Physics Letters A 380 (2016) 3881–3885

Contents lists available at ScienceDirect

Physics Letters A

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# Non-periodic one-dimensional ideal conductors and integrable turbulence

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ARTICLE INFO

*Article history:*
Received 1 March 2016
Received in revised form 13 September 2016
Accepted 22 September 2016
Available online 28 September 2016
Communicated by A.P. Fordy

*Keywords:*
Ideal conductors
Integrability
Solitonic gas

ABSTRACT

To relate the motion of a quantum particle to the properties of the potential is a fundamental problem of physics, which is far from being solved. Can a medium with a potential which is neither periodic nor quasi-periodic be a conductor? That question seems to have been never addressed, despite being both interesting and having practical importance. Here we propose a new approach to the spectral problem of the one-dimensional Schrödinger operator with a bounded potential. We construct a wide class of potentials having a spectrum consisting of the positive semiaxis and finitely many bands on the negative semiaxis. These potentials, which we call primitive, are reflectionless for positive energy and in general are neither periodic nor quasi-periodic. Moreover, they can be stochastic, and yet allow ballistic transport, and thus describe one-dimensional ideal conductors. Primitive potentials also generate a new class of solutions of the KdV hierarchy. Stochastic primitive potentials describe integrable turbulence, which is important for hydrodynamics and nonlinear optics. We construct the potentials by numerically solving a system of singular integral equations. We hypothesize that finite-gap potentials are a subclass of primitive potentials, and prove this in the case of one-gap potentials.



## 1. Introduction

Despite much work spanning almost 90 years, the evolution of a quantum particle in a one-dimensional bounded potential is far from being understood. Depending on the properties of the potential, there is a wide range of possibilities. Many random potentials (but not all, see [1]) display Anderson localization, meaning that the wave packet expands to a bounded size, and the particle does not move freely. In an opposite scenario the wave train propagates ballistically, and the particle can move to infinity in both directions (see [2]). This can happen, for example, in a periodic potential. A number of intermediate possibilities exist, for example the particle can diffuse to infinity, with the diffusion coefficient being a function of energy.

In this letter, we describe a large class of potentials that admit ballistic wave propagation. We give an effective analytic method for constructing such potentials and support this method with numerical computations.

The character of the evolution of a wave train is determined by the spectral properties of the Schrödinger operator

$$L\psi = (-\partial_x^2 + u(x))\psi = E\psi, \quad -\infty < x < \infty \tag{1}$$

A real number $E$ belongs to the spectrum of $L$ if (1) has one (non-degenerate) or two (doubly degenerate) linearly independent bounded solutions. The spectrum of a generic bounded potential can have a very complicated, fractal-like structure. Ballistic transport is possible for energies lying in an *allowed band*, in other words if there is an open interval such that the spectrum is doubly degenerate at each point of the interval.

In what follows we only consider potentials whose spectrum has such a band structure, consisting of a union of intervals on which it is doubly degenerate, separated by forbidden gaps. Periodic potentials, and certain quasi-periodic ones, have such a spectrum (see [3]). A generic periodic potential has infinitely many forbidden gaps, however, a dense subset of potentials has finitely many. Such finite-gap potentials play a fundamental role and can be explicitly described. A finite-gap potential is specified by choosing the gap boundaries on the real axis, a point inside each gap, and a choice of sign at each point. This data determines a hyperelliptic Riemann surface and a divisor on it, and the potential is explicitly given by the Matveev–Its formula in terms of the associated Riemann theta functions (see [4,5]). The resulting potential is quasi-periodic with $k \le N$ periods, and periodic potentials are obtained by imposing $N-1$ additional conditions.

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http://dx.doi.org/10.1016/j.physleta.2016.09.040
0375-9601/

Until recently it was believed that the algebro-geometric $N$-gap potentials are the only ones whose spectrum has a band structure with $N$ gaps. In this letter, we show that this is not the case, and we effectively construct a much wider class of potentials that have such a spectrum. We call such potentials primitive. Unlike $N$-gap potentials, which are determined by finitely many parameters, primitive potentials are determined by an arbitrary continuous function. We hypothesize that all $N$-gap potentials are primitive, but prove this only for $N=1$.

