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Primitive potentials and bounded solutions of the KdV equation

S. Dyachenko [a,*], D. Zakharov [b], V. Zakharov [c,d,e]

[a] Department of Mathematics, University of Illinois, Urbana–Champaign, IL, 61801, United States
[b] Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY, 10012, United States
[c] Department of Mathematics, University of Arizona, Tucson, AZ, 85791, United States
[d] Lebedev Physical Institute RAS, Leninsky 53, Moscow 119991, Russia
[e] Novosibirsk State University, Novosibirsk 630090, Russia

HIGHLIGHTS

- We present an efficient procedure for constructing bounded, non-periodic solutions of the KdV equation.
- The present work is a key ingredient for the study of integrable turbulence and the statistical description of a solitonic gas.
- The analytical procedure is reinforced by numerical simulation that presents some of these potentials.

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ABSTRACT

We construct a broad class of bounded potentials of the one-dimensional Schrödinger operator that have the same spectral structure as periodic finite-gap potentials, but that are neither periodic nor quasi-periodic. Such potentials, which we call primitive, are non-uniquely parametrized by a pair of positive Hölder continuous functions defined on the allowed bands. Primitive potentials are constructed as solutions of a system of singular integral equations, which can be efficiently solved numerically. Simulations show that these potentials can have a disordered structure. Primitive potentials generate a broad class of bounded non-vanishing solutions of the KdV hierarchy, and we interpret them as an example of integrable turbulence in the framework of the KdV equation.

1. Introduction

It has been known since 1967 [1] that the analytic theory of the Korteweg–de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.1}$$

where $u = u(x, t)$, can be substantially advanced by presenting this equation as a compatibility condition for a pair of linear equations imposed on an auxiliary complex function $\psi(x, t)$. One of them is the Schrödinger equation on the real axis

$$-\psi_{xx} + u(x)\psi = E\psi, \quad -\infty < x < \infty, \tag{1.2}$$

and time evolution is given by:

$$\psi_t + \psi_{xxx} + 6u\psi_x + 3u_x\psi = 0. \tag{1.3}$$

The existence of this Lax representation [2] makes it possible to solve the initial value problem for KdV (1.1) by the Inverse Scattering Transform (IST), in the case when the potential $u(x, t)$ is a rapidly decaying function at $|x| \to \infty$:

$$\int_{-\infty}^{\infty} |u(x)|(1 + |x|)dx < \infty. \tag{1.4}$$

In this case, the Schrödinger operator (1.2) has a continuous spectrum on the positive semiaxis $E > 0$ and possibly a finite number of negative discrete energy levels.

The second important step in the development of the theory of the KdV equation was done in the early seventies [3–5], when it was established that this equation has an infinite number of families of exact solutions, called finite-gap solutions. The spectrum of the Schrödinger equation with an N-gap potential consists of $N + 1$ allowed bands separated by N forbidden bands. These finite-gap solutions are explicitly expressed in terms of θ-functions of hyperelliptic algebraic curves. All these solutions are quasi-periodic, and a certain subset of them are periodic. Periodic

* Corresponding author.
E-mail address: sdyachen@illinois.edu (S. Dyachenko).

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finite-gap solutions form a dense subset of all periodic potentials, which enables us to solve the initial value problem in this case.

In spite of all these brilliant achievements, the theory of the KdV equation is not yet developed to a level which would satisfy a pragmatic physicist, who may ask the following question: What happens if the initial data in the KdV equation is neither decaying at infinity nor periodic? Suppose that the initial data is a bounded function

$$u(x) = u(x, 0), \quad |u(x)| < c.$$

Can we extend the IST to this case, which has great practical importance? Moreover, what happens if $u(x)$ is a random statistically uniform field? In this case the KdV equation describes what we call *integrable turbulence* [6]. Can we say anything about the correlation functions and spectra of this turbulence?

These questions are hard to answer and refer to another fundamental mathematical question. What is the spectrum of operator (1.2) if the potential is a bounded function? By definition, a value of E belongs to the spectrum of $u(x)$ if there exists one or two independent bounded wave functions $\psi(x, E)$:

$$|\psi(x, E)| < 1, \quad -\infty < x < \infty.$$

The spectrum is a subset of the axis $-\infty < E < \infty$, and can have a quite complicated structure. We know that the spectrum is Lebesgue measurable, but its measure could be zero, finite or infinite. When the measure is zero, the set could be countable or uncountable. The question formulated above is very difficult.

In this paper we formulate the reverse question. Suppose we know the spectrum. What we can say about the potential $u(x)$? We assert that this question is much easier. First of all, we note that evolving the potential along the KdV equation, or along any equation of the KdV hierarchy, does not change the spectrum set. Hence we have to speak about description of classes of potentials having a given spectral set. This ambitious program must start from the simplest nontrivial case, when the spectrum of the Schrödinger operator with a non-decaying potential consists of the whole positive semiaxis $E > 0$ and one allowed band in the negative semiaxis:

$$-k_2^2 < E < -k_1^2, \quad k_2 > k_1 > 0.$$

Given that the one-gap potential is periodic, it is determined up to translation by the formula

$$u(x) = u_0(x) = 2\wp(x + i\omega' - x_0) + e_3. \tag{1.5}$$

Here $\wp(x)$ is the elliptic Weierstrass function with periods 2ω and $2i\omega'$. We have

$$e_1 - e_3 = k_2^2, \qquad e_2 - e_3 = k_1^2, \qquad e_1 + e_2 + e_3 = 0 \tag{1.6}$$

where $e_1 > e_2 > e_3$.

The spectrum is doubly degenerate and reflectionless, and within the allowed bands a quantum particle moves freely in both directions. We construct potentials that have the same spectrum and that are reflectionless in the infinite band, but that are not periodic. A general one-gap reflectionless potential is determined by two positive Hölder continuous functions $R_1(\kappa)$ and $R_2(\kappa)$, defined inside the allowed gap. For an even potential $u(-x) = u(x)$, we have $R_1 = R_2$.

To construct these potentials, we consider the closure of the set of reflectionless Bargmann potentials, also known as N-soliton potentials, as $N \to \infty$. This problem was posed and formally solved in the works of Marchenko and his students [7–9], but the obtained results are not effective. In this paper we consider a new technique for constructing the closure of the Bargmann potentials, based on "transplanting poles". We show that this technique is quite effective. In particular, we construct the periodic potential (1.5) as a limit of N-soliton solutions.

2. Bargmann potentials via the dressing method

Bargmann potentials were first constructed in 1948 in [10] as a class of potentials of the one-dimensional Schrödinger operator (1.2) having N bound states with negative energy and zero reflection coefficient for all positive energies. From the point of view of the KdV equation, Bargmann potentials correspond to N-soliton solutions at fixed moments of time, and hence can be explicitly constructed using the inverse spectral transform for the operator (1.2). In this section, we give an alternate construction of the Bargmann potentials using the so-called dressing method, following Zakharov and Manakov [11]. Compared with the IST, this method gives us additional flexibility that will later prove crucial when we generalize to the Riemann–Hilbert problem.

