring-shaped and bell-shaped disturbances. Arrows indicate direction of wave propagation (in previous time points). Fig. 5(b) clearly shows processes of collision of convergent and divergent cylindrical disturbances. The maximum splash calculated on the basis of the linearized Eq. (3) takes place later and is 4% lower than calculated by the complete Eq. (3). In Fig. 5(c) there is a splash "fountain" in the center of the tank, and in Fig. 5(d) there is a system of three final centrifugal waves (two first waves are nonlinear undular waves). On these figures influence of the bottom slope becomes even more obvious.

## 4. Conclusion

The main results of this work are as follows. The new special auxiliary function has been introduced, and for this function a nonlinear partial differential equation of the axisymmetric wave type has been obtained. In contrast to the cylindrical Korteweg–de Vries equation this equation allows simultaneously simulating both centrifugal and centripetal waves not only far from the center, but in its vicinity as well. Based on this new equation a number of numerical experiments have been carried out: transformations of the initially quiescent bell-shaped and ring-shaped disturbances as well as their interactions in the shallow liquid layer above the horizontal bottom and above a drowned concave.

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