Unlike finite-gap potentials, primitive potentials are in general neither periodic nor quasi-periodic. Furthermore, numerical experiments show that they can be quite disordered, though we believe that they are not entirely random and have a hidden long-range order. We do not compute their correlation functions analytically, but numerical experiments indicate that these potentials are statistically almost uniform.

We believe that primitive potentials will have wide-ranging applications in diverse areas of physics. Let $u(x)$ be the potential of a one-dimensional medium consisting of irregularly spaced ions and a sea of non-interacting electrons. If Anderson localization holds, then the medium is an ideal dielectric. It seems natural to assume that a medium can be a conductor only if the potential function is periodic or quasiperiodic. We show that this is not the case, and that a much wider class of one-dimensional conductors is possible. If the potential is primitive, and the Fermi level is in one of the allowed bands, then such a medium is an ideal conductor, despite being non-periodic. This may help explain the conductivity of long non-periodic organic molecules, such as DNA.

Our results have important applications for a completely different area of physics. The Schrödinger equation is an auxiliary tool for integrating the Korteweg–de Vries (KdV) equation, which is one of the fundamental models of nonlinear wave dynamics. This procedure is known as the inverse spectral transform, or IST, discovered in 1967 in [7]. Under the IST, the potential is assumed to be time-dependent, and it turns out that KdV evolution does not change the spectrum of the associated stationary Schrödinger operator. Moreover, primitive potentials remain primitive. Hence, our method also constructs a new family of exact solutions of KdV, and the higher KdV hierarchy, which are bounded but non-vanishing as $|x|\to\infty$. Computer simulations show that these solutions are quite irregular.

Integrable nonlinear wave equations, such as KdV, describe a number of important physical systems: waves on shallow water, nonlinear waves in optic fibers, and so on. All of these systems are in need of a statistical description. The first steps in such a theory, known as integrable turbulence [8], have already been made.

We note that, although this letter describes a somewhat complicated mathematical theory, we state most propositions without proof, and we plan to publish them elsewhere. Our method also includes an intricate numerical algorithm, using multiscale accuracy, the details of which will also be published separately.

## 2. Primitive potentials

We give a construction of a wide class of potentials whose spectrum consists of the positive semiaxis and $N$ allowed bands on the negative semiaxis.

Primitive potentials are the continuous limits of reflectionless Bargmann potentials [9], which are also fixed-time slices of $N$-soliton solutions of the KdV hierarchy. We omit the details of this limiting transition and give a direct construction using the dressing method, following [10]. We consider a distribution $T(k)$ on the complex $k$-plane, which we call the dressing function, satisfying the following conditions:

$$T(\bar{k}) = -\overline{T(-k)}, \qquad \int |T(k)| dk\wedge d\bar{k} < \infty, \tag{2}$$

here and now on we assume integration over the entire complex plane unless explicitly specified otherwise. We consider the following integral equation on a function $\chi(x,k)$ defined on the complex $k$-plane (in what follows we write $T(k)$ and $\chi(x,k)$ without assuming either to be analytic):

$$\chi(x,k) = 1 - \frac{1}{2\pi}\int \frac{T(-q)\chi(x,q)e^{-2iqx}}{k+q} dq\wedge d\bar{q}, \tag{3}$$

where $\overline{\chi(x,-k)} = \chi(x,\bar{k})$ and $x$ is a parameter.

Suppose that the dressing function is such that the equation (3) has for all $x$ in an interval $(x_1,x_2)$ a unique solution satisfying $\chi\to 1$ as $|k|\to\infty$. Then the function $\chi$ has the following asymptotic expansion:

$$\chi(x,k) = 1 + \frac{i\chi_0(x)}{k} + \cdots$$

The function $\chi_0(x)$ is real-valued by virtue of (2)–(3). Furthermore, $\chi(x,k)$ is a solution of the equation:

$$\chi_{xx} - 2ik\chi_x - u(x)\chi = 0, \quad u(x) = 2\frac{d}{dx}\chi_0(x),$$

and the function $\psi = \chi e^{ikx}$ is a solution of the Schrödinger equation (1) with $E=k^2$. This, of course, does not mean that $E$ is a point of the spectrum. For this to hold, the following conditions need to be satisfied:

1. Equation (3) must have a solution all $x$, i.e. $x_1=-\infty$ and $x_2=+\infty$. Otherwise, at the boundaries $u(x)$ will have a singularity (generically a pole of order two).
2. The potential $u(x)$ must be bounded for all $x$.
3. At least one solution of the Schrödinger equation must be bounded for all $x$.