We consider a $\bar\partial$-problem on the complex k-plane of the following kind:

$$\frac{\partial \chi}{\partial \bar k} = ie^{2ikx} T(k) \chi(-k, x). \tag{2.1}$$

Here $T(k)$ is a compactly supported distribution called the *dressing function* of the $\bar\partial$-problem. A solution of (2.1) is defined up to multiplication by a function of x, hence if a solution exists we can normalize it by the condition $\chi \to 1$ as $|k| \to \infty$. Such a solution satisfies the integral equation

$$\chi(k, x) = 1 + \frac{i}{\pi} \iint \frac{e^{-2iqx} T(-q) \chi(q, x)}{k + q} dq d\bar q, \tag{2.2}$$

where we normalize the integral in the following way:

$$\frac{1}{k} = \lim_{\varepsilon \to 0} \frac{\bar k}{|k|^2 + \varepsilon^2}, \qquad \frac{\partial}{\partial \bar k}\left(\frac{1}{k}\right) = \pi \delta(k). \tag{2.3}$$

Here $\delta(k)$ is the two-dimensional δ-function.

We assume also that

$$\bar T(\bar k, k) = T(k, \bar k).$$

Note, that the dressing function is not analytic.

We now show that a solution of the $\bar\partial$-problem (2.1) gives rise to a solution of the Schrödinger equation (1.2). Suppose that the dressing function has property that the integral equation (2.2) has a unique solution for x inside a certain interval $x_2 < x < x_1$. Then

$$\chi(k, x) = 1 + \frac{i\chi_0(x)}{k} + O(k^{-2}), \qquad u(x) = 2\frac{d}{dx}\chi_0(x). \tag{2.4}$$

Denote

$$\xi(k, x) = \chi_{xx} - 2ik\chi_x - u(x)\,\chi. \tag{2.5}$$

It is straightforward to check that ξ also satisfies the $\bar\partial$-problem (2.1), and the choice of $u(x)$ guarantees that $\xi \to 0$ as $|k| \to \infty$. By the uniqueness assumption, it follows that ξ is identically equal to zero, hence the function $\chi(k, x)$ is a solution of the differential equation

$$\chi_{xx} - 2ik\chi_x - u(x)\chi = 0, \tag{2.6}$$

and the function $\psi(x, k) = \chi(x, k)e^{-ikx}$ is a solution of the Schrödinger equation (1.2) with $E = k^2$, which completes the proof.

We obtain the class of reflectionless Bargmann potentials by considering a $\bar\partial$-problem whose solution χ is a rational function of k with simple poles along the imaginary axis.

Let $\kappa_1, \dots, \kappa_N$ and $c_1, \dots, c_n$ be a collection of nonzero real numbers satisfying the following properties:

1. $|\kappa_m| < |\kappa_n|$ for all $m < n$,
2. $c_n/\kappa_n > 0$ for all n.

Consider the dressing function

$$T(k) = \pi \sum_{n=1}^{N} c_n \delta(k - i\kappa_n). \tag{2.7}$$

Then the $\overline{\partial}$-problem (2.1) has a unique solution χ satisfying the normalization condition $\chi \to 1$ as $|k| \to \infty$. This solution is a rational function of k having simple poles at the points $k = i\kappa_n$ for $n = 1, \dots, N$, and has the following form:

$$\chi(k, x) = 1 + i \sum_{n=1}^{N} \frac{\chi_n(x)}{k - i\kappa_n}, \tag{2.8}$$

where the $\chi_n(x)$ are real-valued functions. The corresponding potential

$$u(x) = 2\frac{d}{dx} \sum_{n=1}^{N} \chi_n(x)$$

is a reflectionless Bargmann potential having the finite discrete spectrum $-\kappa_1^2, \dots, -\kappa_N^2$, and $\psi_n(x) = \chi_n(x) e^{\kappa_n x}$ are the corresponding eigenfunctions.

Given a reflectionless Bargmann potential $u(x)$ with a finite negative discrete spectrum $-\kappa_1^2, \dots, -\kappa_N^2$, the direct spectral transform proceeds by constructing a solution $\psi(k, x)$ of the Schrödinger equation (1.2) that is analytic in the k-upper half-plane. In the k-lower half-plane, the function $\psi(k, x)$, and hence the function $\chi(k, x) = \psi(k, x) e^{ikx}$, then has poles on the negative imaginary axis at the points $-i|\kappa_1|, \dots, -i|\kappa_N|$ corresponding to the discrete spectrum. To construct $u(x)$ using the dressing method, we can place the poles of χ on both the positive and negative imaginary axes, so long as the poles have distinct absolute values, and every N-soliton Bargmann potential can be constructed in 2^N different ways by arbitrarily choosing the signs of the κ_n.

It is possible to relax the condition that c_n and κ_n have the same sign for each n, but the corresponding potentials $u(x)$ will then be singular functions of x.

Given the dressing function (2.7), the identity (2.1) implies that a solution χ has simple poles at the points $k = i\kappa_n$ and no other singularities. The condition $\chi \to 1$ as $|k| \to \infty$ then implies that χ has the form (2.7). Substituting this into the integral equation (2.2), we obtain a system of linear equations on the residues $\chi_n(x)$:

$$\chi_n(x) = e^{-2\kappa_n x} c_n \chi(-i\kappa_n, x). \tag{2.9}$$

Writing this system out explicitly, and replacing $\chi_n(x) = \psi_n(x) e^{\kappa_n x}$, we obtain the following system:

$$\psi_n(x) + c_n \sum_{m=1}^{N} \frac{e^{-(\kappa_n + \kappa_m)x}}{\kappa_n + \kappa_m} \psi_m(x) = c_n e^{-\kappa_n x}. \tag{2.10}$$

The matrix of this system

$$A_{nm} = \delta_{nm} + \frac{c_n e^{-(\kappa_n + \kappa_m)x}}{\kappa_n + \kappa_m}$$

is the sum of an identity matrix and a Cauchy-like matrix, therefore its determinant is the sum of the principal minors of the Cauchy-like-matrix. This sum is indexed by subsets $I = \{i_1, \dots, i_n\}$ of the index set $\{1, \dots, N\}$ and can be explicitly evaluated as follows:

$$A = \det A_{nm} = \sum_{I \subset \{1, \dots, N\}} \left[\prod_{\{i,j\} \subset I, i<j} \frac{(\kappa_i - \kappa_j)^2}{(\kappa_i + \kappa_j)^2} \prod_{i \in I} \frac{c_i}{2\kappa_i} e^{-2\kappa_i x} \right].$$

By assumption, the quantities c_i/κ_i and $(\kappa_i - \kappa_j)^2$ are all positive, therefore each summand and hence all of A is positive, so the system (2.10) has a unique solution. By our previous statement, χ satisfies Eq. (2.9), and the corresponding potential $u(x)$ is

$$u(x) = 2\frac{d\chi_0}{dx} = 2\frac{d}{dx} \sum_{n=1}^{N} \chi_n(x). \tag{2.11}$$

We evaluate $u(x)$ by Cramer's rule to obtain

$$\begin{aligned} u(x) = 2\frac{d}{dx} \sum_{n=1}^{N} \chi_n(x) &= 2\frac{d}{dx} \sum_{n=1}^{N} \psi_n(x) e^{\kappa_n x} \\ &= 2\frac{d}{dx} \left[-\frac{1}{A} \frac{d}{dx} A \right] = -2\frac{d^2}{dx^2} \ln A. \end{aligned} \tag{2.12}$$

When all the κ_n are positive, this is the familiar formula for the N-soliton reflectionless potentials (see for example [5], formula (1.5)).

