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Lump chains in the KP-I equation

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Abstract

We construct a broad class of solutions of the Kadomtsev–Petviashvili (KP)-I equation by using a reduced version of the Grammian form of the τ-function. The basic solution is a linear periodic chain of lumps propagating with distinct group and wave velocities. More generally, our solutions are evolving linear arrangements of lump chains, and can be viewed as the KP-I analogues of the family of line-soliton solutions of KP-II. However, the linear arrangements that we construct for KP-I are more general, and allow degenerate configurations such as parallel or superimposed lump chains. We also construct solutions describing interactions between lump chains and individual lumps, and discuss the relationship between the solutions obtained using the reduced and regular Grammian forms.

KEYWORDS
Grammian form, line-solitons, lump solutions, tau-function

1 | INTRODUCTION

The Kadomtsev–Petviashvili equation

$$[u_t + 6uu_x + u_{xxx}]_x = -3\alpha^2 u_{yy} \tag{1}$$

was derived in Ref. 1 and was first mentioned by its current name in Ref. 2. The KP equation is the subject of hundreds of research papers and several monographs.[3–7] The KP-I and KP-II forms of the equation are physically distinct and correspond to $\alpha^2 = -1$ and $\alpha^2 = 1$, respectively.

The KP-I and KP-II equations are universal models describing weakly nonlinear waves in media with dispersion of velocity. However, from a mathematical point of view they are quite distinct. They have numerous physical applications, such as the theory of shallow water waves (see, for instance, the monographs[3,4]) and plasma physics (Kadomtsev and Petviashvili were both renowned plasma physicists). Both KP-I and KP-II are Hamiltonian systems. The Cauchy problem for both equations is uniquely solvable for initial data in L^1 (see Refs. 8–10). However, KP-II is completely integrable, while KP-I, in general, is not (see the Ref. 11 for the analysis of the difference between the two equations).

Both versions of the KP equation are solvable using the inverse scattering method. The KP equation is the compatibility condition for an overdetermined linear system

$$\alpha\Psi_y + \Psi_{xx} + u\Psi = 0, \quad \Psi_t + 4\Psi_{xxx} + 6u\Psi_x + (3u_x + 3\alpha w)\Psi = 0, \quad w_x + u_y = 0. \tag{2}$$

The Lax representation for KP was found independently by Zakharov and Shabat[2] and Dryuma.[12] For KP-I we have $\alpha = i$ and Equation (2) is a nonstationary one-dimensional Schrödinger equation with the potential $-u$, while $\alpha = 1$ corresponds to KP-II, and the linear problem is a heat equation with a source term. This fact alone reveals the substantial underlying difference between the theories of KP-I and KP-II.

The KP-I equation has a rich family of rational solutions, describing the interactions of stable, spatially localized solitons known as *lumps*. A lump solution of KP-I was first constructed numerically by Petviashvili,[13] who developed an original method for numerically constructing stationary solutions for a wide class of nonlinear PDEs. Lumps and their interactions were first studied analytically in Ref. 14, and received their name in Ref. 15, where they were constructed using the Hirota transform. Krichever[16,17] showed that the dynamics of the lumps in KP-I is controlled by the Calogero–Moser system. Lumps with distinct asymptotic velocities retain their velocities and phases after scattering, but lumps with the same velocity undergo anomalous scattering, and may form bound states known as multilumps.[18–23] Lump and multilump solutions of KP-I were described in the framework of the inverse scattering method in Refs. 24, 25.

Unlike the KP-I equation, KP-II is not known to have spatially localized solutions, nor does it have nonsingular rational solutions. Instead, the KP-II equation has an interesting family of *line-soliton* solutions. An individual line-soliton is a translation-invariant traveling wave. When several line-solitons interact, they form complicated evolving polyhedral arrangements[26–30] that are described by an elaborate combinatorial theory (see Ref. 31 and the monograph[4]). Line-soliton solutions also exist for KP-I but are unstable with respect to transverse perturbations; this was shown in the original paper[1] for large perturbations and in Refs. 32, 33 for all scales. For stability of three-dimensional solitons, see Ref. 34.

The goal of this paper is to initiate a systematic study of a family of solutions of the KP-I equation, which we call *lump chains*. A simple lump chain (see Figure 1) consists of a sequence of lumps evenly spaced along a line and propagating with a common velocity at an arbitrary angle to the line. Such solutions of KP-I have been described by a number of authors,[5,32,35–41] and are also known as periodic solitons, breathers, and soliton standing waves (interpreting y as a time variable). Lump chains interact with one another by splitting, merging, or interlacing (see Figures 2–8), and may also emit individual lumps (Figure 10), which can escape to infinity or be reabsorbed by other lump chains. The large-scale structure of lump chain solutions of KP-I resembles that of the line-soliton solutions of KP-II, however, lump chains may have degenerate behavior that does not occur with KP-II line-solitons, such as parallel and superimposed chains. The interaction of two lump chains (equivalently, the resonant interaction of two breathers) was

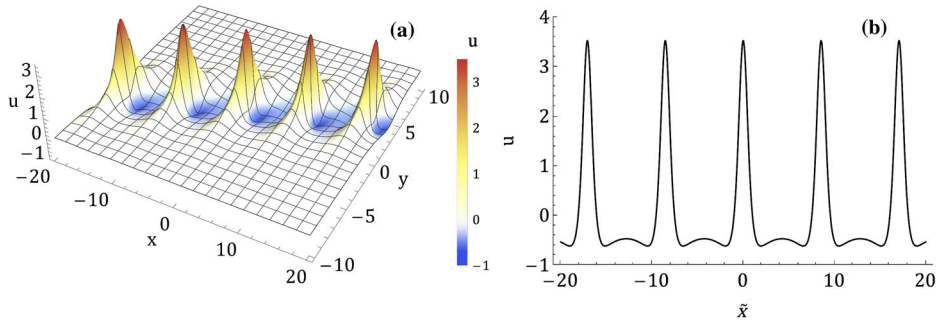

FIGURE 1 Reduced lump chain of order $M = 1$ and rank $N = 2$, given by Equation (11) with $\lambda_1 = 1/2 + i/2$ and $\lambda_2 = 3/8 - i/4$ at $t = 0$. (A) 2D profile of $u(x, y)$. (B) Amplitude of $u(\tilde{x}, y)$ along the line $F_{12} = 0$