The first two of these conditions impose strong restrictions on the dressing function $T(k)$. We choose the dressing function in the following way. Let $0<k_1<k_2$, and let $R_1(\kappa)$ and $R_2(\kappa)$ be two real-valued functions on $[k_1,k_2]$, which we extend by zero to the entire real axis. Let $k=k_R+ik_I$, and define

$$T(k) = i\delta(k_R)[R_1(k_I) - R_2(-k_I)], \tag{4}$$

where $\delta(k_R)$ is the one-dimensional Dirac delta function. The symmetry conditions from (2) follow. A function $\chi(x,k)$ satisfying (3) with such a $T(k)$ is analytic on the $k$-plane away from two cuts $k_1<\operatorname{Im} k<k_2$ and $-k_2<\operatorname{Im} k<-k_1$ on the imaginary axis. It has the following representation:

$$\chi(x,k) = 1 + i\int_{k_1}^{k_2} \frac{\varphi(x,q)e^{-qx}}{k-iq} dq + i\int_{k_1}^{k_2} \frac{\psi(x,q)e^{qx}}{k+iq} dq. \tag{5}$$

Substituting (4) and (5) into (3) gives a system of singular integral equations on $\varphi$ and $\psi$. These equations are equivalent a vector Riemann–Hilbert problem. Denote $\Xi(k) = [\chi(k)\ \chi(-k)]^T$, and let $\Xi^+$ and $\Xi^-$ be the right and left values of $\Xi$ on the cuts. Then the problem is

$$\Xi^+(i\kappa) = M(\kappa)\Xi^-(i\kappa), \quad \Xi^+(-i\kappa) = M^T(\kappa)\Xi^-(-i\kappa) \tag{6}$$

for $\kappa\in[k_1,k_2]$, where the transition matrix is

$$M(x,\kappa) = \frac{1}{1+R_1R_2}\begin{bmatrix} 1-R_1R_2 & 2iR_1e^{-2\kappa x} \\ 2iR_2e^{2\kappa x} & 1-R_1R_2 \end{bmatrix}$$

We claim that if $R_1$ and $R_2$ are non-negative functions satisfying the Hölder condition for some $\alpha>0$, then this Riemann–Hilbert problem has a unique solution for all $x$ with normalization

$\Xi(k) = [1\ 1]^T$ as $k \to \infty$. The solutions $\varphi(x,\kappa)$ and $\psi(x,\kappa)$ are bounded for all $x$ and are eigenfunctions of the Schrödinger operator corresponding to eigenvalue $E = -\kappa^2$, where $k_1 < \kappa < k_2$. The potential $u(x)$ is given by the formula:

$$u(x) = 2\frac{d}{dx}\int_{k_1}^{k_2}\left[\varphi(x,q)e^{-qx} + \psi(x,q)e^{qx}\right]dq.$$

This potential is bounded and satisfies condition $-2k_2^2 < u(x) < 0$ [11]. The functions $\varphi$ and $\psi$ are orthogonal in the following sense:

$$\int_{-\infty}^{\infty}\psi(x,\kappa)\psi(x,\kappa')dx = R_1(\kappa)\delta(\kappa-\kappa'),$$

$$\int_{-\infty}^{\infty}\varphi(x,\kappa)\varphi(x,\kappa')dx = R_2(\kappa)\delta(\kappa-\kappa'),$$

