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Spatial equation for water waves

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We derive compact spatial Hamiltonian equation for the gravity waves on the deep water. The equation is dynamical one, it can describe extreme waves. Also equation for envelope of wave train is obtained.

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1. Introduction. Surface gravity waves generated in the laboratory tank (or flume) is one of the most studied examples of nonlinear wave evolution. Numerical simulation of such wave evolution has to be its integral part. These waves are usually described by classical Hamiltonian system of equations for potential flows with truncated Hamiltonian [1]:

$$H = \frac{1}{2}\int g\eta^2 + \psi\hat{k}\psi dx - \frac{1}{2}\int\{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + \\ + \frac{1}{2}\int\{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}[\eta\hat{k}(\eta\hat{k}\psi)]\}dx \qquad (1)$$

with the Hamiltonian variables $\eta(x,t)$ – surface profile, and $\psi(x,t)$ – potential at the surface. Equations of motions are the following:

$$\frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}, \quad \frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}. \qquad (2)$$

These equations describe Cauchy problem in time, one has to set up initial conditions $\eta(x,0)$ and $\psi(x,0)$ at $t=0$ at all x. However, in the flume situation is different. Typically, at the one end of the flume there is a wavemaker (piston or paddle) which generates (in the ideal case) $\eta(0,t)$ and $\psi(0,t)$. Thus, we have to solve Cauchy problem in space. If we restrict ourselves to an envelope of the wave train, than the equations for spatial Cauchy problem were derived in [2, 3] directly from Zakharov equation. They derived spacial analogies both to Nonlinear Schrödinger and Dysthe equations. Their Hamiltonian structures and new invariants were studied in [4]. However, to study waves with extreme amplitudes, freak-waves, the envelope approximation is not enough. Another words, to simulate real nonlinear waves we *need spatial dynamical* equation for water waves. Below this equation is derived for the case of one horizontal direction (narrow flume).

2. Super compact equation. Let us recall very briefly what is Zakharov equation for water waves. It can be derived by two steps.

1. First, instead of η and ψ, normal canonical variable a_k is introduced:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}), \ \ \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}),$$

$$\omega_k = \sqrt{gk}.$$

2. Canonical transformation from a_k to b_k is chosen to cancel all non resonant terms in the Hamiltonian, both cubic and forth order.

As a result the Hamiltonian acquires the form:

$$H = \int\omega_k b_k b^*_k dk + \\ + \frac{1}{2}\int T^{k_2k_3}_{kk_1} b^*_k b^*_{k_1} b_{k_2} b_{k_3}\delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3. \quad (3)$$

The explicit (and cumbersome) expression for $T^{k_2k_3}_{kk_1}$ can be found in [1, 5]. The motion equation is the following:

$$\frac{\partial b_k}{\partial t} + i\frac{\delta H}{\delta b^*_k} = 0. \qquad (4)$$

For 1-D waves $T^{k_2k_3}_{kk_1}$ has very important for further – it is equal to zero on the four resonant manifold [6]. This property allows to apply another canonical transformation from b_k to c_k, namely

$$b_k = c_k - i\int\tilde{B}^{k_2k_3}_{kk_1} c^*_{k_1} c_{k_2} c_{k_3}\delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 + \ldots$$

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with

$$\tilde{B}_{kk_1}^{k_2k_3} = i\frac{\tilde{T}_{kk_1}^{k_2k_3} - T_{kk_1}^{k_2k_3}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}. \quad (5)$$

This transformation replaces $T_{kk_1}^{k_2k_3}$ by $\tilde{T}_{kk_1}^{k_2k_3}$ in (3). Coefficient $\tilde{T}_{kk_1}^{k_2k_3}$ can be any function having the same values on the four-wave resonant manifold:

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3}. \end{aligned} \quad (6)$$