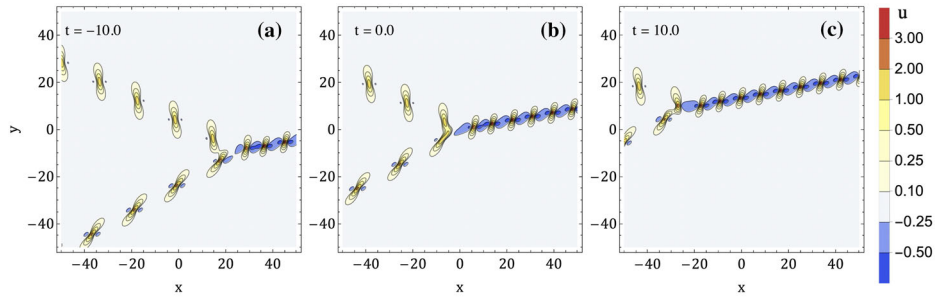

FIGURE 2 $N = 3$ lump chain with $\lambda_1 = 1/2 + i/2$, $\lambda_2 = 3/8 - i/4$ and $\lambda_3 = 1/4 + i/8$ at different moments of time

previously considered in several papers (see eqs. (5) and (6) in Ref. 42, eqs. (7) and (8) in Ref. 43, eqs. (10) and (11) in Ref. 44, and eq. (2.13) in Ref. 45). However, to the best of our knowledge, a general framework for constructing a solution consisting of an arbitrary number of lump chains and individual lumps has not been considered before.

We construct solutions of KP-I using the Grammian form of the τ-function. This form can be derived using a number of methods, such as the dressing method,[2] Sato theory,[46] and the binary Darboux transformation,[47] and is perhaps less known than the Wronskian form. The dressing

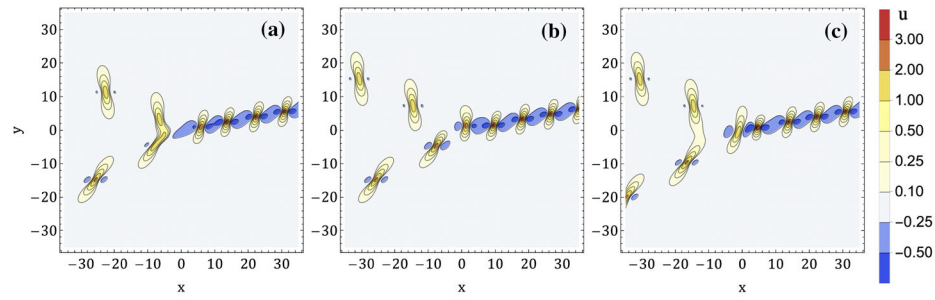

FIGURE 3 $N = 3$ lump chain with $\lambda_1 = 1/2 + i/2$, $\lambda_2 = 3/8 - i/4$ and $\lambda_3 = 1/4 + i/8$ at $t = 0$, and with different relative phases. (A) $\delta_{12} = 0$ and $\delta_{13} = 0$ (B) $\delta_{12} = \pi$ and $\delta_{13} = 0$ (C) $\delta_{12} = 0$ and $\delta_{13} = \pi$

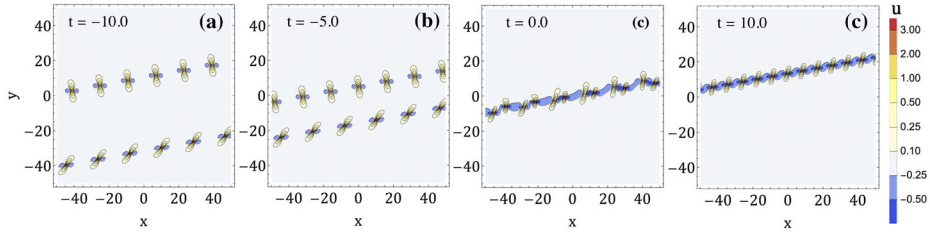

FIGURE 4 Two parallel lump chains merging into one: $N = 3$, $\lambda_1 = 1/2 + i/2$, $\lambda_2 = 3/8 - i/4$ and $\lambda_3 = 3/7 + i/8$ at different moments of time. (A) $t = -15.0$ (B) $t = -10.0$ (C) $t = -5.0$ (D) $t = 0.0$ **(e)** $t = 10.0$ **(a)** $t = 20.0$

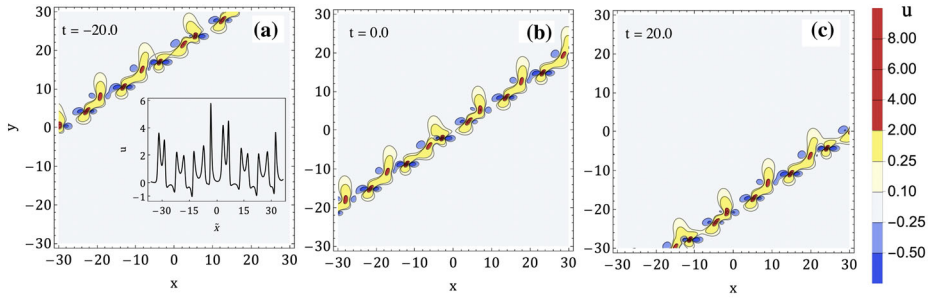

FIGURE 5 Quasi-periodic lump chain of order $M = 1$ and rank $N = 3$, $\lambda_1 = 1 + i/2$, $\lambda_2 = 1/4 - i/4$ and $\lambda_3 = \frac{1}{8}(\sqrt{73} - 5) + \frac{i}{24}(13 - \sqrt{73})$, at times $t = -20.0$, $t = 0.0$, and $t = 20.0$. Inset in (A) shows amplitude along the quasi-periodic lump chain

method was first used to solve the KdV equation in the pioneering paper,[48] and was generalized and applied to the KP-II equation in Ref. 2. A more modern treatment can be found in the papers.[49–51]

As we have noted, individual lump solutions of KP-I are stable, while line-solitons and lump chains are unstable. In Ref. 32, it was shown that a line-soliton can emit a lump chain, hence the latter should be considered as an intermediate stage of the instability development. In the long run, a line-soliton transforms into an expanding cloud of lumps, which can be treated as a model of integrable turbulence.

