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ORIGINAL PAPER

About compact equations for water waves

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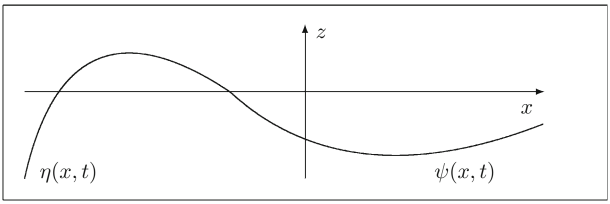
Abstract A simple compact equation for gravity water waves, which includes a nonlinear wave term and advection term, is derived. Numerical simulations in the framework of this equation demonstrate an initial stage of freak wave breaking.

Keywords Nonlinear water waves · Hamiltonian formalism · Modulational instability · Freak waves · Breather

1 Introduction

A one-dimensional potential flow of an ideal incompressible fluid, with a free surface in a gravity field, is a Hamiltonian system



with Hamiltonian

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$$H = \frac{1}{2}\int \mathrm{d}x \int_{-\infty}^{\eta} |\nabla\phi|^2 \mathrm{d}z + \frac{g}{2}\int \eta^2 \mathrm{d}x$$

$\phi(x, z, t)$ is the potential of the fluid, g gravity acceleration, $\eta(x, t)$ surface profile

As it was shown in Zakharov (1968) the Hamiltonian variables are $\eta(x, t)$ and $\psi(x, t) = \phi(x, \eta(x, t), t)$

$$\frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta} \qquad \frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}.$$

In the system there is a natural small parameter, steepness (slope of the surface, η'_x) of the waves-μ. The Hamiltonian can be expanded as infinite series of this small parameter (see Zakharov 1968; Crawford et al. 1980):

$$\begin{aligned} H &= H_2 + H_3 + H_4 + \ldots \\ H_2 &= \frac{1}{2}\int \left(g\eta^2 + \psi\hat{k}\psi\right)\mathrm{d}x, \\ H_3 &= -\frac{1}{2}\int \left\{\left(\hat{k}\psi\right)^2 - (\psi_x)^2\right\}\eta \mathrm{d}x, \\ H_4 &= \frac{1}{2}\int \left\{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}\left(\eta\hat{k}\left(\eta\hat{k}\psi\right)\right)\right\}\mathrm{d}x. \end{aligned} \tag{1}$$

where $\hat{k}$ corresponds to the multiplication by $|k|$ in Fourier space. This truncated Hamiltonian is enough for gravity waves of moderate amplitudes and cannot be reduced.

In the articles (Dyachenko and Zakharov 2011, 2012) we have derived a self-consistent compact equation for water waves moving in one direction. Applying a canonical transformation to the series expansion of the Hamiltonian, we have derived an equation that was written in compact way in X-space. However, it turned out that there is a much more simple and elegant form of water waves equation. We show it next very briefly.

2 Canonical transformation for water waves equation

It is convenient to introduce normal complex variables a_k:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}\left(a_k + a^*_{-k}\right) \quad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}\left(a_k - a^*_{-k}\right) \tag{2}$$

here $\omega_k = \sqrt{gk}$ is the dispersion law for the gravity waves, and Fourier transformations $\psi(x) \rightarrow \psi_k$ and $\eta(x) \rightarrow \eta_k$ are defined as follows:

$$f_k = \frac{1}{\sqrt{2\pi}}\int f(x)e^{-ikx}\mathrm{d}x, \quad f(x) = \frac{1}{\sqrt{2\pi}}\int f_k e^{+ikx}\mathrm{d}k.$$

Using a_k (2) the Hamiltonian takes the form:

$$
\begin{aligned}
H_2 &= \int \omega_k a_k a_k^* \mathrm{d}k, \\
H_3 &= \int V_{k_1 k_2}^{k} \left\{ a_k^* a_{k_1} a_{k_2} + a_k a_{k_1}^* a_{k_2}^* \right\} \delta_{k-k_1-k_2} \mathrm{d}k \mathrm{d}k_1 \mathrm{d}k_2 \\
&\quad + \frac{1}{3} \int U_{k k_1 k_2} \left\{ a_k a_{k_1} a_{k_2} + a_k^* a_{k_1}^* a_{k_2}^* \right\} \delta_{k+k_1+k_2} \mathrm{d}k \mathrm{d}k_1 \mathrm{d}k_2, \\
H_4 &= \frac{1}{2} \int W_{k_1 k_2}^{k_3 k_4} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4} \delta_{k_1+k_2-k_3-k_4} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \\
&\quad + \frac{1}{3} \int G_{k_1 k_2 k_3}^{k_4} \left(a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4} + c.c. \right) \delta_{k_1+k_2+k_3-k_4} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \\
&\quad + \frac{1}{12} \int R_{k_1 k_2 k_3 k_4} \left(a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4}^* + c.c. \right) \delta_{k_1+k_2+k_3+k_4} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4
\end{aligned}
$$

Particular expressions for coefficients of this Hamiltonian are not important here. Nevertheless, they can be found in Dyachenko and Zakharov (2011, 2012), Dyachenko et al. (2016).

Further simplification of the Hamiltonian is based on the absence of the following three- and four-wave resonance of gravity waves:

$$
3 \leftrightarrow 0, \quad 2 \leftrightarrow 1, \qquad \text{and} \qquad 4 \leftrightarrow 0, \quad 3 \leftrightarrow 1.
$$

Now one can apply a canonical transformation from variables a_k to b_k, to exclude nonresonant cubic terms along with nonresonant fourth-order terms with coefficients $G_{k_1 k_2 k_3}^{k_4}$ and $R_{k_1 k_2 k_3 k_4}$, to simplify the term with coefficients $W_{k_1 k_2}^{k_3 k_4}$. This coefficient $W_{k_1 k_2}^{k_3 k_4}$ controls $2 \leftrightarrow 2$ wave interaction and cannot be canceled. This transformation up to the accuracy (b^4) has the form (Zakharov 1968; Krasitskii 1990; Zakharov et al. 1992):

$$
\begin{aligned}
a_k = & b_k + \int \left[2\tilde{V}_{k k_2}^{k_1} b_{k_1} b_{k_2}^* \delta_{k_1-k-k_2} - \tilde{V}_{k_1 k_2}^{k} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} - \tilde{U}_{k k_1 k_2} b_{k_1}^* b_{k_2}^* \delta_{k+k_1+k_2} \right] \mathrm{d}k_1 \mathrm{d}k_2 \\
& + \int \left[A_{k_1 k_2 k_3}^{k} b_{k_1} b_{k_2} b_{k_3} \delta_{k-k_1-k_2-k_3} + \left(A_{k_2 k_3}^{k k_1} + \tilde{\mathbf{B}}_{\mathbf{k_2 k_3}}^{\mathbf{k k_1}} \right) b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} \right. \\
& \left. + A_{k_3}^{k k_1 k_2} b_{k_1}^* b_{k_2}^* b_{k_3} \delta_{k+k_1+k_2-k_3} + A^{k k_1 k_2 k_3} b_{k_1}^* b_{k_2}^* b_{k_3}^* \delta_{k+k_1+k_2+k_3} \right] \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3
\end{aligned} \tag{3}
$$

All coefficients in the canonical transformation (3) provide vanishing of all nonresonant terms in the Hamiltonian and are uniquely defined, except $\tilde{\mathbf{B}}_{\mathbf{k_2 k_3}}^{\mathbf{k k_1}}$. This particular coefficient provides some freedom to make a choice for new coefficients, for $2 \leftrightarrow 2$ wave interaction.

In the Hamiltonian, after the canonical transformation, this freedom can be written in the following way:

$$
\begin{aligned}
H = & \int \omega_k b_k b_k^* \mathrm{d}k + \frac{1}{2} \int \left[T_{k k_1}^{k_2 k_3} - (\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \tilde{\mathbf{B}}_{\mathbf{k_2 k_3}}^{\mathbf{k k_1}} \right] \\
& \times b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} \mathrm{d}k \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 + \ldots.
\end{aligned} \tag{4}
$$

It is $\tilde{\mathbf{B}}_{\mathbf{k_3 k_2}}^{\mathbf{k k_1}}$ which controls $2 \leftrightarrow 2$ coefficient in the new Hamiltonian. It satisfies the following symmetry conditions:

$$\tilde{\mathbf{B}}^{\mathbf{kk_1}}_{\mathbf{k_2k_3}} = \tilde{\mathbf{B}}^{\mathbf{k_1k}}_{\mathbf{k_2k_3}} = \tilde{\mathbf{B}}^{\mathbf{kk_1}}_{\mathbf{k_3k_2}} = -\left(\tilde{\mathbf{B}}^{\mathbf{k_2k_3}}_{\mathbf{kk_1}}\right)^*.$$

