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THE PHILLIPS SPECTRUM AND A MODEL OF WIND-WAVE DISSIPATION

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We consider an extension of the kinetic equation developed by Newell and Zakharov in 2008. The new equation takes not only the resonant four-wave interactions but also the dissipation associated with the wave breaking into account. In the equation, we introduce a dissipation function that depends on the spectral energy flux. This function is determined up to a functional parameter, which should be optimally chosen based on a comparison with experiment. A kinetic equation with this dissipation function describes the usually experimentally observed transition from the Kolmogorov–Zakharov spectrum $E(\omega) \sim \omega^{-4}$ to the Phillips spectrum $E(\omega) \sim \omega^{-5}$. The version of the dissipation function expressed in terms of the energy spectrum can be used in problems of numerically modeling and predicting sea waves.

Keywords: Phillips spectrum, kinetic (Hasselmann) equation for water waves, Kolmogorov–Zakharov spectrum

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1. Introduction

It is well known that the spectra of sea waves in both the presence and absence of wind have power-like tails. The shape of the tails in the short-wave range is universal and is given by the famous Phillips spectrum [1]

$$E(\omega) = \alpha_{\text{Ph}} g^2 \omega^{-5}. \tag{1}$$

Here, $\alpha_{\text{Ph}} = 0.0081$ is the Phillips constant. Phillips expressed the reasonable idea that his spectrum owes its existence and robustness to the phenomenon of wave breaking. But the initial hypothesis that the wave field in this asymptotic range is an ensemble of the Stokes limiting waves [2] is refuted by the fact that the mean-square steepness of the Stokes limiting waves $\mu = \langle \nabla \eta^2 \rangle^{1/2} \approx 0.329$ [3], [4] (η is the free surface elevation, and angle brackets denote averaging in space) significantly exceeds the steepness of even the most severe waves ($\mu \simeq 0.1$) observed in the ocean. In addition, high-amplitude Stokes waves are extremely unstable.

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The correct interpretation of the Phillips spectrum was proposed by Newell and Zakharov in [5], where it was shown that the "Phillips sea" is an ensemble of localized breakers uniformly distributed through inverse scales. At the same time, Phillips himself noted [6] that the maximum breaker scale is approximately an order of magnitude smaller than the length of the dominant wave. Analyzing numerous experiments [7]–[12], Phillips showed that a universal spectrum $E(\omega) \sim \omega^{-4}$ is also realized in the intermediate range of scales and suggested that this spectrum is the result of the simultaneous action of three factors: wind pumping, nonlinear wave interaction, and dissipation. This concept is still quite widespread, but it is erroneous if only because the ω^{-4} spectrum was established in the numerical simulation of swell [13]. In addition, it was definitely shown that in the frequency range $\omega_p < \omega < 3.5\omega_p$, the nonlinear wave–wave interaction is the dominating physical effect [14]–[16]. The theoretical explanation of the spectrum $E(\omega) \sim \omega^{-4}$ is therefore quite simple: this is an exact solution of the stationary Hasselmann equation. This fact was already established by Zakharov and Filonenko in 1966 [17].

The spectrum in the intermediate region has the form

$$E(\omega, \theta) = 2C_p \frac{P^{1/3} g^{4/3}}{\omega^4}. \tag{2}$$

Here, P is the energy flux into the region of large wave numbers, and C_p is the Kolmogorov constant. According to the calculations of Geogjaev and Zakharov [18], $C_p \approx 0.203$. Spectrum (2) is just a special case of the weakly turbulent Kolmogorov–Zakharov (KZ) spectra described in detail in [19].

A remarkable feature of the Phillips dissipation function proposed in [6] is a physically transparent meaning of the function. An attempt to improve the Phillips dissipation function is our starting point here. At the end of the paper, we present the versions of the function that we consider "work horses." An optimal version can be selected based on extensive numerical experiments.