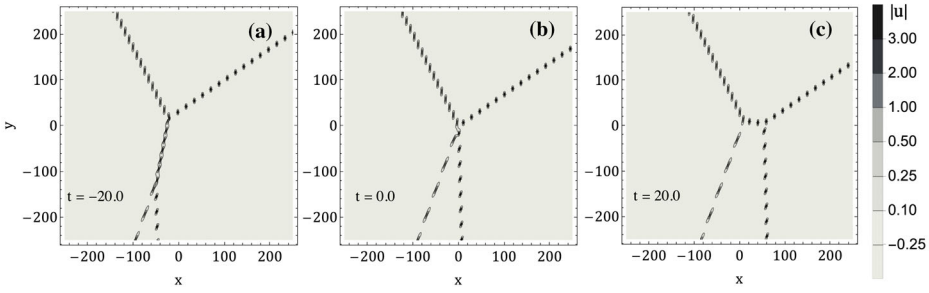

FIGURE 6 H-shaped arrangement of chains of order $M = 1$ and rank $N = 4$, with eigenvalues $\lambda_1 = 17/32 + i/8$, $\lambda_2 = 3/8 - i/8$, $\lambda_3 = 11/32 + i/5$, $\lambda_4 = 3/32 + i/4$ and times $t = -20.0$, $t = 0$ and $t = 20.0$

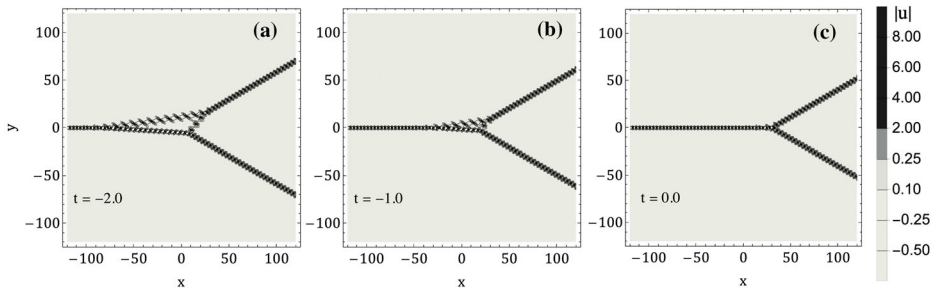

FIGURE 7 Triangular arrangement of chains of order $M = 1$ and rank $N = 4$, with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2 + i\sqrt{3}/2$, $\lambda_3 = 1/2 - i\sqrt{3}/2$, $\lambda_4 = 1/\sqrt{3} - i/(2\sqrt{3})$, and times $t = -2.0$, $t = -1.0$, and $t = 0.0$

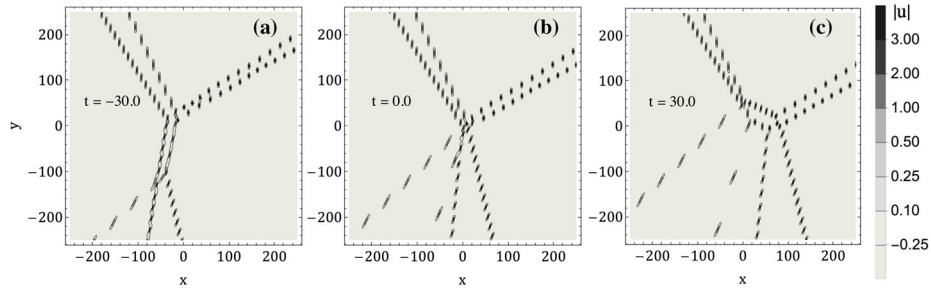

FIGURE 8 The time evolution of a rank 2 order 4 solution with eigenvalues $\lambda_{11} = 1/2 + i/8$, $\lambda_{12} = 3/8 - i/8$, $\lambda_{13} = 1/4 + i/5$, $\lambda_{14} = 1/8 + i/4$, $\lambda_{21} = 4/9 + i/9$, $\lambda_{22} = 3/9 - i/9$, $\lambda_{23} = 2/9 + i/3$, $\lambda_{24} = 1/9 + i/7$

2 | THE GRAMMIAN FORM OF THE τ-FUNCTION

The purpose of this paper is to study a family of solutions of the KP-I equation that can be constructed using the Grammian form of the τ-function, which we now recall.[6,32,46] Fix a positive integer M, which we call the *rank* of the solution. Let $\psi_j = \psi_j^+(x, y, t)$ for $j = 1, \ldots, M$ be a

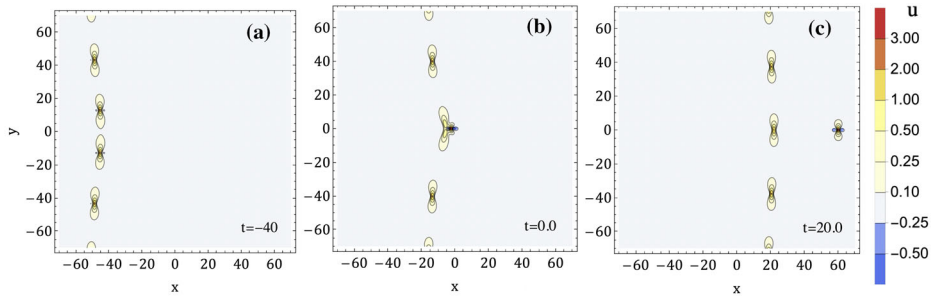

FIGURE 10 A lump chain radiates an individual lump, which propagates away. $\lambda_1 = 1/4$, $\lambda_2 = 1/2$ with multiplicities of 1 and 2, respectively

linearly independent set of solutions to the linear system

$$i\partial_y\psi + \partial_x^2\psi = 0, \quad \partial_t\psi + 4\partial_x^3\psi = 0, \tag{3}$$

and similarly let $\psi_j^-(x,y,t)$ be solutions to the conjugate system

$$i\partial_y\psi - \partial_x^2\psi = 0, \quad \partial_t\psi + 4\partial_x^3\psi = 0.$$

Assume that all $\psi_j^\pm$ lie in $L^2((-\infty, x_0])$ with respect to the variable x for any x_0, and let c_{jk} be an arbitrary constant $M \times M$-matrix. Then the function

$$\begin{aligned} u(x,y,t) = 2\partial_x^2 \log\tau, \quad \tau(x,y,t) = \det\left[c_{jk} + \left\langle \psi_j^+, \psi_k^- \right\rangle\right], \quad \left\langle \psi_j^+, \psi_k^- \right\rangle \\ = \int_{-\infty}^{x} \psi_j^+(x',y,t)\psi_k^-(x',y,t)dx' \end{aligned} \tag{4}$$

is a solution of the KP-I equation (1). To obtain real-valued solutions, we let c_{jk} be real-valued, and we set $\psi_j^- = \overline{\psi}_j$.