In [7, 8] choice of $\tilde{T}_{kk_1}^{k_2k_3}$ has allowed to obtain the Hamiltonian in a compact way. However, it was shown in [9, 10] that the best choice for $\tilde{T}_{kk_1}^{k_2k_3}$ is the following:

$$\tilde{T}_{k_2k_3}^{kk_1} = \frac{(kk_1k_2k_3)^{1/2}}{2\pi}\min(k, k_1, k_2, k_3) \times \theta_k\theta_{k_1}\theta_{k_2}\theta_{k_3}, \quad (7)$$

here θ_k is the step-function, $\theta_k = \theta(k)$.
Hamiltonian can be written in x-space:

$$H = \int c^*\hat{V}c\,dx + $$
$$+ \frac{1}{2}\int\left[\frac{i}{4}\left(c^2\frac{\partial}{\partial x}c^{*2} - c^{*2}\frac{\partial}{\partial x}c^2\right) - |c|^2\hat{K}(|c|^2)\right]dx. \quad (8)$$

Here operator $\hat{V}$ in k-space is so that $V_k = \frac{\omega_k}{k}$. When introducing along with this Gardner–Zakharov–Faddeev bracket

$$\partial_x^+ \Leftrightarrow ik\theta_k \quad (9)$$

equation of motion becomes the following:

$$\frac{\partial c}{\partial t} + \partial_x^+\frac{\delta H}{\delta c^*} = 0. \quad (10)$$

Introducing advection velocity

$$\mathcal{U} = \hat{K}|c|^2 \quad (11)$$

and taking variational derivative one can write the equation (10) in the form:

$$\frac{\partial c}{\partial t} + i\hat{\omega}c - i\partial_x^+\left(|c|^2\frac{\partial c}{\partial x}\right) = \partial_x^+(\mathcal{U}c). \quad (12)$$

3. Derive spatial equation. Equation (12) for water waves can be written in k-space:

$$i\dot{c}_k = \omega_k c_k + $$
$$+ \frac{k\theta_k}{2\pi}\int T_{k_2k_3}^{kk_1}c_{k_1}^*c_{k_2}c_{k_3}\delta_{k+k_1-k_2-k_3}dk_1dk_2dk_3,$$
$$T_{k_2k_3}^{kk_1} = \min(k, k_1, k_2, k_3). \quad (13)$$

Performing Fourier transformation over time and multiplying the result by $\omega + \omega_k$ one can easily get:

$$(\omega^2 - gk)c_{k\omega} = $$
$$= \frac{(\omega+\omega_k)k\theta_k}{(2\pi)^2}\int T_{k_2k_3}^{kk_1}c_{k_1\omega_1}^*c_{k_2\omega_2}c_{k_3\omega_3}\times$$
$$\times\,\delta_{k+k_1-k_2-k_3}\delta_{\omega+\omega_1-\omega_2-\omega_3}dk_1dk_2dk_3d\omega_1d\omega_2d\omega_3. \quad (14)$$

For the waves with small amplitudes all harmonics $c_{k\omega}$ are focused in the vicinity of the dispersion curve:

$$\omega = \sqrt{gk} + \tilde{\omega}_{nl}. \quad (15)$$

Here $\tilde{\omega}_{nl}$ nonlinear frequency shift. Obviously

$$\tilde{\omega}_{nl} \sim c^2.$$

Thus, in the RHS of (14) gk can be replaced by ω^2. Taking into the account of $\tilde{\omega}_{nl}$ would give terms of higher order in (14). Such terms must be dropped out. So:

$$(\omega^2 - gk)c_{k\omega} = \frac{2\omega^3}{g^2}\frac{1}{(2\pi)^2}\int T_{\omega_2^2\omega_3^2}^{\omega^2\omega_1^2}c_{k_1\omega_1}^*c_{k_2\omega_2}c_{k_3\omega_3}\times$$
$$\times\,\delta_{k+k_1-k_2-k_3}\delta_{\omega+\omega_1-\omega_2-\omega_3}dk_1dk_2dk_3d\omega_1d\omega_2d\omega_3. \quad (16)$$