$$\int_{-\infty}^{\infty}\varphi(x,\kappa)\psi(x,\kappa')dx = 0.$$

The spectrum of the corresponding primitive potentials has the following structure. For any non-negative $R_1$ and $R_2$, it includes all positive values of energy. If $R_1(k) > 0$ and $R_2(k) > 0$ for some $k \in [k_1, k_2]$, then the spectrum is doubly degenerate at $E = -k^2$, making the primitive potential behave like an ideal conductor near this energy level. If one of the two functions $R_1$ or $R_2$ vanishes along a subinterval of $[k_1, k_2]$, while the other is positive, then the spectrum for the corresponding values of $E = -k^2$ is non-degenerate. Finally, if both functions vanish along a sub-interval, then that interval forms a spectral gap. Hence, if there are $N-1$ subintervals of $[k_1, k_2]$ along which both $R_1$ and $R_2$ vanish, then the spectrum is $N$-gap and the potential is primitive with $N$ gaps. We prove all of these assertions in [12].

We remark that, unlike many other Riemann–Hilbert problems appearing in scattering theory, (6) involves exponential rather than oscillatory terms. See works [13] and [14] for details of application of the nonlinear steepest descent method in such a setting.

## 3. Periodic one-gap potential

The original goal of our project was to construct the algebro-geometric $N$-gap potentials as limit of soliton potentials using the dressing method. In what follows we give this construction for one-gap potentials. We believe that this construction can be generalized to all finite-gap potentials, which are thus a subclass of primitive potentials.

A periodic reflectionless one-gap potential has the form

$$u(x) = 2\wp(x + i\omega' - x_0) + e_3.$$

Here $\wp$ is the Weierstrass function with periods $2\omega$ and $2\omega'$, where $\omega$ and $\omega'$ are real [6]. Let $e_1$, $e_2$, and $e_3$ be the values of $\wp$ on the half-periods of the lattice, and let

$$k_1^2 = e_2 - e_3, \quad k_2^2 = e_1 - e_3, \quad e_1 + e_2 + e_3 = 0.$$

Here $x_0$ is an arbitrary constant. Let $x_0 = \omega$ and map the $k$-plane onto the period rectangle using the function

$$k^2 = e_3 - \wp(z), \quad z(k) \to -\frac{i}{k} \text{ as } k \to \infty.$$

The Schrödinger equation (1) becomes the Lamé equation:

$$\varphi'' - \left[2\wp(x-\omega-i\omega') + \wp(z)\right]\varphi = 0,$$

which has the following solution:

$$\varphi(x,z) = \frac{\sigma(x-\omega-i\omega'+z)\sigma(\omega+i\omega')}{\sigma(x-\omega-i\omega')\sigma(\omega+i\omega'-z)}\exp^{-\zeta(z)x}.$$

This function is doubly periodic in the period parallelogram. Introduce the new function

$$\xi(x,k) = \left(\frac{k-ik_1}{k-ik_2}\right)^{1/2}\varphi(x,z)e^{ikx}.$$

It satisfies the equation

$$\xi'' - 2ik\xi' - u(x)\xi = 0.$$

It is easy to check that $\Xi(k) = [\xi(k)\ \xi(-k)]^T$ satisfies the Riemann–Hilbert problem (6), where

$$R_1(\kappa) = \frac{1}{R_2(\kappa)} = \sqrt{\frac{(k_2-\kappa)(\kappa+k_1)}{(\kappa-k_1)(\kappa+k_2)}}.$$

Hence a one-gap periodic potential is primitive.

## 4. Solutions of integrable systems

Suppose that the wave function and potential of the Schrödinger equation (1) depend on time $t$ in the following way:

$$\psi_t + 48\psi' + 4\psi''' + 3u_x\psi = 0, \; u_t + 48u' - 6uu' + u''' = 0 \quad (7)$$

In other words $u(x,t)$ is a solution of the KdV equation (see [7]) in a moving frame. The KdV equation preserves the spectrum, and transforms any primitive potential into another. The dressing functions are transformed as follows:

$$R_1(\kappa) \to R_1(\kappa)e^{S(\kappa)t}, \quad R_2(\kappa) \to R_2(\kappa)e^{-S(\kappa)t}. \quad (8)$$