To finish the proof, we consider what happens to formula (2.12) when we change the signs of one of the κ_n. A direct calculation shows that

$$A = \frac{c_n}{2\kappa_n} e^{-2\kappa_n x} \widetilde{A},$$

where $\widetilde{A}$ is the determinant of the matrix A_{nm} corresponding to the data $(\widetilde{\kappa}_i, \widetilde{c}_i)$, where

$$\widetilde{\kappa}_i = \begin{cases} \kappa_i, & i \neq n, \\ -\kappa_n, & i = n, \end{cases} \qquad \widetilde{c}_i = \begin{cases} \left(\dfrac{\kappa_i - \kappa_n}{\kappa_i + \kappa_n} \right)^2 c_i, & i \neq n, \\ -4\kappa_n^2/c_n, & i = n. \end{cases}$$

By formula (2.11), the data (κ_i, c_i) and $(\widetilde{\kappa}_i, \widetilde{c}_i)$ determine the same potential $u(x)$. Hence, starting with an arbitrary (κ_i, c_i), we can make all κ_i positive and make the corresponding changes to the c_i while preserving $u(x)$, so in fact all of the potentials that we obtain in this way are reflectionless Bargmann potentials.

Finally, considering the leading term in Eq. (2.6) near the poles, we see that ψ_n are eigenfunctions of the Schrödinger operator with potential $u(x)$ corresponding to the eigenvalues $-\kappa_n^2$.

3. The symmetric Riemann–Hilbert problem

In this section, we consider a Riemann–Hilbert problem that is a continuous analogue of the finite $\overline{\partial}$-problem that generates the Bargmann potentials.

Let $0 < k_1 < k_2$ be real numbers, and let R_1 and R_2 be two non-negative continuous functions on the interval $[k_1, k_2]$, satisfying the Hölder condition. Consider the dressing function

$$\begin{aligned} T(k) = \pi \int_{k_1}^{k_2} R_1(p) \delta(k - ip) dp \\ - \pi \int_{k_1}^{k_2} R_2(p) \delta(k + ip) dp. \end{aligned} \tag{3.1}$$

Then the corresponding $\overline{\partial}$-problem (2.1) has a unique solution χ satisfying the normalization condition $\chi \to 1$ as $|k| \to \infty$. This function is analytic on the k-plane away from two cuts $[ik_1, ik_2]$ and $[-ik_2, -ik_1]$ on the imaginary axis. Denote by χ^+ and χ^- the right and left boundary values of χ along the cuts:

$$\chi^+(x, ip) = \lim_{\varepsilon \to 0} \chi(x, ip + \varepsilon),$$
$$\chi^-(x, ip) = \lim_{\varepsilon \to 0} \chi(x, ip - \varepsilon),$$

where $k_1 < |p| < k_2$. It is easy to verify that the function $\chi(x, ip)$ satisfies the following contour problem on the cuts:

$$\frac{\chi^+(ip) - \chi^-(ip)}{i\pi} = R_1(p)e^{-2px}\left[\chi^+(-ip) + \chi^-(-ip)\right], \tag{3.2}$$

$$\frac{\chi^+(-ip) - \chi^-(-ip)}{i\pi} = -R_2(p)e^{2px}\left[\chi^+(ip) + \chi^-(ip)\right]. \tag{3.3}$$

Here the dependence of $\chi(x, ip)$ on the x variable is dropped to improve readability.

This is a scalar, but non-local, Riemann–Hilbert problem, and it is equivalent to a local vector Riemann–Hilbert problem. Denote $\Xi(k) = [\chi(k)\ \ \chi(-k)]^T$, and let Ξ^+ and Ξ^- be the right and left values of Ξ on the cuts. Then (3.2)–(3.3) are equivalent to

$$\Xi^+(i\kappa) = M(\kappa)\Xi^-(i\kappa), \qquad \Xi^+(-i\kappa) = M^T(\kappa)\Xi^-(-i\kappa) \tag{3.4}$$

for $\kappa \in [k_1, k_2]$, where the transition matrix is

$$M(x, \kappa) = \frac{1}{1 + R_1R_2}\begin{bmatrix} 1 - R_1R_2 & 2iR_1e^{-2\kappa x} \\ 2iR_2e^{2\kappa x} & 1 - R_1R_2. \end{bmatrix}$$

A function χ, analytic on the k-plane, having jumps along the cuts $[ik_1, ik_2]$ and $[-ik_2, -ik_1]$, and satisfying the normalization $\chi \to 1$ as $|k| \to \infty$ can be written as

$$\chi(x, k) = 1 + i\int_{k_1}^{k_2} \frac{f(x, p)}{k - ip}dp + i\int_{k_1}^{k_2} \frac{g(x, p)}{k + ip}dp, \tag{3.5}$$

where $f(x, p)$ and $g(x, p)$ are real-valued functions defined for real x and for $p \in [k_1, k_2]$. The corresponding potential of the Schrödinger operator (1.2) is

$$u(x) = 2\frac{d}{dx}\int_{k_1}^{k_2} [f(x, p) + g(x, p)]dp.$$

Given R_1 and R_2, we look for a solution of the $\overline{\partial}$-problem (2.1) in the form (3.5), where f and g are unknown functions of x and $p \in [k_1, k_2]$. The jumps of χ along the cuts are then equal to

$$\chi^+(x, ip) - \chi^-(x, ip) = 2\pi i f(x, p),$$
$$\chi^+(x, -ip) - \chi^-(x, -ip) = 2\pi i g(x, p).$$

Plugging (3.5) into (2.1), we see that χ satisfies the Riemann–Hilbert problem (3.2)–(3.3) if f and g satisfy the following system of singular integral equations:

$$f(x, p) + R_1(p)e^{-2px}\left[\int_{k_1}^{k_2} \frac{f(x, q)}{p + q}dq + \fint_{k_1}^{k_2} \frac{g(x, q)}{p - q}dq\right] = R_1(p)e^{-2px} \tag{3.6}$$

$$g(x, p) + R_2(p)e^{2px}\left[\fint_{k_1}^{k_2} \frac{f(x, q)}{p - q}dq + \int_{k_1}^{k_2} \frac{g(x, q)}{p + q}dq\right] = -R_2(p)e^{2px}. \tag{3.7}$$

We note that the Riemann–Hilbert problem (3.2)–(3.3) is a continuous generalization of Eq. (2.9). We need to show that the system (3.6)–(3.7) has a unique solution on the entire real axis. Let us formulate the idea of the proof. A more rigorous proof will be published separately.