Now we can perform backward Fourier transformation of the Eq. (16) over space and get spatial equation for water waves:

$$\frac{\partial}{\partial x}c_\omega - i\frac{\omega^2}{g}c_\omega = $$
$$= -\frac{2\omega^3}{g^3}\frac{i}{2\pi}\int T_{\omega_2^2\omega_3^2}^{\omega^2\omega_1^2}c_{\omega_1}^*c_{\omega_2}c_{\omega_3}\delta_{\omega+\omega_1-\omega_2-\omega_3}d\omega_1d\omega_2d\omega_3.$$

This equation can be written in the Hamiltonian form:

$$\frac{\partial}{\partial x}c_\omega = i\omega^3\frac{\delta H}{\delta c_\omega^*}$$

with the third order bracket

$$i\omega^3 \leftrightarrow \frac{\partial^3}{\partial t^3}$$

and the following Hamiltonian:

$$H = \frac{1}{g}\int\frac{1}{\omega}|c_\omega|^2d\omega - $$
$$- \frac{1}{2\pi}\frac{1}{g^3}\int T_{\omega_2^2\omega_3^2}^{\omega^2\omega_1^2}c_\omega^*c_{\omega_1}^*c_{\omega_2}c_{\omega_3}\delta_{\omega+\omega_1-\omega_2-\omega_3}d\omega d\omega_1d\omega_2d\omega_3.$$

Explicit form of $T_{\omega_2^2\omega_3^2}^{\omega^2\omega_1^2}$ in (13) is the following:

$$T_{\omega_{k_2}^2\omega_{k_3}^2}^{\omega_k^2\omega_{k_1}^2} = \frac{1}{4}(\omega_k^2 + \omega_{k_1}^2 + \omega_{k_2}^2 + \omega_{k_3}^2 - $$
$$- |\omega_k^2 - \omega_{k_2}^2| - |\omega_k^2 - \omega_{k_3}^2| - |\omega_{k_1}^2 - \omega_{k_2}^2| - |\omega_{k_1}^2 - \omega_{k_3}^2|)$$

and it allows compact form of the quartic part $H_{\rm int}$ of the Hamiltonian:

$$H_{\text{int}} = \frac{1}{2g^3}\int |c|^2(\ddot{c}^* c + \ddot{c}c^*)dt + \frac{i}{g^2}\int |c|^2\hat{\omega}(\dot{c}c^* - c\dot{c}^*)dt.$$

Using the relation

$$\hat{\omega} = \hat{H}\frac{\partial}{\partial t} \ (\hat{H} \text{ is the Hilbert transformation}),$$

fourth order part of the Hamiltonian can written as following:

$$H_{\text{int}} = \frac{1}{2g^3}\int |c|^2(\ddot{c}^* c + \ddot{c}c^*)dt + \frac{i}{g^2}\int |c|^2\hat{H}(\ddot{c}c^* - c\ddot{c}^*)dt.$$

Equation of motion is:

$$\frac{\partial}{\partial x}c = \frac{\partial^3}{\partial t^3}\frac{\delta H}{\delta c^*} \quad (17)$$

or in t-space:

$$\frac{\partial}{\partial x}c + \frac{i}{g}\frac{\partial^2}{\partial t^2}c = \\ = \frac{1}{2g^3}\frac{\partial^3}{\partial t^3}\left[\frac{\partial^2}{\partial t^2}\left(|c|^2 c\right) + 2|c|^2\ddot{c} + \ddot{c}^* c^2\right] + \\ + \frac{i}{g^3}\frac{\partial^3}{\partial t^3}\left[\frac{\partial}{\partial t}\left(c\hat{\omega}|c|^2\right) + \dot{c}\hat{\omega}|c|^2 + c\hat{\omega}\left(\dot{c}c^* - c\dot{c}^*\right)\right]. \quad (18)$$

So Eq. (18) with the Hamiltonian

$$H = \frac{1}{g}\int \frac{1}{\omega}|c_\omega|^2 d\omega + \\ + \frac{1}{2g^3}\int |c|^2(\ddot{c}^* c + \ddot{c}c^*)dt + \frac{i}{g^2}\int |c|^2\hat{\omega}(\dot{c}c^* - c\dot{c}^*)dt \quad (19)$$

solves the spatial Cauchy problem for surface gravity wave on the deep water.