2. Phillips spectrum and the asymptotic theory of water waves

The Hasselmann kinetic equation [20] for a spatial spectrum $N_{\mathbf{k}}$ of the wave action of wind-driven waves is written in the form

$$\frac{\partial N_{\mathbf{k}}}{\partial t} + \nabla_{\mathbf{k}} \omega_{\mathbf{k}} \nabla_{\mathbf{r}} N_{\mathbf{k}} = S_{\mathrm{nl}} + S_{\mathrm{in}} + S_{\mathrm{diss}}. \tag{3}$$

The subscripts $\mathbf{k}$ and $\mathbf{r}$ for ∇ are used for the respective gradients in the wave vector $\mathbf{k}$ and the coordinate $\mathbf{r}$. For $N_{\mathbf{k}}(\mathbf{x}, t)$ and $\omega_{\mathbf{k}}$, the subscript $\mathbf{k}$ means dependence on the wave vector. The term S_{nl} in (3) is responsible for four-wave resonant interactions. The terms S_{in} and S_{diss} represent the respective wave action inputs from wind and dissipation. In contrast to the theoretically based term S_{nl} derived from first principles, the description of S_{in} and S_{diss} is mostly based on phenomenological parameterizations [21]. It gives very high dispersion of estimates of S_{in} and S_{diss} in wave modeling and forecasting [22], [23], [16]. The validity and physical correctness of the empirically based terms S_{in} and S_{diss} are generally beyond critical consideration: quantitative aspects dominate the obvious questions of physical relevance. In many cases, these assumptions can be validated in comparison with results of direct simulations using dynamical phase-resolving models [24]–[28].

The collision integral

$$S_{\mathrm{nl}}(\mathbf{k}, \mathbf{x}, t) = \pi g^2 \int |T_{0123}|^2 (N_0 N_1 N_2 + N_1 N_2 N_3 - N_0 N_2 N_3 - N_0 N_1 N_3) \times \\ \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \, d\mathbf{k}_1 \, d\mathbf{k}_2 \, d\mathbf{k}_3 \tag{4}$$

plays a central role in our study. Explicit formulas can be found in many papers (see, e.g., [22]). Crucial is the homogeneity of the power-law dispersion dependence $\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}|}$ and consequently the homogeneity of the interaction coefficient T_{0123} on the wave vector $\mathbf{k}$,

$$|T(\kappa\mathbf{k}_0,\kappa\mathbf{k}_1,\kappa\mathbf{k}_2,\kappa\mathbf{k}_3)|^2=\kappa^6|T(\mathbf{k}_0,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)|^2, \tag{5}$$

and of the collision integral itself on the wave vector,

$$S_{\mathrm{nl}}[\kappa\mathbf{k},\nu N_{\mathbf{k}}]=\kappa^{19/2}\nu^3 S_{\mathrm{nl}}[\mathbf{k},N_{\mathbf{k}}], \tag{6}$$

or the frequency ω,

$$S_{\mathrm{nl}}[\upsilon\omega,\nu N_{\omega}]=\upsilon^{11}\nu^3 S_{\mathrm{nl}}[\omega,N_{\omega}]. \tag{7}$$

Here, κ, υ, and ν are arbitrary positive coefficients. The basic assumption of the weak-nonlinearity theory that the wave period T is small compared with the time scale T_{nl} of nonlinear interactions,

$$\frac{T}{T_{\mathrm{nl}}}=\frac{1}{\omega_{\mathbf{k}}N_{\mathbf{k}}}\frac{dN_{\mathbf{k}}}{dt}=\frac{S_{\mathrm{nl}}}{\omega_{\mathbf{k}}N_{\mathbf{k}}}\ll 1, \tag{8}$$

can be violated at long times and/or for sufficiently short waves. This is not the case with special distributions, the so-called generalized Phillips spectra, where the ratios in (8) are independent of the wave scale, i.e., the weak-nonlinearity assumption inherits the property of the initial wave field [5]. For deep water waves, the classic Phillips spectrum written for the wave energy