We approximate these equations by Riemann sums. Fix an integer N, and let $\Delta = (k_2 - k_1)/2N$. We subdivide the segment $[k_1, k_2]$ into $2N$ equal parts and denote

$$\lambda_1 = k_1, \quad \mu_1 = k_1 + \Delta, \quad \lambda_2 = k_1 + 2\Delta, \quad \mu_2 = k_1 + 3\Delta, \ldots$$

Denoting

$$f_n(x) = f(x, \lambda_n), \qquad g_n(x) = g(x, \mu_n),$$
$$\alpha_n = \Delta R_1(\lambda_n), \qquad \beta_n = -\Delta R_2(\mu_n)$$

we approximate the Riemann–Hilbert problem (3.2)–(3.3) by replacing the integrals containing f with their Riemann sums at the λ_n and the integrals containing g with their Riemann sums at the μ_n:

$$f_n(x) + \alpha_n e^{-2\lambda_n x}\left(\sum_{m=1}^{N+1} \frac{f_m(x)}{\lambda_n + \lambda_m} + \sum_{m=1}^{N} \frac{g_m(x)}{\lambda_n - \mu_m}\right) = \alpha_n e^{-2\lambda_n x}, \tag{3.8}$$

$$g_n(x) + \beta_n e^{2\mu_n x}\left(\sum_{m=1}^{N+1} \frac{f_m(x)}{-\mu_n + \lambda_m} + \sum_{m=1}^{N} \frac{g_m(x)}{-\mu_n - \mu_m}\right) = \beta_n e^{2\mu_n x}. \tag{3.9}$$

We see that this system is equivalent to the system (2.9) on the eigenfunctions of a Bargmann potential having $2N + 1$ solitons corresponding to the poles $(\lambda_1, \ldots, \lambda_{N+1}, -\mu_1, \ldots, -\mu_N)$ and the constants $(\alpha_1, \ldots, \alpha_{N+1}, -\beta_1, \ldots, -\beta_N)$.

According to the results of the last paragraph, this system has a unique solution for all x and gives a Bargmann potential with $2N + 1$ solitons, which is bounded uniformly in N. We can solve these equations iteratively:

$$\widetilde{f}_n(x) = 1 + \widetilde{f}_n^{(1)}(x) + \cdots, \qquad \widetilde{g}_n(x) = 1 + \widetilde{g}_n^{(1)}(x) + \cdots. \tag{3.10}$$

We get

$$\widetilde{f}_n^{(1)}(x) = \sum_{m=1} \frac{\alpha_m e^{-2\alpha_m x}}{\lambda_n + \lambda_m} + \sum_{m=1} \frac{\beta_m e^{2\beta_m x}}{\lambda_n - \mu_m} \tag{3.11}$$

and a similar expression for $\tilde{g}_n^{(1)}(x)$. The question is: can we turn the sum (3.11) into an integral? Notice that x in (3.8), (3.9) is a parameter, which we assume to be contained inside a certain interval. The transition to an integral holds for $-L < x < L$, where

$$2\Delta R e^{\kappa_2 L} \ll 1, \quad R = \max(R_1(p), R_2(p)). \tag{3.12}$$

To increase L, we need to exponentially increase N:

$$N \simeq e^{\kappa_2 L}. \tag{3.13}$$

Nevertheless, for N sufficiently large, it is possible to include any point of the real axis in the interval $(-L, L)$. Since the systems (3.2)–(3.3) describe $2N + 1$-soliton solutions, Eqs. (3.8)–(3.9) are uniquely solvable for all x. The resulting potentials are bounded in both directions. According to results obtained by A. Shabat [12] the potential $u(x)$ is strictly negative and satisfies condition

$$-2k_2^2 < u(x) < 0. \tag{3.14}$$

Proceeding as before, we determine that the eigenfunctions

$$\varphi(x, \kappa) = f(x, \kappa)e^{\kappa x}, \qquad \psi(x, \kappa) = g(x, \kappa)e^{-\kappa x} \tag{3.15}$$

are bounded and orthonormal:

$$\int_{-\infty}^{\infty} \varphi(x, \kappa)\varphi(x, \kappa')dx = R_1(\kappa)\delta(\kappa - \kappa'), \tag{3.16}$$

$$\int_{-\infty}^{\infty} \psi(x, \kappa)\varphi(x, \kappa')dx = 0, \tag{3.17}$$

$$\int_{-\infty}^{\infty} \psi(x, \kappa)\psi(x, \kappa')dx = R_2(\kappa)\delta(\kappa - \kappa'). \tag{3.18}$$

The functions $f(x, \kappa)$ and $g(x, \kappa)$ grow exponentially as $x \to -\infty$ and $x \to \infty$, respectively. However, the functions $\phi(x, k)$, $\psi(x, k)$ remain bounded on $-\infty < x < \infty$.

The function $\chi_0(x)$ grows linearly in both directions:

$$\chi_0 = -c_1 x + \chi_0(x) \quad \text{as } x \to -\infty; \tag{3.19}$$

$$\chi_0 = -c_2 x + \chi_0(x) \quad \text{as } x \to +\infty, \tag{3.20}$$

where $|\chi_0(x)| < const$ for $-\infty < x < \infty$.

We note that in the κ-region, in which both functions $R_1(\kappa)$ and $R_2(\kappa)$ are strictly positive, the spectrum is doubly degenerate, and the potential is primitive. All these statements are supported by numerical experiments.

If we assume that $R_1(\kappa) = R_2(\kappa)$, then

$$g(x, \kappa) = -f(-x, \kappa), \tag{3.21}$$

and the potential is symmetric $u(-x) = u(x)$. We note that for $R_1(\kappa) = R_2(\kappa)$ every finite approximation only gives an approximately symmetric potential, however, the accuracy of symmetry grows exponentially as $N \to \infty$.

Let us now assume that $R_2(\kappa) \equiv 0$. Then Eq. (3.7) is no longer singular and is a regular Fredholm equation of the second kind:

$$f(x, \kappa) + R_1(\kappa) e^{-2\kappa x} \int_{k_1}^{k_2} \frac{f(x, q)}{x + q} dq = R_1(\kappa) e^{-2\kappa x}. \tag{3.22}$$

It follows from what we said above that

$$f(x, \kappa) \to R_1(\kappa) e^{-2\kappa x}, \text{ as } x \to +\infty. \tag{3.23}$$

However, studying the asymptotic behavior as $x \to -\infty$ is a very difficult problem, because taking the naive limit $x \to -\infty$ gives a Fredholm equation of the first kind

$$\int_{k_2}^{k_1} \frac{f(x, q)}{x + q} dq = 1, \tag{3.24}$$

which has no solutions. Therefore, we cannot discard the term $f(x, \kappa) e^{2\kappa x}$ as $x \to \infty$, since it may not be small. Studying the asymptotic behavior as $x \to -\infty$ is a difficult and so far unsolved problem. An even more difficult problem is the asymptotic behavior of the general Eqs. (3.6) and (3.7) as $x \to \pm\infty$.