It is customary to choose $c_{jk} = \delta_{jk}$ to ensure that the solution (4) is nonsingular; we call solutions of KP-I obtained in this way *regular*. In this paper, however, we are more interested in the case $c_{jk} = 0$; we call such solutions *reduced*. We note that if the solutions ψ_j are linearly independent, then the reduced τ-function (4) is the determinant of a Gram matrix, and hence the solution is nonsingular. We discuss the relationship between regular and reduced solutions of KP-I in Section 4, for now we note that the latter can be obtained from the former by setting $\psi_j^+ = C\psi_j$, where C is a real constant, and taking the limit $C \to +\infty$. It would also be interesting to consider solutions where the matrix c_{jk} is nonzero but does not have maximal rank, however this is beyond the scope of our paper.

In this paper, we restrict our attention to functions ψ_j with finite spectral support. Fix a positive integer N, called the *order* of the solution, and fix distinct eigenvalues $\lambda_1, \dots, \lambda_N$ with positive real parts. Denote

$$\phi(x,y,t,\lambda) = \lambda x + i\lambda^2 y - 4\lambda^3 t,$$

and let $p_s(x,y,t,\lambda)$ denote the polynomial (homogeneous of degree s in x, y, and t) defined by

$$p_s(x,y,t,\lambda) = e^{-\phi(x,y,t,\lambda)}\partial_\lambda^s e^{\phi(x,y,t,\lambda)},$$

so that, for example,

$$p_0 = 1, \quad p_1 = x + 2i\lambda y - 12\lambda^2 t, \quad p_2 = p_1^2 + 2iy - 24\lambda t, \dots$$

Any function of the form $\partial_\lambda^s e^\phi = p_s e^\phi$ is a solution of (3).

We now consider solutions of KP-I given by the tau-function (4), where the eigenfunctions ψ_j are given by (see eqs. (3.3.13) and (3.3.18) in Ref. 47)

$$\psi_j(x,y,t) = \sum_{n=1}^{N}\sum_{s=0}^{S} C_{jns}p_s(x,y,t,\lambda_{jn})e^{\phi(x,y,t,\lambda_{jn})}. \tag{5}$$

The highest degree S of a polynomial p_s that occurs in any of the ψ_j is called the *depth* of the solution. The complex constants C_{jns} are required to satisfy a nondegeneracy condition to ensure that the functions ψ_j are linearly independent. We do not spell out this condition, and instead verify it in each particular example.

An exhaustive classification of the solutions of KP-I obtained in this manner is far beyond the scope of this paper. Instead, our goal is to describe several interesting families of solutions that illuminate the behavior of the generic solution.

1. *Line-solitons*. The simplest solution of KP-I, called a *line-soliton*, is the regular solution obtained from (4) and (5) for $M = 1$, $N = 1$, and $S = 0$, in other words by setting $\psi(x, y, t) = Ce^{\phi(x,y,t,\lambda)}$. This solution is a translation-invariant traveling wave, and a similar solution exists for KP-II. However, unlike the KP-II case, a line-soliton solution of KP-I is unstable (see Refs. 32, 33).
2. *Rational solutions: lumps and multilumps*. A distinguishing feature of the KP-I equation is the existence of rational, spatially localized solutions, which are not known for KP-II. Consider the solution of KP-I given by (4) and (5), where each function ψ_j is a polynomial multiple of a single exponential $e^{\phi(x,y,t,\lambda_j)}$ (the eigenvalues λ_j corresponding to the ψ_j may or may not be distinct). In this case the integral $\langle \psi_j^+, \psi_k^- \rangle$ occurring in (4) is a polynomial multiple of $e^{(\lambda_j+\overline{\lambda}_k)x}$. In the regular case (when $c_{jk} = \delta_{jk}$), the τ-function is a sum of distinct exponentials. However, in the reduced case (when $c_{jk} = 0$), the τ-function is a polynomial multiple of a single exponential term $\exp \sum(\lambda_j + \overline{\lambda}_j)x$, and the exponential disappears when taking the second logarithmic derivative. Therefore, the corresponding solution u is a rational function of x, y, and t. These are the so-called *lump* and *multilump* solutions of KP-I. Corresponding to each distinct eigenvalue λ_j there is a lump, or, more generally, a collection of lumps, whose number is related to the depth S. The lumps in each collection are either bounded or undergo anomalous scattering, while the collections of lumps corresponding to different λ_j undergo normal scattering without phase shifts. Multilump solutions of KP-I were obtained in a number of papers (see, for example, Refs. 16–23). The most general Grammian form of the multilump solutions of KP-I was considered in Ref. 52.
3. *Lump chains*. In this paper, we are mostly concerned with reduced solutions of depth $S = 0$, in other words when each function ψ_j is a linear combination of exponentials. For the solution to be nonsingular, we require $N \geq M$. As we will see, the corresponding reduced solution u of KP-I is an arrangement of *lump chains*, which are sequences of lumps moving along parallel trajectories (the group velocity of the chain is in general distinct from the velocity of the individual lumps). The time evolution of the underlying linear arrangement supporting the lumps is very similar to that of the line-soliton solutions of KP-II (see Ref. 4). However, the linear arrangements that can occur for lump chains are more general than those of KP-II line-solitons, and allow for various degenerate configurations such as parallel or superimposed lump chains. The regular solution of KP-I of depth $S = 0$ and rank $M = 1$ consists of a linear arrangement of lump chains interacting with a single line-soliton of KP-I. We give a detailed description of certain families of reduced lump chain solutions in Section 3, and we give a single example of a regular solution of depth $S = 0$ in Section 4.
4. *Lumps and lump chains*. In Section 4, we also construct an example of a reduced solution of depth $S > 0$ that is not rational. The solution consists of a chain of lumps that emits, at a certain moment of time, a single lump, which propagates away from the chain. We conjecture that the general reduced solution of KP-I with depth $S > 0$ consists of an arrangement of lump chains, with an additional number of individual lumps being either emitted or reabsorbed by

the chains. The general regular solution consists of such an arrangement, and additional line-solitons.

3 | LUMP CHAINS: REDUCED SOLUTIONS OF DEPTH $S = 0$

In this section, we consider solutions of KP-I given by (4) that are reduced ($c_{jk} = 0$), and where the auxiliary ψ-functions are given by (5) with depth $S = 0$, in other words are sums of pure exponential terms with no polynomial multiples. We mostly focus on solutions of rank $M = 1$, in other words having the form

$$u(x,y,t) = 2\partial_x^2 \log\tau, \quad \tau(x,y,t) = \int_{-\infty}^{x} |\psi(z,t,y)|^2\, dz, \tag{6}$$

where the ψ-function is a sum of $N \geq 2$ exponentials (the solution is trivial when $N = 1$). We will see that such a solution is an arrangement of linear *lump chains*, with the individual lumps moving with constant velocity along the chains, and the entire assembly evolving with time. Such lump chain solutions of KP-I bear a strong resemblance to the well-known line-soliton solutions of KP-II, which are the subject of an elaborate combinatorial theory (see Ref. 4).