$$E_{\mathbf{k}}\sim|\mathbf{k}|^{-4}\qquad\text{or}\qquad E_{\omega}\sim\omega^{-5} \tag{9}$$

or the wave action

$$N_{\mathbf{k}}\sim|\mathbf{k}|^{-9/2}\qquad\text{or}\qquad N_{\omega}\sim\omega^{-6} \tag{10}$$

satisfies condition (8) for any stretching parameters κ, υ, and ν in (6) and (7). In other words, the asymptotic approach seems formally valid at any wave scale. Moreover, it can be proved that condition (8) holds for every term $S_{\mathrm{nl}}^{(n)}$ that represents the resonant interaction of n waves in the asymptotic series of collision integral (3) [5],

$$S_{\mathrm{nl}}=\sum_{n=4}^{\infty}S_{\mathrm{nl}}^{(n)}. \tag{11}$$

Generalized Phillips spectrum (9), (10) does not satisfy conservative kinetic equation (3) and can hence be realized only as a balance of an external forcing (dissipation) and wave–wave resonant interactions. In this regard, solution (9), (10) differs from the classic KZ solutions for direct and inverse cascading [17], [29] (see [22] for the notation)

$$N^{(1)}(\mathbf{k})=C_pP^{1/3}g^{-2/3}|\mathbf{k}|^{-4},\qquad N^{(1)}(\omega,\theta)=2C_pP^{1/3}g^{4/3}\omega^{-5}, \tag{12}$$

$$N^{(2)}(\mathbf{k})=C_qQ^{1/3}g^{-1/2}|\mathbf{k}|^{-23/6},\qquad N^{(2)}(\omega,\theta)=2C_qQ^{1/3}g^{4/3}\omega^{-14/3}. \tag{13}$$

Here,

$$Q=\int_0^{\omega}\int_{-\pi}^{\pi}S_{\mathrm{nl}}\,d\omega\,d\theta,\qquad P=-\int_0^{\omega}\int_{-\pi}^{\pi}\omega S_{\mathrm{nl}}\,d\omega\,d\theta \tag{14}$$

are the wave action and energy fluxes, and C_q and C_p are the corresponding Kolmogorov constants. The collision integral S_{nl} for solutions (12) and (13) vanishes (the fluxes are constant), and estimates of the applicability criteria for kinetic equation (8) requires great care. Most simply (but nontrivially) [14], [15], we can split the nonlinear transfer term $S_{\rm nl}$ into two parts: the nonlinear forcing $F_{\mathbf{k}}$ and the positive-definite nonlinear damping term $\Gamma_{\mathbf{k}} N_{\mathbf{k}}$ ($\Gamma_{\mathbf{k}}$ is the nonlinear damping rate) as

$$S_{\rm nl} = F_{\mathbf{k}} - \Gamma_{\mathbf{k}} N_{\mathbf{k}}.$$

The relaxation rate $\Gamma_{\mathbf{k}}$ gives a physically correct estimate of the time scale of nonlinear wave–wave interactions in kinetic equation (3). In accordance with (8), the asymptotic approximation becomes inapplicable if (see Eq. (17) in [14])

$$\Gamma_{\mathbf{k}}\omega \simeq 4\pi g|\mathbf{k}|^9 N_{\mathbf{k}}^2 = \pi\omega^{12} g^{-4} N_{\omega}^2 \simeq 1. \tag{15}$$