If $\tilde{\mathbf{B}}^{\mathbf{kk_1}}_{\mathbf{k_2k_3}} = 0$, Eq. (4) is known as Zakharov equation (Zakharov 1968) where $T^{k_2k_3}_{kk_1}$ is very cumbersome function, satisfying the symmetry conditions:

$$T^{k_2k_3}_{kk_1} = T^{k_2k_3}_{k_1k} = T^{k_3k_2}_{kk_1} = T^{kk_1}_{k_2k_3}. \tag{5}$$

The possibility to change the resonant fourth-order term in Hamiltonian (4) is based on choosing the coefficient $\tilde{\mathbf{B}}^{\mathbf{k_2k_3}}_{\mathbf{kk_1}}$ as follows:

$$\tilde{\mathbf{B}}^{\mathbf{k_2k_3}}_{\mathbf{kk_1}} = \frac{T^{k_2k_3}_{kk_1} - \tilde{T}^{k_2k_3}_{kk_1}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}} \tag{6}$$

It makes the four-wave coefficient in (4) equal to $\tilde{T}^{kk_1}_{k_2k_3}$. Obviously $\tilde{T}^{k_2k_3}_{kk_1}$ must satisfy the same symmetry conditions as $T^{k_2k_3}_{kk_1}$ (5) and must have the same values on the resonant

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3}, \end{aligned} \tag{7}$$

as $T^{k_2k_3}_{kk_1}$. In this case $\tilde{B}^{k_2k_3}_{kk_1}$ has no singularities, which is important [$\tilde{B}^{k_2k_3}_{kk_1}$ is part of canonical transformation (3)].

There are two resonant manifolds for $2 \leftrightarrow 2$ wave interaction (7):

1. Trivial manifold with solution

$$k_2 = k_1, \quad k_3 = k, \quad or \quad k_2 = k, \quad k_3 = k_1, \tag{8}$$

2. Nontrivial manifold with solution

$$\begin{aligned} k &= a(1+\zeta)^2, \\ k_1 &= a(1+\zeta)^2\zeta^2, \\ k_2 &= -a\zeta^2, \\ k_3 &= a\left(1+\zeta+\zeta^2\right)^2 \quad \text{here } 0 < \zeta < 1, \text{and } a > 0. \end{aligned} \tag{9}$$

Coefficient of $2 \leftrightarrow 2$ wave interaction $T^{k_2k_3}_{kk_1}$ has the following properties:

1. Value of the four-wave coefficient on the trivial manifold (8) is Zakharov (1968):

$$T^{kk_1}_{kk_1} = |k||k_1|\frac{1}{2\pi}\min(|k|,|k_1|) \tag{10}$$

2. Value of the four-wave coefficient on the nontrivial manifold (9) is identically equal to zero (Dyachenko and Zakharov 1994). This statement means that system initially consisting of the following waves

$$\simeq \mathrm{e}^{i(kx-\omega t)} \quad \text{with positive } k \text{ and } \omega$$

will not produce waves with negative k. In other words, we can consider a system of unidirectional waves (moving in one direction, similar to KdV equation).

3 Super compact equation

The choice of $\tilde{T}_{kk_1}^{k_2k_3}$ is not unique. Unlike in Dyachenko and Zakharov (2011, 2012), we choose another expression for $\tilde{T}_{kk_1}^{k_2k_3}$, which is the most symmetrical one.