Now assume that both functions $R_1 = R_2$ are zero on the set of N subintervals inside the interval $k_1 < \kappa < k_2$. In this case the spectrum consists of $N + 1$ allowed bands, including the positive semi-axis, separated by N lacunae. We hypothesize that all N-gap potentials can be obtained in this way, and prove this for $N = 1$.

4. Periodic one-gap potentials

In this section, we show that periodic one-gap potentials of the Schrödinger operator can be constructed from the symmetric Riemann–Hilbert problem.

Let ω and ω' be positive real numbers, and consider the elliptic curve $E = \mathbb{C}/\Lambda$, where Λ is the period lattice generated by 2ω and $2i\omega'$. Denote by $\wp(z)$ the Weierstrass elliptic function associated to the lattice Λ (see Fig. 1). It satisfies the differential equation

$$\begin{aligned}[\wp'(z)]^2 &= 4\wp(z)^3 - g_2\wp(z) - g_3 \\ &= 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3),\end{aligned}$$

where the zeros e_1, e_2, e_3 are real-valued, satisfy $e_1 + e_2 + e_3 = 0$, and we assume that $e_3 < e_2 < e_1$.

The function

$$u(x) = 2\wp(x - \omega - i\omega') + e_3 \tag{4.1}$$

is a real-valued potential of the Schrödinger operator (1.2) with period 2ω. Our goal is to construct a solution of (1.2) that gives a solution of the symmetric Riemann–Hilbert problem.

We consider the following function $\varphi(x, z)$, where x is real and z is defined on the curve E:

$$\varphi(x, z) = \frac{\sigma(x - \omega - i\omega' + z)\sigma(\omega + i\omega')}{\sigma(x - \omega - i\omega')\sigma(\omega + i\omega' - z)} \exp[-\zeta(z)x]. \tag{4.2}$$

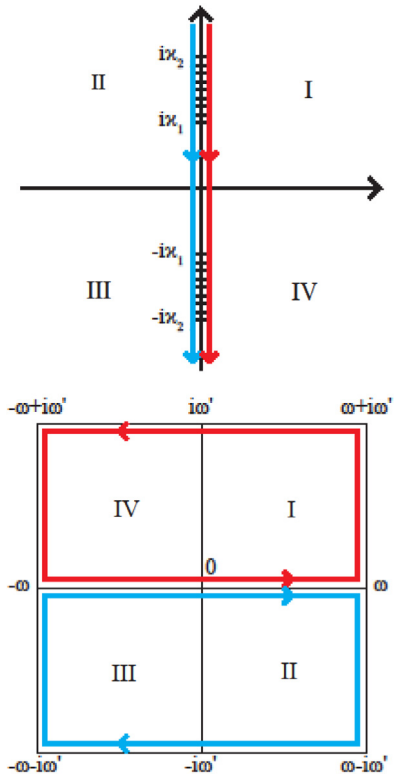

Fig. 1. (Top) k-plane and (Bottom) z-plane.

Here σ and ζ are the Weierstrass elliptic functions. A direct calculation shows that φ satisfies the Lamé equation

$$\varphi'' - [2\wp(x - \omega - i\omega') + \wp(z)]\varphi = 0.$$

Hence we see that φ is a solution of the Schrödinger equation (1.2) with potential (4.2) if the parameter z satisfies the relation

$$k^2 = e_3 - \wp(z). \tag{4.3}$$

The Weierstrass function $\wp$ has degree two, hence for a generic complex value of k there are two values of z on E that satisfy (4.3). In order to make the function (4.2) a single-valued function of k, we need to choose a branch of z. We choose the solution $z(k)$ of (4.3) that satisfies

$$z(k) = \frac{i}{k} + O\left(\frac{1}{k^2}\right) \quad \text{as } |k| \to \infty. \tag{4.4}$$

This branch defines a single-sheeted map from the complex k-plane with two cuts on the imaginary axis to a period rectangle of the lattice Λ centered at 0. The cuts on the imaginary axis are $[-ik_2, -ik_1]$ and $[ik_1, ik_2]$, where

$$k_1 = \sqrt{e_2 - e_3}, \qquad k_2 = \sqrt{e_1 - e_3}.$$

The right and left sides of the top cut $[ik_1, ik_2]$ are mapped to the line segments joining ω to $\omega + i\omega'$ and $\omega - i\omega'$, respectively, and the right and left sides of the bottom cut $[-ik_2, -ik_1]$ are respectively mapped to the segments joining $-\omega$ to $-\omega + i\omega'$ and $-\omega - i\omega'$.

The function φ satisfies the following properties:

$$\varphi(x, z + 2\omega) = \varphi(x, z), \qquad \varphi(x, z + 2i\omega') = \varphi(x, z),$$
$$\overline{\varphi}(x, z) = \varphi(x, z) \quad \text{when } \overline{z} = z, \overline{x} = x.$$

Also

$$\overline{\varphi}(x, z) = \varphi(x, \overline{z})$$

for all z having real part ω.

Let $z(k)$ be the branch of the solution of (4.3) satisfying (4.4). Let $f(k)$ be the branch of the function

$$f(k) = \sqrt{\frac{k+ik_1}{k+ik_2}}$$

satisfying $f(k) \to 1$ as $|k| \to \infty$. On the complex k-plane with two cuts $[ik_1, ik_2]$ and $[-ik_2, -ik_1]$ along the imaginary axis, define the function

$$\xi(x,k) = f(k)\varphi(x, z(k))e^{-ikx}.$$

Then the function $\xi(x,k)$ satisfies the equation

$$\xi'' + 2ik\xi' - u(x)\xi = 0, \qquad \xi \to 1 \text{ as } |k| \to \infty$$

with potential $u(x)$ given by (4.1). On the cuts, the function $\xi(x,k)$ satisfies the Riemann–Hilbert problem

$$\frac{\xi^+(iq) - \xi^-(iq)}{i\pi} = R_1(q)e^{2qx}\left[\xi^+(-iq) + \xi^-(-iq)\right],$$
$$\frac{\xi^+(-iq) - \xi^-(-iq)}{i\pi} = -R_2(q)e^{-2qx}\left[\xi^+(iq) + \xi^-(iq)\right],$$

where dependence of ξ on its first variable is omitted to improve readability. Here $q \in [k_1, k_2]$, and $\xi^+(x, \pm iq)$ are the right hand values of the upper and lower cuts, and $\xi^-(x, \pm iq)$ are the left hand values on the upper and lower cuts. The functions R_1 and R_2 are

$$R_1(q) = \frac{1}{\pi}h(q), \qquad R_2(q) = \frac{1}{\pi h(q)},$$
$$h(q) = \sqrt{\frac{(q-k_1)(q+k_2)}{(k_2-q)(q+k_1)}}. \tag{4.5}$$