For Phillips spectrum (9), (10), dimensionless rate (15) is determined by only the spectrum magnitude and is independent of the wave scale. For direct cascade KZ solution (12), the applicability criterion becomes

$$4\pi C_p^2 g^{-1/3} P^{2/3} |\mathbf{k}_{\rm br}| = 4\pi C_p^2 g^{-4/3} P^{2/3} \omega_{\rm br}^2 \simeq 1 \tag{16}$$

and can be expressed in terms of the wave scale and the wind speed using an empirical parameterization of wind–wave spectra in the form [30]

$$E(\omega) = \int_{-\pi}^{\pi} E(\omega,\theta)\, d\theta = \beta g u_* \omega^{-4}, \tag{17}$$

where u_* is the friction velocity, g is the acceleration of gravity, and the empirical coefficient $\beta \approx 0.13$ [8], [31], [32]. This gives the estimate (cf. [5])

$$\omega_{\rm br} \approx 0.9 \frac{u_*}{g}. \tag{18}$$

For the wind speed $U_{10} = 15\,\mathrm{m/s}$ (at the standard height 10 m above the sea surface), break (16) occurs for a wave length of about 20 cm, which is quite close to the conventional range of wind-driven waves. This fact leads to the idea of relating the balance of wave–wave interactions and nonlinear dissipation to the Phillips spectrum that holds formally in the whole range of wave scales.

3. A flux-based model of the Phillips spectrum

The formal applicability criterion for weakly nonlinear approximation (15), (16) can be satisfied by a dissipation function that absorbs the spectral flux. A one-dimensional version of the kinetic equation in the flux form (see, e.g., [33])

$$\frac{dE(\omega)}{dt} = -\frac{\partial P}{\partial \omega} + S_{\rm diss}(P,\omega) \tag{19}$$

describes a balance of the energy spectral flux divergence (the nonlinear transfer term $S_{\rm nl}$ given by (4)) and the dissipation function $S_{\rm diss}$ that depends on only the flux P and frequency ω. A dimensional analysis gives the expression

$$S_{\rm diss} = -\Psi\left(\frac{P\omega^3}{g^2}\right)\frac{P}{\omega}. \tag{20}$$

The term P/ω has the same homogeneity properties as the nonlinear transfer term $S_{\rm nl}$ (see (6)), i.e., realizes the general principle "like cures like." With the same homogeneity properties (6), the dimensionless argument of Ψ can be related to the Phillips saturation function [34] and the energy (action) spectrum. In the isotropic case, we have

$$B(\omega) = \frac{\mu_{\rm d}^2}{2} = \frac{\omega^6 N(\omega)}{2g^2} = \frac{\omega^5 E(\omega)}{2g^2} \sim \left(\frac{P\omega^3}{g^2}\right)^{1/3}, \tag{21}$$

i.e., $B(\omega)$ is proportional to the squared differential wave steepness μ_{d}. The corresponding integral

$$s^2 = 2\int_0^{\omega} B(\omega)\,\frac{d\omega}{\omega} \tag{22}$$

is called the mean-square slope of the field of surface waves. For the Phillips spectrum, the mean-square slope s in (22) increases logarithmically with frequency. In the Phillips model [6], $B(\omega)$ is used as an indicator of the degree of saturation of the wave field tending to a finite limit for the spectrum ω^{-5}. The quantity $B(\omega)$ for (1) as a model of a fully developed sea [35] can be easily estimated (cf. Eq. (7) in [36])

$$\lim_{\omega\to\infty} B(\omega) = \frac{\alpha_{\mathrm{Ph}}}{2} \approx 4.05\cdot 10^{-3}. \tag{23}$$

A similar effect of saturation can be found in an explicit form for stationary solutions of (20) with the power-law dependences

$$\Psi = a\left(\frac{P\omega^3}{g^2}\right)^R. \tag{24}$$

The stationary solution of (20) is not unique. The simplest solution corresponds to saturation of the dissipation function in the whole range of wave frequencies,

$$\Psi = a\left(\frac{P\omega^3}{g^2}\right)^R = 3 \tag{25}$$

for arbitrary parameters a and R. The second solution describes a transition from a finite flux P_0 as $\omega \to 0$ to a vanishingly small flux with the same limit of the dissipation function $\Psi \to 3$ at high frequencies:

$$\frac{P}{P_0} = \left(1 + \frac{a}{3}\left(\frac{P_0\omega^3}{g^2}\right)^R\right)^{-1/R}. \tag{26}$$