The coefficient $\tilde{T}_{kk_1}^{k_2k_3}$ has the following properties:

1. $\tilde{T}_{kk_1}^{k_2k_3}$ is identically equal to zero on the nontrivial manifold (9), so it is proportional to the product of step functions:

$$\tilde{T}_{kk_1}^{k_2k_3} \sim \theta(k)\theta(k_1)\theta(k_2)\theta(k_3)$$

2. New $\tilde{T}_{kk_1}^{k_2k_3}$ is proportional to $\min(k, k_1, k_2, k_3)$, so it satisfies the symmetry conditions (5) and coincides to $T_{kk_1}^{kk_1}$ on the trivial manifold (8):

$$\tilde{T}_{kk_1}^{k_2k_3} \sim \min(k, k_1, k_2, k_3)\theta(k)\theta(k_1)\theta(k_2)\theta(k_3),$$

3. Because $\tilde{T}_{kk_1}^{k_2k_3}$ is a homogeneous function of degree 3:

$$\tilde{T}_{\epsilon k\epsilon k_1}^{\epsilon k_2\epsilon k_3} = \epsilon^3 \tilde{T}_{kk_1}^{k_2k_3}$$

and it is equal to (10) on the trivial manifold (8); then,

$$\tilde{T}_{kk_1}^{k_2k_3} = \frac{(kk_1k_2k_3)^{\frac{1}{2}}}{2\pi}\min(k, k_1, k_2, k_3)\theta(k)\theta(k_1)\theta(k_2)\theta(k_3) \tag{11}$$

One can check that for $k, k_1 > 0$ the following formula is valid:

$$\min(k, k_1) = \frac{1}{2}(k + k_1 - |k - k_1|) \tag{12}$$

Obviously, for $k, k_1, k_2, k_3 > 0$ which satisfy condition $k + k_1 = k_2 + k_3$

$$\begin{aligned}\min(k, k_1, k_2, k_3) &= \min(\min(k, k_2), \min(k_1, k_3))\\ &= \frac{1}{4}(k + k_1 + k_2 + k_3 - |k - k_2| - |k - k_3| - |k_1 - k_2| - |k_1 - k_3|)\end{aligned} \tag{13}$$

Now H_4 takes the following form:

$$H_4 = \frac{1}{4\pi}\int \min(k, k_1, k_2, k_3)\left(k^{\frac{1}{2}}\theta_k b_k^*\right)\left(k_1^{\frac{1}{2}}\theta_{k_1} b_{k_1}^*\right)\left(k_2^{\frac{1}{2}}\theta_{k_2} b_{k_2}\right)\left(k_3^{\frac{1}{2}}\theta_{k_3} b_{k_3}\right) \delta_{k+k_1-k_2-k_3}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$$

and the variation of H is

$$i\dot{b}_k = \frac{\delta H}{\delta b_k^*} = \omega_k b_k + \frac{k^{\frac{1}{2}}\theta_k}{2\pi}\int \min(k, k_1, k_2, k_3) c_{k_1}^* c_{k_2} c_{k_3} \delta_{k+k_1-k_2-k_3}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 \tag{14}$$

where

$$c_k = k^{\frac{1}{2}}\theta_k b_k$$

is Fourier image of analytical (in the upper half-plane) function. Note, nonlinear term in (14) preserves this property. Multiplying (14) by $ik^{\frac{1}{2}}$ one can easily get:

$$\dot{c}_k + ik\theta_k\left[\frac{\omega_k}{k}c_k + \frac{1}{2\pi}\int \min(k, k_1, k_2, k_3)c^*_{k_1}c_{k_2}c_{k_3}\delta_{k+k_1-k_2-k_3}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\right] = 0 \quad (15)$$

The expression in square brackets of (15) is the variational derivative of the following Hamiltonian:

$$H = \int \frac{\omega_k}{k}|c_k|^2\mathrm{d}k + \frac{1}{4\pi}\int \min(k, k_1, k_2, k_3)c^*_k c^*_{k_1}c_{k_2}c_{k_3}\delta_{k+k_1-k_2-k_3}\mathrm{d}k\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3 \quad (16)$$

Using the following relations between k-space and x-space

$$kc^*_k \Leftrightarrow i\frac{\partial}{\partial x}c^*(x), \quad kc_k \Leftrightarrow -i\frac{\partial}{\partial x}c(x),$$
$$|k - k_2|c^*_k c_{k_2} \Leftrightarrow \hat{K}\left(|c(x)|^2\right), \quad (k + k_1)c_k c_{k_1} \Leftrightarrow -i\frac{\partial}{\partial x}\left(c(x)^2\right),$$