5. Solutions of integrable systems

Suppose that the wave function and potential of the Schrödinger equation (1.2) depend on time t in the following way:

$$\psi_t + 48\psi_x + 4\psi_{xxx} + 3u_x\psi = 0, \tag{5.1}$$
$$u_t + 48u_x - 6uu_x + u_{xxx} = 0.$$

In other words $u(x,t)$ is a solution of the KdV equation (see [1]) in a moving frame. The KdV equation preserves the spectrum, and transforms any primitive potential into another. The solution $u(x,t)$ is obtained by modifying the dressing problem (2.1) by replacing the exponent with $2ix + 8ik^3t$ (see [11]). The dressing functions are transformed as follows:

$$R_1(\kappa) \to R_1(\kappa)e^{S(\kappa)t}, \qquad R_2(\kappa) \to R_2(\kappa)e^{-S(\kappa)t}. \tag{5.2}$$

Here $S(\kappa) = 8(\kappa^3 - 12\kappa)$. The time evolution (5.2) transforms the Schrödinger operator into a unitary equivalent one with a different potential, and the same is true for evolution under higher KdV flows. For higher KdV flows, $S(\kappa)$ must be replaced by some odd polynomial on κ. But as far as any continuous function on a finite interval can be approximated by odd polynomials, one can consider that $S(\kappa)$ in (5.2) is an arbitrary continuous function. Then we come to an important conclusion: The unique invariant of unitary equivalence is the product $\omega(\kappa) = R_1(\kappa)R_2(\kappa)$. It means that:

1. All Bargmann potentials with the same energy levels are unitary equivalent to each other. Moreover, if the spectrum is non-degenerate (single), all operators with the same spectrum are unitary equivalent to each other.
2. All finite-gap potentials with the same band structure are unitary equivalent to each other.

In particular, if we put $h(\kappa) = 1$ in (4.5) we get $R_1(\kappa) = R_2(\kappa) = 1/\pi$. Again, this dressing gives a periodic potential.

6. Numerical experiments

We solve the system of integral equations associated to (2.1) numerically for $k_1 = 2$ and $k_2 = 4$. Denote $\kappa = p + 3$, $(-1 < p < 1)$. It is convenient to replace $\phi(x,\kappa)$ and $\psi(x,\kappa)$ with the following functions:

$$P(x,p) = \sqrt{1-p^2}\phi(x, p+3),$$
$$Q(x,p) = \sqrt{1-p^2}\psi(x, p+3).$$

The system of integral equations (3.6), (3.7) becomes

$$P(x,p) + r_1(p)e^{-2(3+p)x}\left[\int_{-1}^{1}\frac{P(x,q)e^{-qx}}{(6+p+q)\sqrt{1-q^2}}\,dq + \fint_{-1}^{1}\frac{Q(x,q)e^{qx}}{(k-q)\sqrt{1-q^2}}dq\right] = r_1(p)e^{-2(3+p)x}, \tag{6.1}$$

and

$$Q(x,p) + r_2(p)e^{2(3+p)x}\left[\fint_{-1}^{1}\frac{P(x,q)e^{qx}}{(6+p+q)\sqrt{1-q^2}} + \int_{-1}^{1}\frac{Q(x,q)e^{-qx}\,dq}{(k-q)\sqrt{1-q^2}}\right] = -r_2(p)e^{2(3+p)x}. \tag{6.2}$$

Here we denote

$$r_{1,2}(p) = \sqrt{1-q^2}\,R_{1,2}(p+3) = \sqrt{1-p^2}\,\tilde{R}(p).$$

The continuous functions $P(x,q)$ and $Q(x,q)$ are discretized at Chebyshev nodes $q_k = \cos\frac{(2k-1)\pi}{2M}$ with $k = 1, 2, \ldots, M$. The integrals are evaluated via Gauss–Chebyshev quadrature that is exact for polynomials of degree less than $2M-1$. Note that each equation of the system contains a Cauchy principal value integral denoted by $\fint$, and that integration in the vicinity of singularity at $q = k$ requires a shift from the real axis.

The spatial variable x appears as a parameter in (6.1)–(6.2) and the x-dependence of r_1 and r_2 becomes a major obstacle, since the condition number of the discretized system is exponential in x and requires usage of multiprecision arithmetics.

It is convenient to choose Chebyshev nodes to discretize the parameter x and have efficient high-order polynomial interpolation. Lagrange interpolation is used to determine values of $P(x,q)$ and $Q(x,q)$ at intermediate points in x. Having simulations performed with arbitrary precision, we typically use interpolating polynomials up to degree 200 to have an accurate approximation of the solution in the range $|x| < 10$ without any loss of precision.

We performed numerical experiments using different choices of the dressing functions $\tilde{R}_1(p)$, $\tilde{R}_2(p)$.

6.1. Simulation 1. Periodic potential

To obtain a periodic potential we choose dressing functions in the following form:

$$R_1(p) = \frac{1}{\pi} \quad \text{and} \quad r_1(p) = \frac{1}{\pi}\sqrt{1-p^2}e^{-(R+p)x}$$
$$R_2(p) = \frac{1}{\pi} \quad \text{and} \quad r_2(p) = \frac{1}{\pi}\sqrt{1-p^2}e^{+(R+p)x}$$

where $-1 \le p \le 1$.

A simple modification of dressings above:

$$h(p) = \sqrt{\frac{(1-p)(p+5)}{(1+p)(p+7)}}$$
$$R_1(p) = \frac{1}{\pi}h(p) \quad \text{and} \quad r_1(p) = \frac{1}{\pi}\sqrt{1-p^2}h(p)e^{-(R+p)x}$$

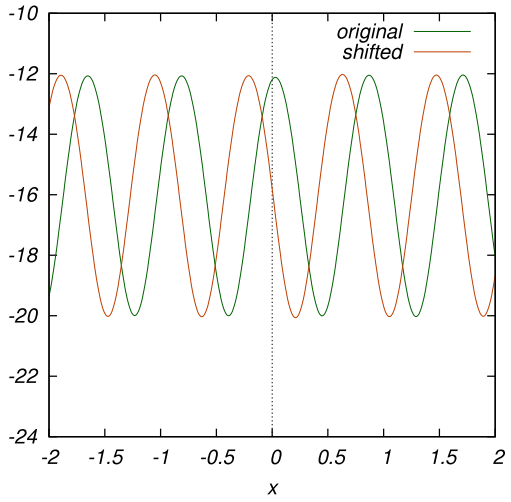

Fig. 2. Potential $U(x)$ corresponding to the periodic case (green) and shifted periodic (dark-orange). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$R_2(p) = \frac{1}{\pi}\frac{1}{h(p)} \quad \text{and} \quad r_2(p) = \frac{1}{\pi}\sqrt{1-p^2}\frac{1}{h(p)}e^{+(R+p)x}$$

leads to a shifted periodic potential. See illustration in Fig. 2.

6.2. Simulation 2. Non-periodic potentials

In this subsection we present results of numerical simulations with dressings that do not result in periodic potentials. We also evolve the resulting potentials in time by evolving the dressing functions under KdV flow.