Here $S(\kappa) = 8(\kappa^3 - 12\kappa)$. The time evolution (8) transforms the Schrödinger operator into a unitary equivalent one with a different potential, and the same is true for evolution under higher KdV flows. For higher KdV flows, $S(\kappa)$ must be replaced by some odd polynomial on $\kappa$. But because any continuous function on a finite interval not containing $\kappa = 0$ can be approximated by odd polynomials, one can consider that $S(\kappa)$ in (8) is an arbitrary odd continuous function. Then we come to an important conclusion: The unique invariant of unitary equivalence is the product $\omega(\kappa) = R_1(\kappa)R_2(\kappa)$. It means that:

1. All Bargmann potentials with the same energy levels are unitary equivalent to each other. Moreover, if the spectrum is non-degenerate (single), all operators with the same spectrum are unitary equivalent to each other.
2. All finite-gap potentials with the same band structure are unitary equivalent to each other.

## 5. Numerical solution

We solve the system of integral equations associated to (6) numerically for $k_1 = 2$ and $k_2 = 4$. Denote $\kappa = p + 3$, $(-1 < p < 1)$. It is convenient to replace $\varphi(x,\kappa)$ and $\psi(x,\kappa)$ with the following functions:

$$\Phi(x,p) = \sqrt{1-p^2}\varphi(x,p+3) \quad \text{and}$$

$$\Psi(x,p) = \sqrt{1-p^2}\psi(x,p+3).$$

The resulting system of coupled equations for $\Phi$ and $\Psi$:

$$\Phi(x,p)+r_1(p)\left[\int_{-1}^{1}\frac{\Phi(x,q)e^{-qx}\,dq}{(6+p+q)\sqrt{1-q^2}}+\fint_{-1}^{1}\frac{\Psi(x,q)e^{qx}\,dq}{(p-q)\sqrt{1-q^2}}\right]$$
$$=r_1(p), \tag{9}$$
$$\Psi(x,p)+r_2(p)\left[\int_{-1}^{1}\frac{\Psi(x,q)e^{qx}\,dq}{(6+p+q)\sqrt{1-q^2}}+\fint_{-1}^{1}\frac{\Phi(x,q)e^{-qx}\,dq}{(p-q)\sqrt{1-q^2}}\right]$$
$$=-r_2(p), \tag{10}$$

holds true for $-1<p<1$ and we denote

$$r_1(p)=\sqrt{1-q^2}\,R_1(p+3)e^{-2(3+p)x},$$
$$r_2(p)=\sqrt{1-q^2}\,R_2(p+3)e^{2(3+p)x},$$

for the sake of brevity.

The continuous functions $\Phi(x,q)$ and $\Psi(x,q)$ are discretized at Chebyshev nodes $q_k=\cos\frac{(2k-1)\pi}{2M}$ with $k=1,2,\dots,M$. The spatial variable $x$ appears as a parameter in (9)–(10) and the $x$-dependence of $r_1$ and $r_2$ becomes a major obstacle, since the condition number of the discretized system is exponential in $x$, it requires usage of multiprecision arithmetics to find an accurate solution even in a moderate interval in $x$.

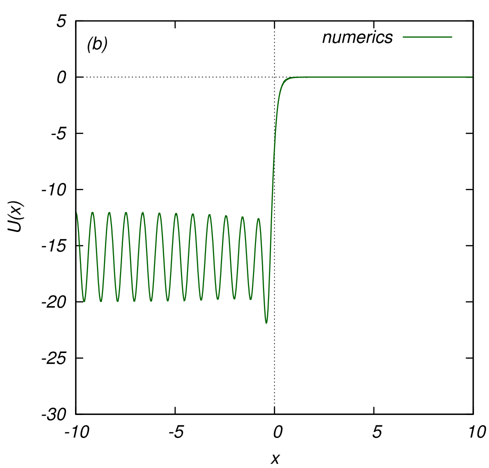


**Fig. 1.** Simulation A: dressing functions are $R_1=1$ and $R_2=0(a)$.

The Chebyshev grid is used for the spatial coordinate $x$ which gives optimal polynomial interpolation between the grid points, for interpolation we use Lagrange basis.