The ψ-function defining a lump chain solution of rank $M = 1$ and order $N \geq 2$ is defined by $2N$ complex parameters. It is convenient to introduce them as follows. Let

$$\lambda_n = a_n + ib_n, \quad \theta_n = \rho_n + i\varphi_n, \quad n = 1, \dots, N,$$

be complex constants, where we assume that $a_n > 0$ and that $\lambda_n \neq \lambda_m$ for $n \neq m$. Define the functions

$$\Phi_n(x,y,t) = \lambda_n x + i\lambda_n^2 y - 4\lambda_n^3 t + \theta_n,$$

then the function

$$\psi(x,y,t) = \sum_{n=1}^{N} \sqrt{2a_n}\, e^{\Phi_n(x,y,t)}$$

satisfies the linear system (3). Plugging ψ into (6), we obtain the following formula for the τ-function:

$$\tau(x,y,t) = \sum_{n=1}^{N} e^{2F_n} + \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} 2\mu_{nm} e^{F_n + F_m} \cos(G_n - G_m - \varphi_{nm}), \tag{7}$$

where we have denoted

$$F_n(x,y,t) = \operatorname{Re}\Phi_n(x,y,t), \quad G_n(x,y,t) = \operatorname{Im}\Phi_n(x,y,t),$$

and the constants μ_{nm} and φ_{nm} are given by

$$\mu_{nm} = 2\sqrt{\frac{a_n a_m}{(a_n + a_m)^2 + (b_n - b_m)^2}}, \quad \varphi_{nm} = \tan^{-1}\left(\frac{b_n - b_m}{a_n + a_m}\right).$$

We first observe that the large-scale structure of the solution (6) given by the τ-function (7) is determined by the relative values of the linear functions $F_n(x,y,t)$. Fix a moment of time t, and consider the following sets:

$$\Delta_n = \Delta_n(t) = \{(x,y) \in \mathbb{R}^2 \,:\, F_n(x,y,t) \geq F_m(x,y,t) \text{ for all } m \neq n\}, \quad n = 1, \dots, N. \tag{8}$$

The regions $\Delta_1(t), \dots, \Delta_N(t)$ form a partition of the (x,y)-plane into finitely many polygons (some of the $\Delta_n(t)$ may be finite or empty for some or all values of t). The partition evolves linearly with t, and some of the $\Delta_n(t)$ may appear and disappear.

We claim that the solution (6) determined by the τ-function (7) is exponentially small in the interiors of the Δ_n, and is therefore supported on the union of narrow strips along the boundaries where two of the Δ_n meet. Indeed, elementary linear algebra shows that there exists a constant $c > 0$ with the following property: for all n, if (x,y) is a point in $\Delta_n(t)$ located at a distance d from the boundary of $\Delta_n(t)$ (in other words, at a distance of at least d from all other $\Delta_m(t)$), then

$$F_n(x,y,t) - F_m(x,y,t) \geq cd \quad \text{for all } m \neq n.$$

It follows that near such a point (x,y) the exponential term e^{2F_n} in the τ-function (7) is dominant, in other words

$$\tau(x,y,t) = e^{2F_n}(1 + f(x,y,t)), \quad |f| < Ce^{-cd}.$$

The term e^{2F_n} disappears when taking the second logarithmic derivative, hence the solution $u(x,y,t)$ is exponentially small near (x,y).

For future use, we introduce the following notation:

$$F_n - F_m = A_{nm}x + B_{nm}y + C_{nm}t + D_{nm}, \quad G_n - G_m - \varphi_{nm} = \alpha_{nm}x + \beta_{nm}y + \gamma_{nm}t + \delta_{nm},$$

where the constants are given by the following formulas:

$$A_{nm} = a_n - a_m, \quad B_{nm} = 2(a_m b_m - a_n b_n), \quad C_{nm} = -4\left(a_n^3 - a_m^3\right) + 12\left(a_n b_n^2 - a_m b_m^2\right), \tag{9}$$

and

$$\alpha_{nm} = b_n - b_m, \quad \beta_{nm} = a_n^2 - a_m^2 - b_n^2 + b_m^2, \quad \gamma_{nm} = 4\left(b_n^3 - b_m^3\right) - 12\left(a_n^2 b_n - a_m^2 b_m\right). \tag{10}$$

The phases D_{nm} and γ_{nm} play an auxiliary role and we will not require their precise formulas.

We now describe the structure of the corresponding solution $u(x,y,t)$ for $N \geq 2$. We note that adding a common complex constant to the θ_n multiplies τ by a real constant, and hence does not change u, therefore the solution is in fact determined by $2N-1$ complex parameters.

3.1 | Lump chain of rank $M = 1$ and order $N = 2$

The solution of KP-I with τ-function (7) of rank $M = 1$ and order $N = 2$ is the basic building block for solutions of order $N \geq 3$, so we study it in detail. This solution is a linear traveling wave consisting of an infinite chain of lumps, and is analogous to the simple line-soliton solution of

KP-II. In the $N = 2$ case, the τ-function (7) can be simplified by factoring out the exponential term $e^{F_1+F_2}$. The corresponding solution of KP-I is given by

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \log[\cosh(F_2 - F_1) + \mu_{12} \cos(G_2 - G_1 - \varphi_{21})]. \tag{11}$$

The regions Δ_1 and Δ_2 (8) are complementary half-planes separated by the line $F_1 = F_2$, and the solution is supported on a narrow strip along this line where the argument of the hyperbolic cosine is small. The vector

$$\mathbf{U}_{21} = (A_{21}, B_{21}) = (a_2 - a_1, 2a_1b_1 - 2a_2b_2)$$

is a normal direction vector for the line $F_1 = F_2$, and it is clear that the direction of the line may be arbitrary. In particular, the line is parallel to the x-axis if $a_1 = a_2$, which cannot happen for a line-soliton of KP-II.