In the case (A) we consider one-sided dressings [13]:

$$R_1(p) = \frac{1}{\pi}e^{tS_0(p)} \quad \text{and} \quad R_2(p) = 0,$$
$$r_1(p) = \frac{1}{\pi}\sqrt{1-p^2}e^{-(R+p)x+tS_0(p)},$$
$$r_2(p) = 0,$$

where $-1 \le p \le 1$, and the time evolution of the potential $u(x, t)$ is given by adding:

$$S_0(p) = 8(p+3)\left[(p+3)^2 - 12\right] \tag{6.3}$$

to the exponent (see Fig. 3).

In the case (B) we investigate the dynamics of a potential with a flat plateau for small x, found by the following dressing:

$$R_1(p) = \frac{10^{-3}}{\pi}e^{tS_0(p)} \quad \text{and} \quad R_2(p) = \frac{10^{-6}}{\pi}e^{-tS_0(p)},$$
$$r_1(p) = \frac{10^{-3}}{\pi}\sqrt{1-p^2}e^{-(R+p)x+tS_0(p)},$$
$$r_2(p) = \frac{10^{-6}}{\pi}\sqrt{1-p^2}e^{+(R+p)x-tS_0(p)},$$

where $-1 \le p \le 1$ (see Fig. 4).

The third case (C) that we treat involves the dynamics of a modulated periodic potential given by the following dressing:

$$R_1(p) = \frac{1}{\pi}e^{\lambda S(p)+tS_0(p)} \quad \text{and} \quad R_2(p) = \frac{1}{\pi}e^{\lambda S(p)-tS_0(p)},$$
$$r_1(p) = \frac{1}{\pi}\sqrt{1-p^2}e^{-(R+p)x+\lambda S(p)+tS_0(p)},$$
$$r_2(p) = \frac{1}{\pi}\sqrt{1-p^2}e^{+(R+p)x+\lambda S(p)-tS_0(p)},$$

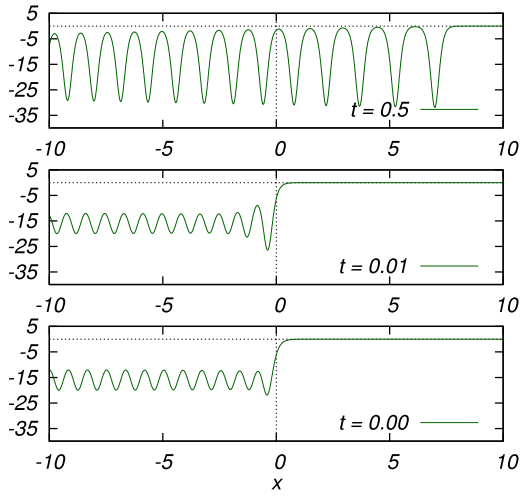

Fig. 3. Potential $U(x)$ corresponding to dressing case (A) at time $t = 0$, $t = 0.01$ and $t = 0.5$.

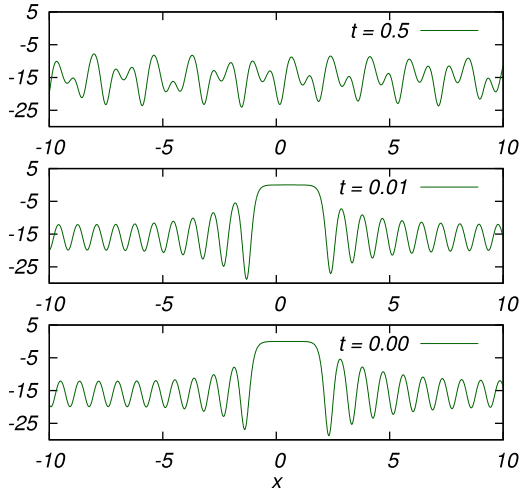

Fig. 4. Potential $U(x)$ corresponding to dressing case (B) at time $t = 0$, $t = 0.01$ and $t = 0.5$.

where $-1 \le p \le 1$ and

$$S(p) = \prod_{n=1}^{N_{\max}}(p - r_n),$$

where $r_1 = -1$, $r_{N_{\max}} = 1$ and r_n is a sequence of randomly generated real numbers in the interval $-1 \le r_n \le 1$ for $n = 2 \ldots N_{\max} - 1$. In this simulation $N_{\max} = 10$ (see Fig. 5), also see Fig. 11 for time evolution of this potential.

6.2.1. Wave functions ψ and φ

The wavefunctions $\psi(p, x)$ and $\varphi(p, x)$ for the simulation (C) are presented in Figs. 6 and 7.

6.2.2. Integrals of motion

In addition, it is worthwhile to investigate the time dependence of integrals of motion associated with KdV flow, note however that we can only observe the potential on a subinterval in variable x, hence the integrals will not be conserved in the presented interval. Instead there will be fluctuations in the integrals of motion due to flow through the boundaries on the left and on the right.

However, we note that for a statistically developed state flow through the right boundary is in dynamic balance with the flow through the left boundary, and hence integrals will be conserved

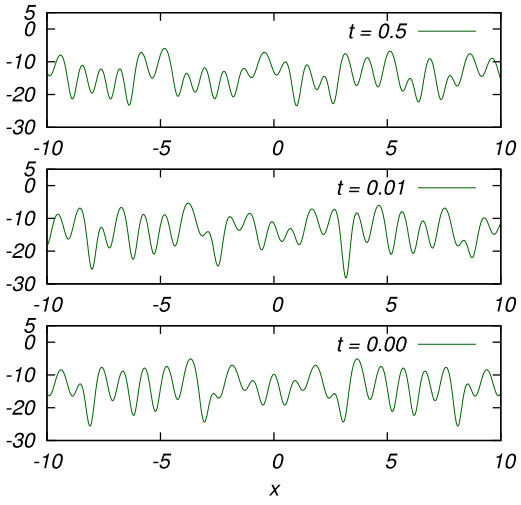

Fig. 5. Potential $U(x)$ corresponding to dressing case (C) at time $t = 0, t = 0.01$ and $t = 0.5$ with $\lambda = 4096$.

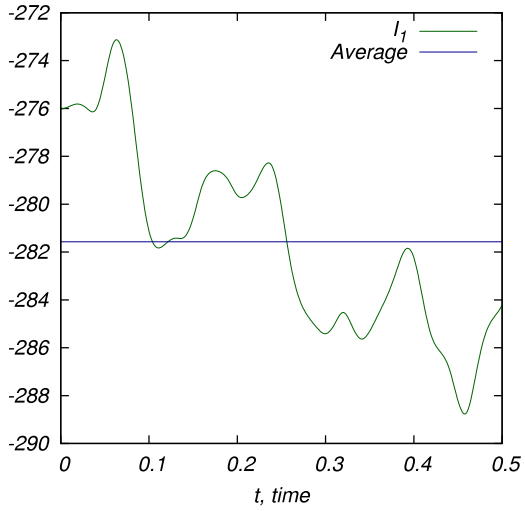

Fig. 8. Time dependence of integral I_1, the root mean square deviation is 3.9471113 (1.40182%).