Solutions (25) and (26) are shown in Fig. 1 as functions of the dimensionless frequency

$$\Omega = \left(\frac{\omega^3 P_0}{g^2}\right)^{1/3}\left(\frac{a}{3}\right)^{1/(3R)}. \tag{27}$$

The degenerate solution $\Psi = 3$ given by (25) corresponds to infinitely large energy fluxes as $\omega \to 0$ (see the solid lines in Figs. 1a and 1b). Solutions (26) for different exponents R show a transition from a finite energy flux at low frequencies to the power-law flux decay as $\omega \to \infty$. The dissipation rate $\Psi(P\omega^3/g^2)$ manifests a step-like behavior for high exponents R near a characteristic dimensionless frequency $\Omega = 1$ (Fig. 1c). The energy flux P in (26) with (6) taken into account can be converted into a spectral density, which also shows a transition from the KZ solution ω^{-4} to the Phillips spectrum ω^{-5} (Fig. 1d). This transition is obviously sharper with higher R.

Solution (26) allows relating the transition parameters to the available experimental data. For the transition frequency, Forristall's data [11] give the estimate $\omega_{\mathrm{tr}} = g\omega_{\mathrm{tr}}/U_{10} \approx 4$ to 5. For typical inverse wave age of wind-driven waves less than 2, this implies the ratio of the transition frequency to the peak frequency $\omega_{\mathrm{tr}}/\omega_p \approx 2$ to 3, which agrees well with the Hwang's observations [37]. With the experimental parameterization of wave spectra (17), we have [30]

$$P_0 = 0.12\frac{\rho_a}{\rho_w}\frac{u_*^3}{g} \tag{28}$$

and an estimate of the unknown coefficient in dissipation function (24)

$$a = 3\left(0.06\frac{\omega_{\mathrm{tr}}u_*}{g}\right)^{-3R}. \tag{29}$$

A nonzero R means that the dissipation in terms of fluxes is nonlinear, while S_{diss} given by (20) remains inherently nonlinear as a function of the spectral density $E(\omega)$ even for $R = 0$.