The Hamiltonian can be written in x-space:

$$H = \int c^*\hat{V}c\mathrm{d}x + \frac{1}{2}\int\left[\frac{i}{4}\left(c^2\frac{\partial}{\partial x}c^{*2} - c^{*2}\frac{\partial}{\partial x}c^2\right) - |c|^2\hat{K}\left(|c|^2\right)\right]\mathrm{d}x \quad (17)$$

Here the operator $\hat{V}$ in K-space is so that $V_k = \frac{\omega_k}{k}$. If along with this, one introduces the Gardner-Zakharov-Faddeev bracket (for the analytic in the upper half-plane function)

$$\partial^+_x \Leftrightarrow ik\theta_k \quad (18)$$

then equation of motion is the following:

$$\frac{\partial c}{\partial t} + \partial^+_x\frac{\delta H}{\delta c^*} = 0. \quad (19)$$

Introducing advection velocity

$$\mathcal{U} = \hat{K}|c|^2 \quad (20)$$

and taking variational derivative one can write Eq. (19) in the form:

$$\frac{\partial c}{\partial t} + i\hat{\omega}c - i\partial^+_x\left(|c|^2\frac{\partial c}{\partial x}\right) = \partial^+_x(\mathcal{U}c) \quad (21)$$

one can recognize two terms in the equation:

- nonlinear wave term: $i\hat{\omega}c - i\partial^+_x\left(|c|^2\frac{\partial c}{\partial x}\right)$
- advection term: $\partial^+_x(\mathcal{U}c)$.

Along with usual quantities such as energy and both momenta Eq. (21) conserves action or number of waves:

$$N = \int \frac{|c|^2}{k} \mathrm{d}x.$$

Introducing the complex potential Φ, which is an analytical function in the upper half-plane,

$$\Phi = |c|^2 - i\hat{H}|c|^2$$

the nonlinear part of the Hamiltonian becomes the following:

$$H = \frac{i}{8} \int \left[c^2 \frac{\partial}{\partial x} c^{*2} - c^{*2} \frac{\partial}{\partial x} c^2 - \Phi \frac{\partial \Phi^*}{\partial x} + \Phi^* \frac{\partial \Phi}{\partial x} \right] \mathrm{d}x \tag{22}$$

4 Back to η and ψ

According to the canonical transformation (3), η_k and ψ_k are power series of b_k (or c_k) up to the third order:

$$\eta_k = \eta_k^{(1)} + \eta_k^{(2)} + \eta_k^{(3)}, \quad \psi_k = \psi_k^{(1)} + \psi_k^{(2)} + \psi_k^{(3)}. \tag{23}$$

Details of the recovering physical quantities $\eta(x,t)$ and $\psi(x,t)$ are given in Dyachenko et al. (2016). Obviously

$$\eta_k^{(1)} = \sqrt{\frac{\omega_k}{2g}} \left[b_k + b_{-k}^* \right], \quad \psi_k^{(1)} = -i\sqrt{\frac{g}{2\omega_k}} \left[b_k - b_{-k}^* \right]. \tag{24}$$

or

$$\eta^{(1)}(x) = \frac{1}{\sqrt{2}g^{\frac{1}{4}}} \left(\hat{k}^{\frac{1}{4}} b(x) + \hat{k}^{\frac{1}{4}} b(x)^* \right), \quad \psi^{(1)}(x) = -i \frac{g^{\frac{1}{4}}}{\sqrt{2}} \left(\hat{k}^{-\frac{1}{4}} b(x) - \hat{k}^{-\frac{1}{4}} b(x)^* \right). \tag{25}$$

Operators $\hat{k}^{\alpha}$ act in Fourier space as multiplication by $|k|^{\alpha}$.

$$\begin{aligned} \eta^{(2)}(x) =& \frac{\hat{k}}{4\sqrt{g}} \left[\hat{k}^{\frac{1}{4}} b(x) - \hat{k}^{\frac{1}{4}} b^*(x) \right]^2, \\ \psi^{(2)}(x) =& \frac{i}{2} \left[\hat{k}^{\frac{1}{4}} b^*(x) \hat{k}^{\frac{3}{4}} b^*(x) - \hat{k}^{\frac{1}{4}} b(x) \hat{k}^{\frac{3}{4}} b(x) \right] \\ &+ \frac{1}{2} \hat{H} \left[\hat{k}^{\frac{1}{4}} b(x) \hat{k}^{\frac{3}{4}} b^*(x) + \hat{k}^{\frac{1}{4}} b^*(x) \hat{k}^{\frac{3}{4}} b(x) \right]. \end{aligned} \tag{26}$$

Here $\hat{H}$ - is the Hilbert transformation with eigenvalue $i\,\mathbf{sign}(k)$.