## 6. Simulation A

In this simulation we solve the coupled integral system (9)–(10) to obtain a one-sided primitive potential. The choice of dressing functions is $R_1=1$ and $R_2=0$ (see the illustration in Fig. 1).

## 7. Simulation B

In this simulation we solve the coupled integral system (9)–(10) where the dressing functions are:

$$R_1(p)=\frac{1}{\pi}e^{\lambda S(p)+tS_0(p)}, \tag{11}$$
$$R_2(p)=\frac{1}{\pi}e^{\lambda S(p)-tS_0(p)}, \tag{12}$$

where $S(p)$ is a polynomial of degree 16 with 14 roots chosen randomly from a uniform distribution on the interval $-1<p<1$ and the remaining two are $p=\pm1$. The function $S_0(p)$ is a cubic polynomial associated with KdV time evolution in the moving frame of reference:

$$S_0(p)=8(p+3)\left[(p+3)^2-12\right].$$

The illustration of the random polynomial is in Fig. 2 and Fig. 3, the position of the roots are $-1$, $-0.9424$, $-0.8123$, $-0.7551$, $-0.5190$, $-0.4894$, $-0.2893$, $0.3376$, $0.4271$, $0.4493$, $0.4756$, $0.5128$, $0.6258$, $0.7349$, $0.9452$ and $1$. And the resulting polynomial is:

$$S(p)=\prod_{j=1}^{16}(p-p_j),$$

where the $p_j$ are listed above.

## 8. Conclusion

The procedure of closing the set of Bargmann potentials described above can be generalized to a wide class of linear operators, such as the Dirac operator. This is especially important for various nonlinear integrable systems, such as the Nonlin-

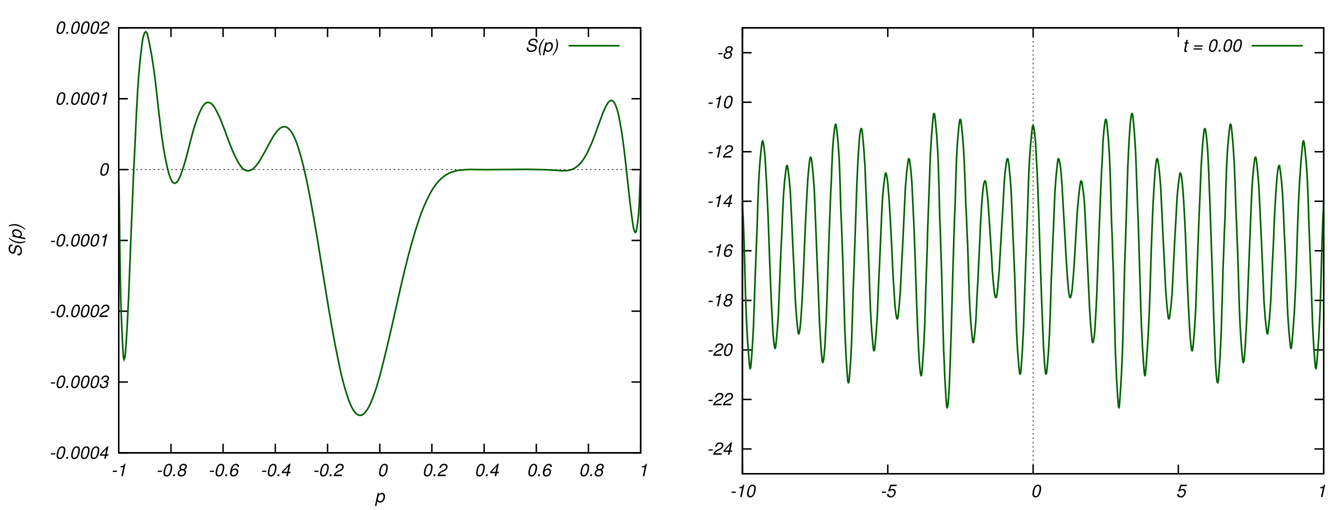


**Fig. 2.** The random polynomial $S(p)$ of degree 16 plotted versus $p$ (left); primitive potential computed with dressing functions (11)–(12) for the value $\lambda=8192$ and $t=0$.