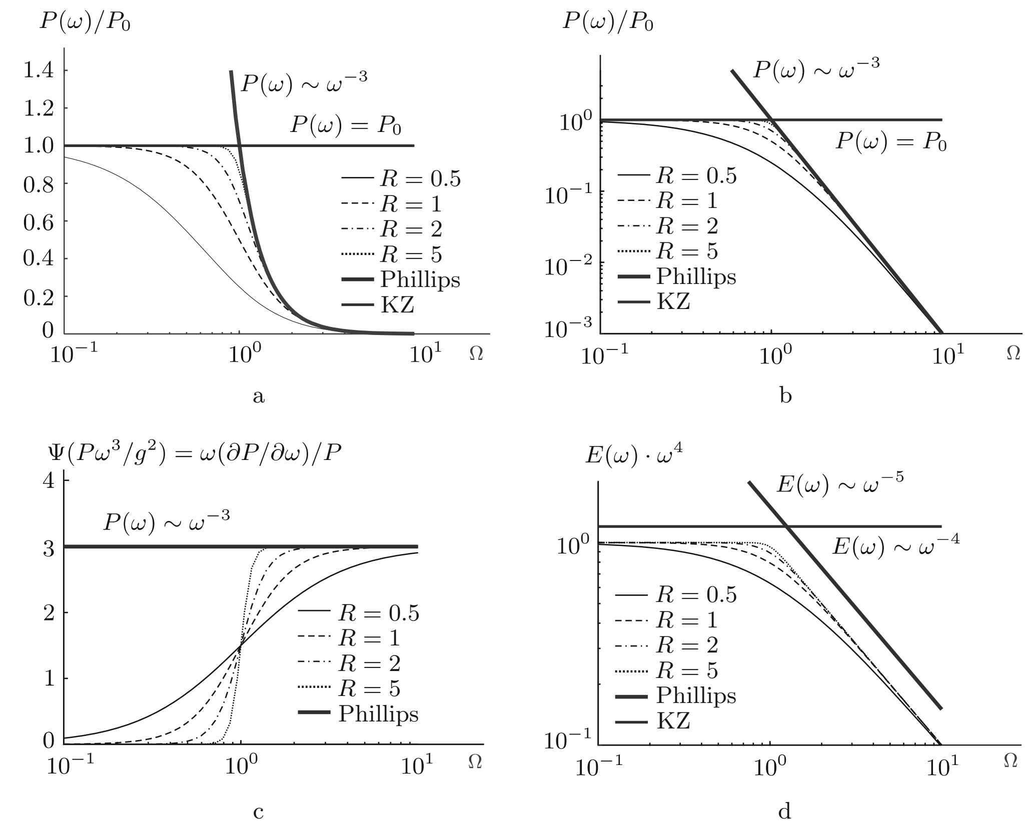

Fig. 1. Stationary solutions for model (20): (a, b) The dimensionless spectral flux for the solutions with different exponents R plotted on semi-log and log–log axes. (c) Dissipation functions with different R: $\Psi = 3$ for degenerate solution (25) and one corresponding to power-law dependence (26) are shown. (d) Compensated spectra obtained with homogeneity relations (6) for spectral fluxes and spectra taken into account. For reference, asymptotic KZ spectra (12), (13) and Phillips spectra (9) are shown.

4. A local substitute for the dissipation function

The proposed dissipation function (20) is nonlocal because it depends on the spectral flux P given by (14). It is therefore difficult to use in solving problems of modeling and predicting waves. In this section, we show a way to construct a "local substitute" for the dissipation function S_{diss} in the spirit of widely used parameterizations [21]. We consider power-law distributions of the form

$$N(\mathbf{k}) = b|\mathbf{k}|^{-x} \qquad \text{or} \qquad E(\omega) = 4\pi b\omega^{4-2x}g^{x-2}. \tag{30}$$

The energy flux for (30) can be calculated analytically [18],

$$P = -\frac{2\pi b^3 g^{3x-10}}{12-3x}\omega^{24-6x}F(x), \tag{31}$$

where the dimensionless function $F(x)$ depends only on the exponent x. For the Phillips spectrum ω^{-5}, the exponent $x = 9/2$ gives

$$\frac{P\omega^3}{g^2} = \frac{F(9/2)}{48\pi^2}\left(\frac{E\omega^5}{g^2}\right)^3. \tag{32}$$

For the saturated state, the dimensionless energy dissipation rate $\Psi = 3$ in (25) for the Phillips spectrum with (21) and $F(9/2) \approx 327$ [18] becomes

$$\gamma_{\mathrm{E}} = \frac{S_{\mathrm{diss}}}{\omega E} = \frac{3P}{\omega^2 E} = \frac{F(9/2)}{16\pi^2} B^2(\omega) \approx 2.07 \left(\frac{E\omega^5}{g^2} \right)^2. \tag{33}$$

Similar estimates for the experimentally based dissipation function developed by Donelan [38] in terms of the Phillips saturation function

$$S_{\mathrm{diss}} = 36\omega E(\mathbf{k})(B(\mathbf{k}))^n \tag{34}$$

with $n = 2.5$ ($R = 0.5$ in our flux representation) give 4 times lower values than the theoretical estimates given by (33) and (21):

$$\gamma_{\mathrm{E}} = 1.36 \cdot 10^{-4} \gg \gamma_{\mathrm{E}}^{\mathrm{Donelan}} = 36 \cdot B^{2.5}(\omega) \approx 3.8 \cdot 10^{-5}. \tag{35}$$