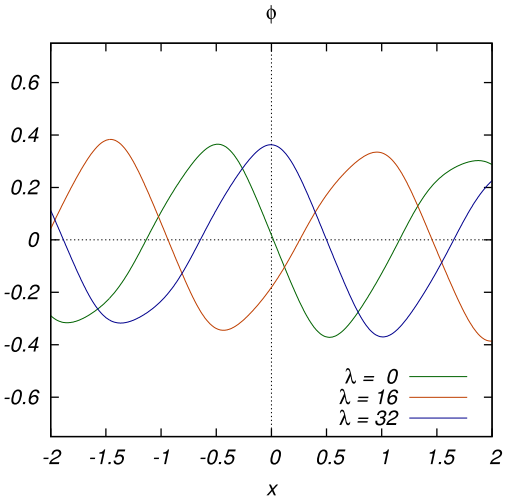

Fig. 6. Wave function φ in case (C) for distinct values of λ and $p = 0$.

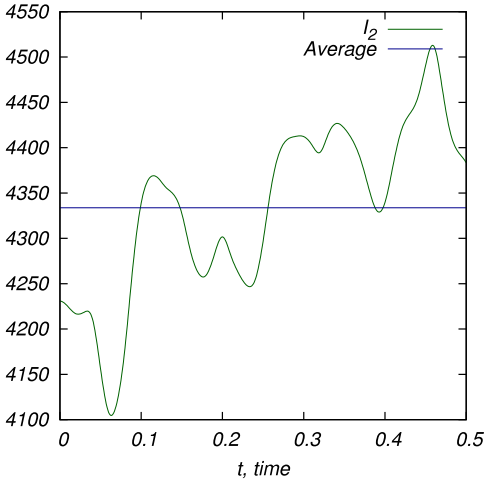

Fig. 9. Time dependence of integral I_2, the root mean square deviation is 94.134925 (2.17212%).

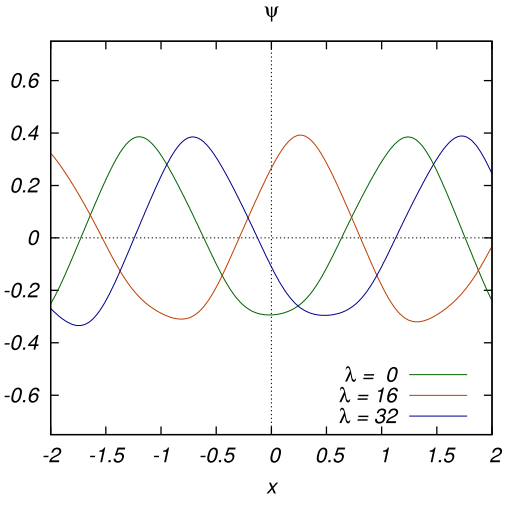

Fig. 7. Wave function ψ in case (C) for distinct values of λ and $p = 0$.

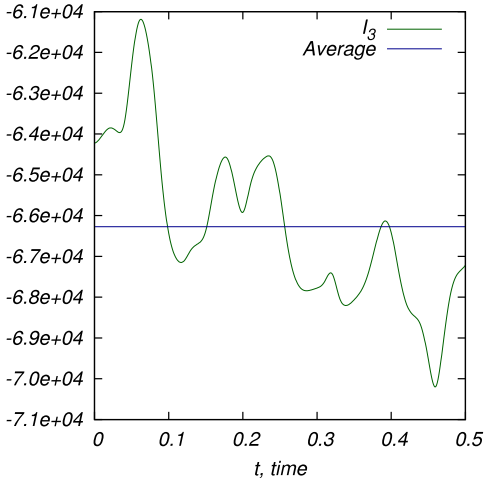

Fig. 10. Time dependence of integral I_3, the root mean square deviation is 1994.2285 (3.00909%).

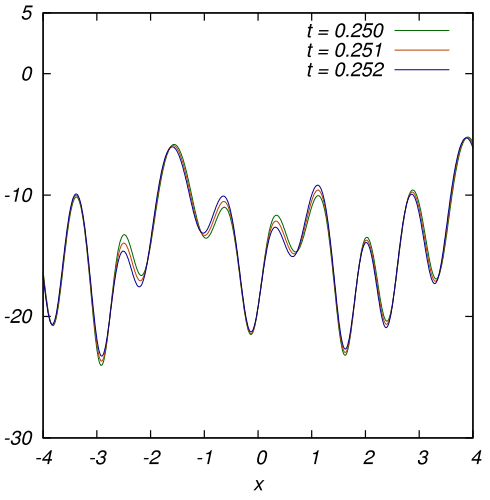

Fig. 11. Details of time evolution of potential $u(x, t)$ for times $t = 0.25$ (green), $t = 0.251$ (dark-orange) and $t = 0.252$ (blue). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

when averaged over a long period of time. In the three figures (Figs. 8–10) we present time dependence of:

$$I_1 = \int_{-10}^{10} u(x)\, dx \tag{6.4}$$

$$I_2 = \int_{-10}^{10} u^2\, dx \tag{6.5}$$

$$I_3 = \int_{-10}^{10} \left(u^3 + \frac{1}{2}u_x^2\right) dx \tag{6.6}$$

corresponding to case (C).

7. Conclusions

We describe a procedure for taking the closure as $N \to \infty$ of the set of N-soliton solutions of the KdV equation, or, equivalently, the closure of the set of reflectionless rapidly vanishing potentials of the Schödinger operator. The resulting solutions are bounded, non-periodic, and non-vanishing as $x \to \pm\infty$. This procedure can be generalized to a wide class of scalar or matrix linear operators, such as the Dirac operator. This would allow us to construct non-vanishing, non-periodic solutions to the various associated nonlinear integrable systems, such as the Nonlinear Schödinger Equation and the Kadomtsev–Petviashvili equation.

An outstanding problem in the theory of nonlinear equations is the development of a statistical theory of integrable systems having infinitely many degrees of freedom and infinitely many conserved quantities. Such a theory, which we may call a theory of integrable turbulence, is beginning to be developed, the first steps having been suggested in [6]. Physical examples of integrable turbulence include coastal areas of seas, and effects occurring in optical fibers. A basic problem in such a theory is to specify a sufficiently large space of functions undergoing evolution (according to KdV, for example), over which we would be able to take statistical averages. The space of rapidly vanishing potentials of the Schödinger operator is parametrized by a function, namely the reflection coefficient, but the conditions on this coefficient are very restrictive. To the best of our knowledge, primitive potentials are the first class of solutions of the KdV equation parametrized by functions without any additional conditions.

One can treat the kinetics of integrable turbulence described in our experiments as the kinetics of a very dense soliton gas. We plan to perform a careful measurement of a pair of correlation functions

$$R_{kk'} = \langle u_k\, u_{k'}^* \rangle,$$

where u_k is the Fourier transform, and brackets mean averaging over time, and compare results with the theory of soliton gas presented in papers [14,15].

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