5 Numerical simulation

5.1 Breather

A breather is a localized solution of the following type:

$$c(x,t) = C(x - Vt)e^{i(k_0 x - \omega_0 t)} \quad \text{or} \quad c_k = e^{i(\Omega + Vk)t}\phi_k$$

where ϕ_k satisfies the equation:

$$(\Omega + Vk - \omega_k)\phi_k = \frac{1}{2}\int T_{kk_1}^{k_2k_3}\phi_{k_1}^*\phi_{k_2}\phi_{k_3}\delta_{k+k_1-k_2-k_3}\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3$$

It can be found by the Petviashvili method

$$\phi_k^{n+1} = \frac{NL_k^n}{M_k}\left[\frac{<\phi^n \cdot NL(\phi^n)>}{<\phi^n \cdot \hat{M}\phi^n>}\right]^\gamma, \quad M_k = \Omega + Vk - \omega_k,$$
$$NL(\phi^n) = -P^+\frac{\partial}{\partial x}\left(|\phi^n|^2\frac{\partial\phi^n}{\partial x}\right) + iP^+\frac{\partial}{\partial x}\left(\hat{k}\left(|\phi^n|^2\right)\phi^n\right)$$

Here $< \cdot >$ means integral over periodic domain in x-space, and $\hat{M}$ is the linear operator with eigenvalue in k-space, M_k.

A breather solution of this equation, in the periodic domain 2π, with $k_0 = 100$, is shown in Fig. 1. The breather is a very stable structure. Collision of two breathers, moving with different velocities (or with $k_0 = 100$ and $k_0 = 200$), is shown in Fig. 2. Clip with the breathers collision (Fourier spectrum and surface) can be found at http://alexd.itp.ac.ru/SCEWWNH/breatherscollision.avi.

It should be mentioned that the compact equation derived in Dyachenko and Zakharov (2011, 2012) also has breather solutions. In Fedele and Dutykh (2012a, b), Dyachenko et al. (2012) collisions of breathers seem to be elastic. Thus, it suggests the integrability of the compact 1D Zakharov equation. However, in these papers, very few collisions were simulated, and radiation was hardly observed.

In Dyachenko et al. (2013a, b) numerical experiments show that the multiple collisions of breathers are not purely elastic. It has been found in Dyachenko et al. (2013a) that the six wave amplitude is not canceled for this equation. Thus, the 1D Zakharov equation is not integrable and breathers collisions are not elastic.