A "correction" (34) proposed by Donelan that, in his opinion, takes the effect of long waves on the short-wave range into account [38],

$$S_{\mathrm{diss}} = 36\omega E(\mathbf{k})(1 + 500 \cdot s^2)^2 (B(\mathbf{k}))^n \tag{36}$$

changes estimate (35) dramatically because of the large multiplier (500 !!!) of the formally small value s given by (22). The conservative estimate $s^2 = 0.02$ [36] gives $\gamma_{\mathrm{E}}^{\mathrm{Donelan}} \approx 46 \cdot 10^{-4}$, which is now more than an order of magnitude higher than theoretical value (35).

The considered example demonstrates the previously mentioned problems of experimental estimates of the dissipation rates, which even in the framework of a single paper [38] can yield a range of two orders of magnitude. Moreover, we note the qualitative similarity of dependences (35) and (36) and some prognostic parameterizations (see, e.g., [39]) with their theoretical counterparts developed here. Formulas (35) and (36) operate exclusively with parameters of the wave field and therefore reflect an inherent physical link between the breaking phenomenon and the intrinsic wave dynamics. The effects of wind do not appear in these dependences explicitly.

Simple dissipation model (19) thus shows its consistency with the experimental results. The key physical effect of saturation of dissipation (25) taken into account by empirical dependences [38] makes the issue of the particular dependence of the function on the sea state less important. We can propose an ansatz for the dissipation function that reflects its rather general features:

- a characteristic scale (frequency) of transition from the KZ spectrum to the Phillips spectrum associated with the dimensionless frequency $\Omega = 1$ in (26) and (27) found experimentally at $\omega_{\mathrm{tr}} \approx 3$ to $4\omega_p$ and

- a nonlinear dependence on dimensionless energy spectrum (33), which is responsible for the saturation effect of the dissipation function and is expressed in terms of dimensionless energy, the differential steepness μ_{d} given by (21), or the Phillips saturation function $B(\omega)$ given by (34).

The result can be written as

$$S_{\mathrm{diss}}(\omega) = C_{\mathrm{Phillips}}\, \omega \mu_{\mathrm{d}}^4 E(\omega) \Theta(\omega - \omega_{\mathrm{tr}}), \tag{37}$$

where Θ is the Heaviside function determining the transition from the KZ spectrum to the Phillips spectrum. We previously showed the correspondence between dissipation function (37) and the problem of saturation of the Phillips spectrum [40]. An alternative version of the dissipation function was recently used in [28], where the transition to the Phillips spectrum was provided by a threshold value for the wave steepness μ_{d}. The choice between these two options of the KZ-to-Phillips transition can be made based on extensive simulations, which we plan to conduct in the nearest future.

5. Conclusions

We list our main results.

- We proposed a simple model of wind-wave dissipation. This model realizes the classic Phillips spectrum ω^{-5} as a balance of a nonlinear transfer due to wave–wave resonant interactions and a nonlinear dissipation.

- Stationary solutions of the simple model describe a saturation of the nonlinear dissipation function for an arbitrary dependence of the function on the dimensionless spectral flux. These solutions correspond to a transition from the KZ spectrum ω^{-4} to the dissipative Phillips spectrum ω^{-5}.

- The parameters of the transition from the KZ spectrum to the Phillips spectrum are quantitatively consistent with experimental findings [11].

- We developed a theoretically consistent substitute for the proposed nonlinear dissipation function that is local (in spectral scales). A comparison with the experimental nonlinear parameterization of the dissipation function by Donelan [38] shows a qualitative agreement in the form of dependences. The dissipation function appears to be almost linear in terms of spectral flux and heavily nonlinear (the dependence is stronger than cubic) for the wave energy spectrum. The possibility of a quantitative comparison is substantially complicated by the large scatter of experimental estimates.

Conflicts of interest. The authors declare no conflicts of interest.

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