The solution itself is a traveling wave $u(x, y, t) = U(x - Xt, y - Yt)$, where the function U satisfies the Boussinesq equation. The velocity vector (X, Y) is given by the formulas

$$X = \frac{B_{21}\gamma_{21} - C_{21}\beta_{21}}{A_{21}\beta_{21} - B_{21}\alpha_{21}} = 4\frac{a_1^3 + a_2^3 + 2a_1a_2(a_1 + a_2) + a_2b_2(2b_1 + b_2) + a_1b_1(b_1 + 2b_2)}{a_1 + a_2},$$

$$Y = \frac{C_{21}\alpha_{21} - A_{21}\gamma_{21}}{A_{21}\beta_{21} - B_{21}\alpha_{21}} = 4\frac{a_1(2b_1 + b_2) + a_2(b_1 + 2b_2)}{a_1 + a_2},$$

and may form an arbitrary angle to the normal vector $\mathbf{U}_{21}$ (note that we have assumed that all $a_n > 0$, which implies that the denominators do not vanish). The line $F_1 = F_2$ supporting the solution propagates with normal velocity vector

$$\mathbf{V}_{21} = -\frac{C_{21}}{A_{21}^2 + B_{21}^2}(A_{21}, B_{21}), \tag{12}$$

which is in general distinct from the velocity vector (X, Y) of the solution itself. Furthermore, the line is stationary if $C_{21} = 0$, which also cannot happen for a KP-II line-soliton. Note, however, that $\mathbf{V}_{21} \neq \mathbf{0}$ if the line $F_1 = F_2$ is vertical.

Along the line $F_1 = F_2$, the phase of the solution (11) is determined by the argument of the cosine function. The solution is periodic along the line (see Figure 1), and consists of a sequence of lumps each propagating with velocity vector (X, Y) (in fact, Zaitsev showed in Ref. 40 that this solution can be obtained by a nonlinear superposition of lumps). The distance between two consecutive lumps is equal to (see eq. (2.11) in Ref. 45)

$$L_{21} = 2\pi\frac{\sqrt{A_{21}^2 + B_{21}^2}}{A_{21}\beta_{21} - B_{21}\alpha_{21}} = 2\pi\frac{\sqrt{(a_1 - a_2)^2 + 4(a_1b_1 - a_2b_2)^2}}{(a_1 + a_2)((a_1 - a_2)^2 + (b_1 - b_2)^2)}. \tag{13}$$

To see that the individual peaks are indeed KP-I lumps, we note that the distance L_{21} between two consecutive lumps diverges as $\lambda_2 \to \lambda_1$. Setting

$$a_1 = a - \varepsilon, \quad b_1 = b - \varepsilon\mu, \quad a_2 = a + \varepsilon, \quad b_2 = b + \varepsilon\mu, \quad \theta_1 = \theta_2 = 0,$$

in the $\varepsilon \to 0$ limit we obtain (for arbitrary μ) the standard lump solution of KP-I (see Refs. 14, 15, 37 and other papers):

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log[1 + 576a^6t^2 + 16a^4(72b^2t^2 - 12bty - 6tx + y^2) + 4a^2(12b^2t - 2by + x)^2]. \tag{14}$$

The KP-I equation has infinitely many integrals of motion, the simplest being $\int_{-\infty}^{\infty} u(x,y,t)dx$ (in general, this integral is a linear function of y, but for our solutions it is in fact constant). It is easy to verify that for a lump chain of order $N = 2$ we have

$$\frac{1}{4}\int_{-\infty}^{\infty} u(x,y,t)\,dx = A_{21}.$$

We call the quantity A_{21} the *flux* of the lump chain.

We note that the integral $\int_{-\infty}^{\infty} u(x,y,t)\,dx$ is equal to zero for a one-lump solution (14), because $u(x,y)$ is the x-derivative of a rational function that vanishes at infinity. This agrees with the limiting procedure, because $A_{21} \to 0$ as $\lambda_2 \to \lambda_1$. However, for the one-lump solution $u(x,y)$ given by (14) the integral over the entire plane is nonzero:

$$\int_{\mathbb{R}^2} u(x,y)\,dx \wedge dy = \frac{4\pi}{a} > 0.$$

There is no contradiction here, because $u(x,y)$ does not vanish sufficiently rapidly as $x^2 + y^2 \to \infty$, and this improper integral cannot be evaluated using Fubini's theorem.

It has already been observed by a number of authors that a linear chain of lumps can occur as part of a solution of the KP-I equation. A chain of lumps appears in Ref. 41, and formula (11) occurs in Ref. 5 (see p. 74), but is not analyzed in detail. In Ref. 32, chains of lumps parallel to the y-axis are shown to result from the decay of an unstable line-soliton. Zaitsev[40] developed a procedure for constructing stationary wave solutions of integrable systems out of spatially localized solitons, and constructed a lump chain for KP-I in this manner. Burtsev showed in Ref. 35 that a lump chain is unstable with respect to transverse perturbations, as is the case for a line-soliton. The development of the instability of the lump chain was studied in Ref. 32.

3.2 | Lump chains of rank $M = 1$ and order $N = 3$

We now consider the reduced solutions $u(x,y,t)$ of KP-I with τ-function given by (7) in the case $N = 3$. As we see, a generic solution of this form consists of three lump chains meeting at a triple point, and a number of degenerate configurations are also possible. Such solutions, interpreted as resonant interactions of breathers, were observed in Refs.[42–44], and a pair of merging lump chains was obtained in Ref. 45 (see Figure 3).

The solution is supported on the common boundaries of the three regions Δ_1, Δ_2, and Δ_3 defined in (8). The boundary $\Delta_n \cap \Delta_m$ lies on the line $F_n = F_m$ for $n, m = 1, 2, 3$, and the normal vectors to these lines are given by

$$\mathbf{U}_{nm} = (A_{nm}, B_{nm}) = (a_n - a_m, 2a_mb_m - 2a_nb_n).$$

The three normal vectors satisfy $\mathbf{U}_{31} = \mathbf{U}_{21} + \mathbf{U}_{32}$, and hence are either collinear or pairwise linearly independent. The collinearity is controlled by the quantity

$$\eta_{123} = A_{21}B_{31} - A_{31}B_{21} = a_1b_1(a_2 - a_3) + a_2b_2(a_3 - a_1) + a_3b_3(a_1 - a_2). \tag{15}$$

For generic values of λ_1, λ_2, and λ_3 we have $\eta_{123} \neq 0$, and no two of the three vectors $\mathbf{U}_{mn}$ are collinear. The three lines $F_m = F_n$ for $m, n = 1, 2, 3$ intersect at the triple point where $F_1 = F_2 = F_3$, and the three regions Δ_1, Δ_2, and Δ_3 are sectors with common vertex at the triple point.