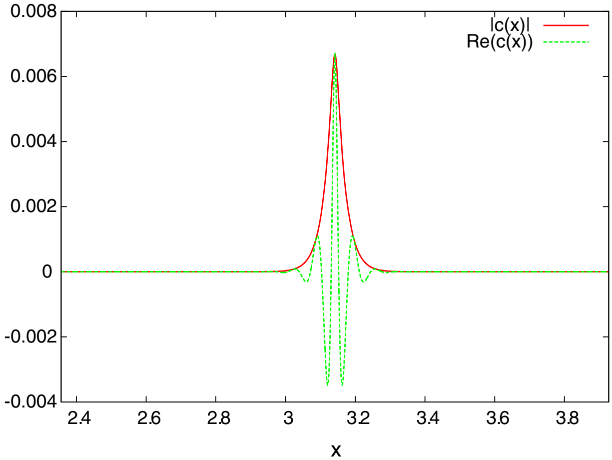

Fig. 1 Narrow breather with three crests. $\mathrm{Re}(c(x,0))$ and $|c(x, 0)|$

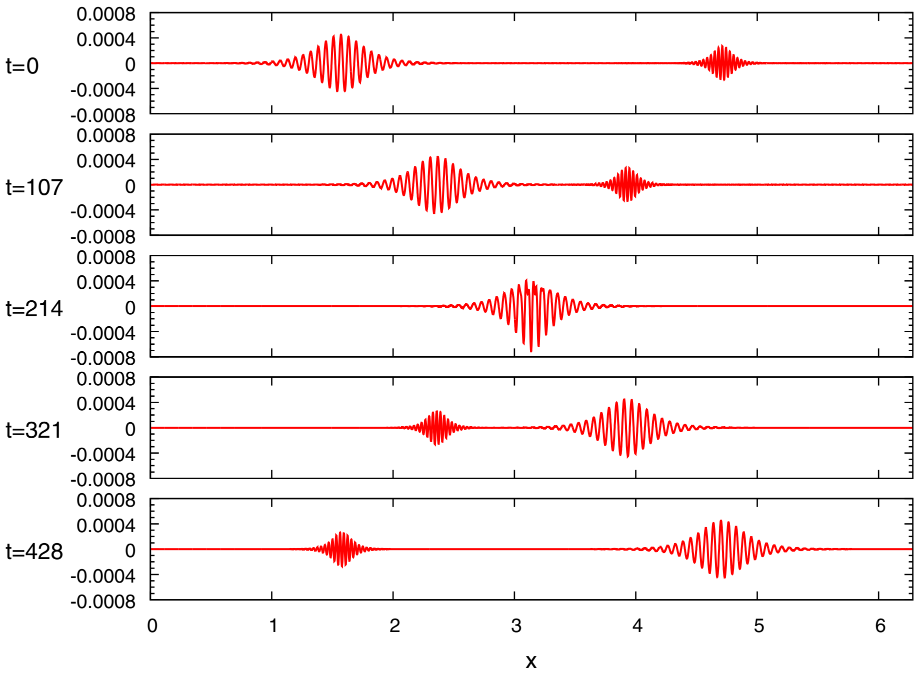

Fig. 2 Snapshots of breather collision

5.2 Modulational instability

Freak wave appearing from homogeneous sea, with $k_0 = 100$ and steepness $\mu = 0.085$ (Figs. 3, 4).

One can see the beginning of wave breaking: Front of the crest becomes steeper and steeper, and some mechanism of dissipation of such waves has to be introduced. Clip with

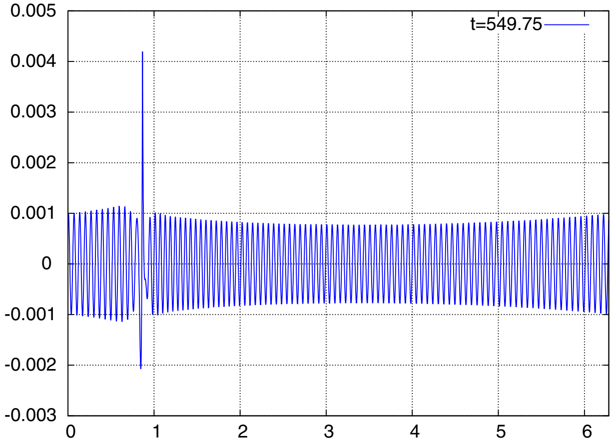

Fig. 3 Freak wave

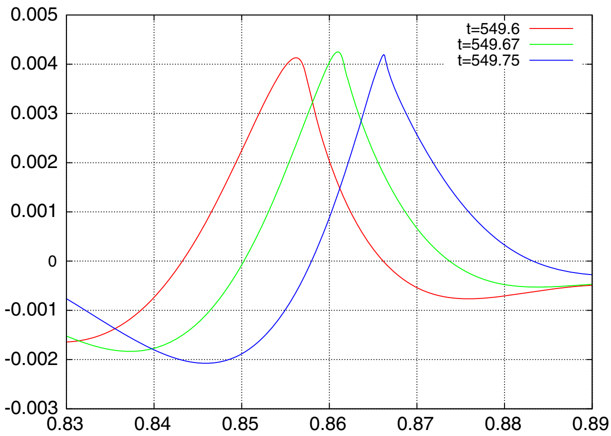

Fig. 4 Three snapshots showing the beginning of wave breaking

freak wave formation (with less initial steepness) is at http://alexd.itp.ac.ru/SCEWWNH/freakwaveformation.avi .