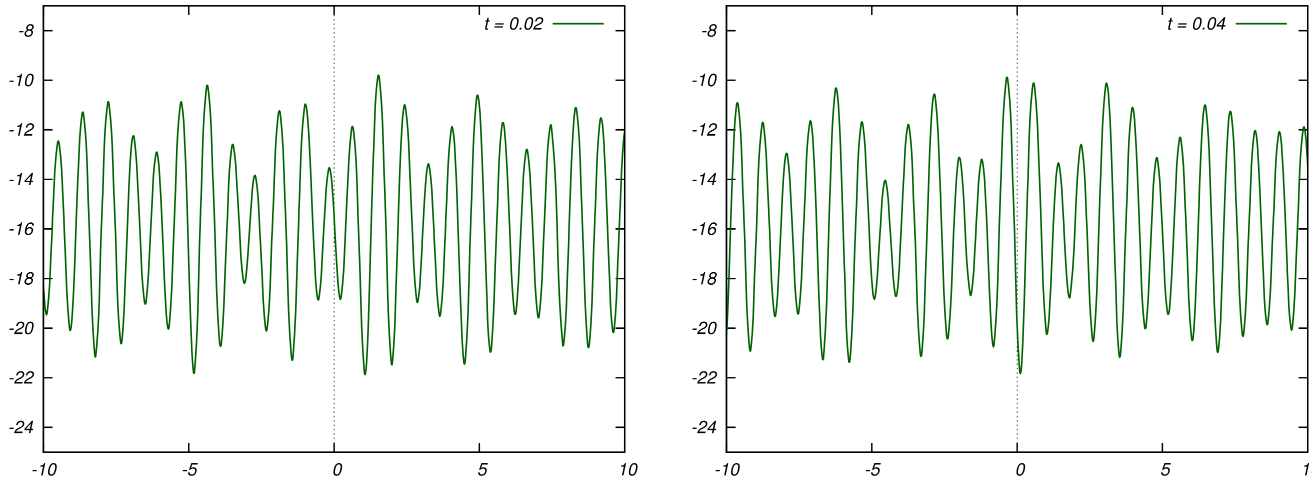


**Fig. 3.** The primitive potential computed with dressing functions (11)–(12) for the value $\lambda = 8192$, $t = 0.02$ (left) and $t = 0.04$ (right).

ear Schödinger Equation and the Kadomtsev–Petviashvili equation, where these linear operators play a fundamental role and which are solved using the IST.

The term "integrable terbulence" is rapidly becoming popular and there has been numerous papers devoted to this new and exciting field, see e.g. [15–18] and [8]. The present work contributes to understanding the effects randomness in integrable systems.

We hesitate to claim the potentials that we have constructed numerically to be the snapshots of integrable turbulence because there is no theoretical proof that the resulting fields are statistically homogeneous. The role of the numerical experiment here is twofold: firstly, they show the strength of dressing method formulation, in particular its usefulness for numerical simulation; secondly, our simulations give a hint that at least some of the dressings may result in a statistically uniform potentials.

To apply the theory that has been developed for these systems to real physical problems, it is necessary to develop a statistical theory of integrable systems with an infinite number of degrees of freedom. This theory of integrable turbulence is still very much in its infancy, the first stages have been suggested in [8]. The technique described in this letter shows an approach to constructing strongly nonlinear statistically homogeneous solutions to integrable systems such as KdV and the nonlinear Schrödinger equation. In fact, integrable turbulence is a common physical phenomenon. This turbulence takes place in the coastal areas of seas, and describes effects occurring in optical fibers. Thus the natural extension and our next step will be the study of random nonperiodic potentials with a continuous spectrum, with this work being at the core of the theory of integrable turbulence.

## Acknowledgements

The authors would like to thank Harry Braden, Percy Deift, Igor Krichever and Thomas Trogdon for insightful discussions. The authors gratefully acknowledge the support of Russian Science Foundation grant number 14-22-00174 for analytic work and developing numerical simulations.

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