We claim that along the common boundary of any two sectors, the solution is exponentially close to the order $N = 2$ lump chain solution given by (11). Consider, for example, the ray $\Delta_1 \cap \Delta_2$ lying on the line $F_1 = F_2$. Along this ray we have $F_1 = F_2 \geq F_3$. Because the functions F_n are linear, it is clear that there is a constant $c > 0$ such that for any point on this ray we in fact have $F_1 - F_3 = F_2 - F_3 \geq cd$, where d is the distance to the triple point. The τ-function (7) for $N = 3$ consists of six terms: three terms whose exponentials do not involve F_3 and whose sum is the $N = 2$ τ-function (7), and three terms that are multiples of e^{F_3}. Factoring out $e^{F_1+F_2}$ as in the $N = 2$ case, we see that the $N = 3$ τ-function is equal to

$$\tau = e^{F_1+F_2}[\cosh(F_2 - F_1) + \mu_{12}\cos(G_2 - G_1 - \varphi_{21}) + f],$$

where $|f| \leq e^{-cd}$ along the ray $\Delta_1 \cap \Delta_2$. It follows that the $N = 3$ solution is exponentially close to the $N = 2$ lump chain (11) along this ray.

We see that when $\eta_{123} \neq 0$ the $N = 3$ solution consists of three lump chains meeting at the triple point. Each chain lies on a ray supported on a line $F_n = F_m$, and we call this the $[n, m]$-chain. Depending on the values of the spectral parameters, there are two possibilities. In the first, shown on Figure 2, the [2,1]- and [3,2]-lump chains meet at the triple point. The individual lumps from the two chains interlace one by one and form the new [3,1]-chain. Conversely, the lumps on the [3,1]-chain may split at the triple point into two new chains. In either case, individual lumps are preserved, and the fluxes of the three chains satisfy the local conservation law $A_{31} = A_{21} + A_{32}$ (see eq. (12) in Ref. 42). In addition to the orientation of the chains, the position of the triple point, and the velocities of the lumps along the chains, there are two free parameters that determine the solution, namely, the relative phases $\delta_{mn} = \mathrm{Im}(\theta_m - \theta_n)$. Figure 3 shows the solution for three different sets of values of the relative chain phases.

There are additionally a number of degenerate configurations, corresponding to $\eta_{123} = 0$. In this case the three vectors $\mathbf{U}_{21}$, $\mathbf{U}_{31}$, and $\mathbf{U}_{32}$ are collinear, and hence so are the lines $F_n = F_m$. Depending on the values of the λ_n, there are two possibilities for a fixed value of t. It may happen that all three regions Δ_1, Δ_2, and Δ_3 are nonempty, in which case two of them are half-planes (say Δ_1 and Δ_3) and the third is a strip of finite width. The solution then consists of two infinite [2,1]- and [3,2]-lump chains supported on the lines $\Delta_1 \cap \Delta_2 = \{F_1 = F_2\}$ and $\Delta_2 \cap \Delta_3 = \{F_2 = F_3\}$. It is also possible that one of the regions (say Δ_2) is empty, so that $F_2 \leq \max(F_1, F_3)$ everywhere on the (x, y)-plane. The solution is then supported near the line $F_1 = F_3$, and is generically a simple [3,1]-lump chain (but see below).

If we now turn on time, then for generic values of λ_n satisfying $\eta_{123} = 0$ the intermediate region Δ_2 either disappears or appears at a finite moment of time. In the former case, the solution consists of two parallel [2,1]- and [3,2]-chains merging into the [3,1]-chain, as shown on Figure 4. The opposite case is also possible: a single [3,1]-chain may, at a certain moment of time, split into two lump chains, both parallel to the original chain. A similar splitting process was observed in[32] (see also fig. 3 in Ref. 45).

Imposing the condition $A_{21}C_{31} - A_{31}C_{21} = 0$ in addition to $\eta_{123} = 0$, we obtain a further degeneration: the three lines $F_n = F_n$ that can support the chains move are not only parallel, but move with equal velocity. Depending on the values of the phases, the intermediate region Δ_2 either exists for all t (and has constant width) or is empty for all t. The solution consists either of two parallel lump chains propagating at a fixed distance, or of a single lump chain (in the latter case, the solution may be visually indistinguishable from a lump chain of order $N = 2$ if the linear function F_2 is sufficiently small compared to F_1 and F_3).

Finally, it is possible that the three lines $F_n = F_m$ are the same for all values of t. The three lump chains merge into a complex, periodic or quasi-periodic chain supported along the common line, which propagates linearly (see Figure 5).

3.3 | Lump chains of rank $M = 1$ and order $N \geq 4$

We now discuss the general form of the solution (7) for arbitrary N, which is determined by the spectral parameters λ_n. As discussed before, the linear functions F_n determine a polygonal partition Δ_n (see (8)) of the (x, y)-plane. The boundaries $F_m = F_n$ that determine the partition evolve linearly with time, and finite polygonal regions may appear or disappear. Along a boundary $\Delta_n \cap \Delta_m$, we have $F_m = F_n \geq F_k$ for $k \neq n, m$. The τ-function has three dominant exponential terms (involving e^{2F_m}, $e^{F_m+F_n}$, and e^{2F_n}) and can be approximated with exponential accuracy by the $N = 2$ τ-function, and hence the boundary $\Delta_n \cap \Delta_m$ supports an $[n, m]$-lump chain. At the points where three or more of the boundaries $\Delta_n \cap \Delta_m$ meet, the corresponding lump chains join or split, with the individual lumps interlacing, and the flux A_{mn} is locally conserved. The structure of the lump chains closely resembles the arrangement of line-solitons in KP-II (see Refs. 4, 26).

We do not develop a general theory describing the line structure of the solutions. Instead, we give two generic examples of order $N = 4$, and discuss the possible degenerate behavior. The first example, shown on Figure 6, may be called an H-configuration. It consists two triple points that are separated by a lump chain bridge. The bridge contracts and disappears at $t = 0$, and the triple points scatter along a different bridge. A similar configuration appears in the KP-II equation (see Refs. 4, 26).

The second example, shown on Figure 7, has three triple points bounding a finite triangular region. The region shrinks and disappears at $t = 0$, and the solution henceforth resembles a solution of order $N = 3$. This configuration can be reversed in time, with a triangular region appearing out of a triple point. We stress that both these examples are generic, in other words the structure of the lump chains does not change under small perturbations of the λ_n.

The reader may recognize that the structure of lump chain solutions of KP-I is very similar to the structure of line-soliton solutions of KP-II. We point out that the line structure in the KP-I case may in fact be more complex. Specifically, the following kinds of behavior, all of them forbidden for KP-II line-solitons, can occur for KP-I lump chains of rank $M = 1$ and order N.