6 Self-similar solution

Equation (21) has exact self-similar substitution

$$c(x,t) = g(t_0 - t)^{\frac{3}{2}} C\left(\frac{x}{g(t_0 - t)^2}\right).$$

It is easy to check that $C(\xi)$ satisfies the following equation:

$$\frac{3}{2}C - 2\xi\frac{\partial C}{\partial \xi} + i\hat{K}^{\frac{1}{2}}C - i\frac{\partial}{\partial \xi}\left(|C|^2\frac{\partial C}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left((\hat{K}|C|^2)C\right) \tag{27}$$

where $C(\xi)$ is a dimensionless function, which is analytic in the upper half-plane, and $\hat{K}$ is dimensionless operator.

In k-space Eq. (15) has the following solution:

$$c(k,t) = g^2(t_0 - t)^{\frac{7}{2}} F\left(gk(t_0 - t)^2\right) \tag{28}$$

It is easy to check that the dimensionless function $F(\xi)$ satisfies the following equation:

$$\frac{7}{2}F + 2\xi\frac{\partial F}{\partial \xi} = i\xi^{\frac{1}{2}}F + \frac{i\xi}{2\pi}\int \min(\xi, \xi_1, \xi_2, \xi_3)F^*(\xi_1)F(\xi_2)F(\xi_3)\delta_{\xi+\xi_1-\xi_2-\xi_3}d\xi_1 d\xi_2 d\xi_3 \tag{29}$$

First we solve the linear part of the equation and then treat the nonlinear part of (29) as a correction. Equation

$$\frac{7}{2}F + 2\xi\frac{\partial F}{\partial \xi} = i\xi^{\frac{1}{2}}F \tag{30}$$

has the following solution:

$$F = \alpha \xi^{\frac{-7}{4}} \mathrm{e}^{i\sqrt{\xi}} \tag{31}$$

Let us plug this solution into the nonlinear part of (29):

$$\begin{gathered} i\alpha\xi^{\frac{-5}{4}}\mathrm{e}^{i\sqrt{\xi}}\left[\frac{|\alpha|^2}{2\pi}\int \min\left(1,\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3\right)\tilde{\xi}_1^{\frac{-7}{4}}\tilde{\xi}_2^{\frac{-7}{4}}\tilde{\xi}_3^{\frac{-7}{4}} \right. \\ \left. \mathrm{e}^{-i\sqrt{\xi}(1+\sqrt{\tilde{\xi}_1}-\sqrt{\tilde{\xi}_2}-\sqrt{\tilde{\xi}_3})}\delta_{1+\tilde{\xi}_1-\tilde{\xi}_2-\tilde{\xi}_3}d\tilde{\xi}_1 d\tilde{\xi}_2 d\tilde{\xi}_3\right] \end{gathered} \tag{32}$$

Obviously the expression in the square brackets weakly depends on ξ (due to $\mathrm{e}^{-i\sqrt{\xi}(1+\sqrt{\tilde{\xi}_1}-\sqrt{\tilde{\xi}_2}-\sqrt{\tilde{\xi}_3})}$). If we consider it as a constant, then nonlinearity just leads to small correction for the exponent in the solution of linear Eq. (30).

However, this self-similar solution describes trivial behavior (wedge formation), due to linear dispersion.

In the paper (Fedele and Dutykh 2012a), the existence of so-called peakon-type solutions was shown. They are traveling waves, with wedge. The self-similar solution derived above may describe formation of such peakons.

7 Conclusion

We derived a new compact and elegant form of the Hamiltonian and equation for the gravity waves, at the surface of deep water. Main features of the equation are:

- It is written for the complex normal variable $c(x, t)$, which is analytic in the upper half-plane
- The Hamiltonian in both k-space (16) and x-space (17) is very simple
- The equation itself is very straightforward, consisting of only two terms—nonlinear waves and advection
- It can easily be implemented for numerical simulation

The equation can be generalized for "almost" 2D waves, like KdV is generalized to KP:

$$\begin{aligned} H = & \int c^* \hat{V} c \,\mathrm{d}x\mathrm{d}y + \\ & + \frac{1}{2}\int \left[\frac{i}{4}\left(c^2 \frac{\partial}{\partial x} c^{*2} - c^{*2}\frac{\partial}{\partial x} c^2\right) - |c|^2 \hat{K}_x\left(|c|^2\right)\right]\mathrm{d}x\mathrm{d}y \end{aligned} \tag{33}$$

Here operator $\hat{V}$ in K-space is $V_{\mathbf{k}} = \frac{\omega_{\mathbf{k}}}{k_x}$.

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