4. Back to $\boldsymbol{\eta}$ and $\boldsymbol{\psi}$. According to canonical transformation η_k and ψ_k are power series of c_k up to the third order:

$$\eta_k = \eta_k^{(1)} + \eta_k^{(2)} + \eta_k^{(3)}, \quad \psi_k = \psi_k^{(1)} + \psi_k^{(2)} + \psi_k^{(3)}. \quad (20)$$

Details of the recovering physical quantities $\eta(x,t)$ and $\psi(x,t)$ are given in [9, 11]. Here we focus on the η only. Obviously

$$\eta_k^{(1)} = \frac{1}{2\omega_k}[c_k + c_{-k}^*],$$

or

$$\eta^{(1)}(x,t) = \frac{1}{\sqrt{2}g^{1/4}}[\hat{k}^{-1/4}c(x,t) + \hat{k}^{-1/4}c(x,t)^*].$$

Operators $\hat{k}^\alpha$ act in Fourier space as multiplication by $|k|^\alpha$. Following [9, 11] let us consider transformation for η taking into account only first and second order terms. Then

$$\eta^{(2)}(x,t) = \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{-1/4}c(x,t) - \hat{k}^{-1/4}c^*(x,t)]^2. \quad (21)$$

Using approximate relation (15) one can get the following compact formula to get physical observed value η:

$$\eta(x,t) = \frac{\hat{\omega}^{-1/2}}{\sqrt{2}}[c(x,t) + c^*(x,t)] - \\ - \frac{\partial^2}{\partial t^2}\frac{1}{4g}\left[\hat{\omega}^{-1/2}[c(x,t) - c^*(x,t)]\right]^2 + \ldots$$

5. Frequency narrow band approximation. From Eq. (18) one can easily derive equation for envelope of modulated wave train. Obviously such a wave train propagates with the group velocity and it is convenient to introduce reference system moving with this velocity. So, let $c(x,t)$ is almost monochromatic wave with the frequency ω_0:

$$c(x,t) = C\left(x, t - \frac{x}{v_g}\right)e^{i(k_0 x - \omega_0 t)}, \\ \omega_0 = \sqrt{gk_0}, \quad v_g = \frac{\omega_0}{2k_0}, \quad (22)$$

where capital $C(x,t)$ is a slowly varying function. Plugging (22) into the motion equation (18), and keeping in the nonlinear part of the equation term with no more the first time derivative, one can derive the following equation:

$$\frac{\partial}{\partial x}C + \frac{i}{g}\frac{\partial^2}{\partial t^2}C + \frac{2i\omega_0^5}{g^3}|C|^2 C = \\ = \frac{4\omega_0^4}{g^3}\left[4|C|^2\dot{C} + \frac{3}{2}C^2\dot{C}^* + iC\hat{\omega}|C|^2\right]. \quad (23)$$

This is Dysthe equation for spatial Cauchy problem. Dropping the small corrections, namely the RHS, we end up just with Nonlinear Schrödinger equation. So, we have now both full dynamical equation (18) and envelope approximation (23). Hamiltonian of the NLSE is the following:

$$H = \frac{1}{g}\int \left[|\dot{C}|^2 - \frac{\omega_0^5}{g^2}|C|^4\right]dt$$

and equation of motion is:

$$\frac{\partial}{\partial x}C = i\frac{\partial H}{\partial C^*}.$$

6. Conclusion. The spatial compact equation is the most convenient tool for comparison of the theory of nonlinear gravity waves on deep water and their experimental study in laboratory wave tanks. It can be easily solved numerically by the use of spectral code. We plan to present the results of our numerical simulations shortly.

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