3.3.1 | Generic solutions: The number of chains at infinity and forbidden configurations

We first consider the case when the eigenvalues λ_n are sufficiently generic. A natural first question is to determine the linear configurations of chains that may occur, in particular, the number of lump chains extending to infinity. An order N line-soliton of KP-II always has N solitons extending

to infinity, but a generic order N solution of KP-I may have anywhere between 3 and N infinite lump chains. Similarly, certain configurations of lines are forbidden for KP-II line-solitons but may occur for KP-I lump chain solutions. For example, the solution given on Figure 7 has $N = 4$ and three infinite chains, and represents a line arrangement that cannot occur in KP-II (see Exercise 4.6 in Ref. 4).

3.3.2 | Degenerate solutions: Stable points, parallel chains, and higher order chains

Various degenerate configurations may be achieved by imposing appropriate conditions on the eigenvalues λ_n. The triple points where lump chains meet may be stationary relative to one another, and may even coincide for all times, producing stable quadruple points and points of higher multiplicity. A solution may have sets of parallel lump chains, in which case the number of chains at infinity may be greater than the order N. Finally, lump chains may coincide, producing quasiperiodic chains of higher order.

3.4 | Lump chains of rank $M \geq 2$

The structure of reduced solutions of KP-I of depth $S = 0$ and higher rank $M \geq 2$ is broadly similar to the $M = 1$ case. The τ-function (4) is a sum of purely exponential terms and mixed terms involving trigonometric multipliers. For a given moment of time t, the (x, y)-plane is partitioned into finitely many polygons, in the interior of which the τ-function has a single dominant exponential term and hence produces an exponentially small solution. This decomposition evolves linearly with time, and finite polygonal regions may appear and disappear. The boundaries of the polygons support lump chains, and the total flux of the lump chains arriving at a given vertex is equal to the flux of the chains that are leaving. In degenerate cases, there may be coinciding polygonal boundaries supporting quasiperiodic superpositions of lump chains. We give a single example of such a solution with rank $M = 2$ and order $N = 4$ in Figure 8.

4 | REGULAR SOLUTIONS AND SOLUTIONS OF DEPTH $S > 0$: LINE-SOLITONS AND INDIVIDUAL LUMPS

We now discuss the relationship between regular and reduced solutions of depth $S = 0$, and solutions of positive depth. We first consider regular solutions, and for simplicity restrict our attention to rank $N = 1$. The τ-function of such a solution is nearly identical to that of the reduced solution (7), and has the form

$$\tau(x, y, t) = 1 + \sum_{n=1}^{N} e^{2F_n} + \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} 2\mu_{nm} e^{F_n + F_m} \cos(G_n - G_m - \varphi_{nm}). \tag{16}$$

As before, the (x, y)-plane is partitioned into polygonal regions $\Delta_1, \ldots, \Delta_N$ in each of which one of the terms e^{2F_n} in (16) is dominant. However, there is now a new region Δ_0, on which the dominant term in the τ-function is the constant 1. Because $a_n = \mathrm{Re}\,\lambda_n > 0$, this region contains, for a given

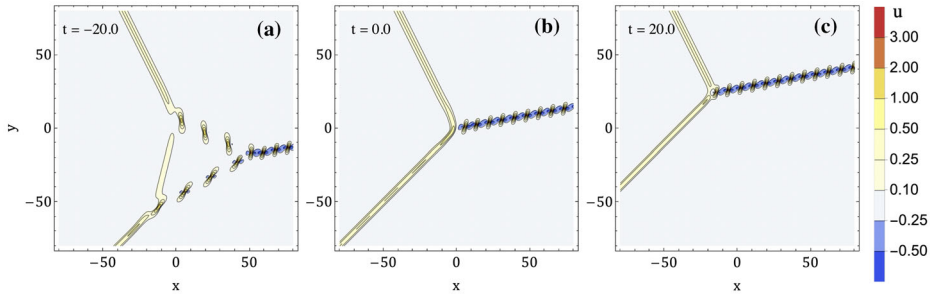

FIGURE 9 Regular solution of rank $M = 1$ and order $N = 3$, with eigenvalues $\lambda_1 = 1/2 + i/2$, $\lambda_2 = 3/8 - i/4$, and $\lambda_3 = 1/4 + i/8$, at different moments of time

fixed y, all points (x, y) with sufficiently large negative x. Inside this region the τ-function is approximately constant, and the solution u is exponentially small. At the boundary of this region, the two dominant terms in the τ-function are the 1 and one of the exponentials e^{2F_n}. Hence the boundary of the region where 1 dominates is a line-soliton of KP-I, instead of a lump chain. In other words, the solution consists of an infinite line-soliton of KP-I coupled with an arrangement of lump chains (see Figure 9).

It is possible to degenerate a regular solution to a reduced solution by replacing the 1 in Equation (16) with an ε and taking the limit $\varepsilon \to 0$. The line-soliton occurs on the boundary of the region where the ε is the dominant term, and this region moves in the negative x-direction as $\varepsilon \to 0$. In the limit, the line-soliton disappears to infinity, and we are left with a solution consisting entirely of lump chains. Therefore, the limiting procedure that produces reduced solutions out of regular solutions has the effect of removing the line-soliton and isolating the lump chain structure.

We also briefly consider the structure of reduced solutions of depth $S > 0$. Consider again the general form of the τ-function (4), where $c_{jk} = 0$ and the ψ_j are given by (5). As discussed in Section 2, the τ-function is rational if each ψ_j is a polynomial multiple of a single exponential term $e^{\phi(x,y,t,\lambda_j)}$. The corresponding solution is localized in the (x, y)-plane and represents the normal (if all λ_j are distinct) or anomalous scattering of lumps, or even bound states of lumps. We now consider what happens in general, when each ψ_j is a multiple of several exponentials. For sufficiently large x and y, the polynomial terms are negligible compared to the exponentials, and the ψ_j can be assumed to be purely exponential. Hence the solution can be assumed to have depth $S = 0$ and is an arrangement of lump chains, as described in Section 3. In the finite part of the (x, y)-plane, however, the polynomial terms in the ψ_j produce individual lumps. Hence, the overall structure of the solution is an arrangement of lump chains interacting with finitely many individual lumps: A lump chain may emit or absorb an individual lump, and the lumps may scatter on one another. A detailed classification of such solutions appears to be a challenging combinatorial problem. In Figure 10, we give a single example of such a solution, consisting of a lump chain emitting an individual lump. We note that the local number of lumps is conserved: Two lumps from the chain meet and scatter, with one lump propagating away and the other filling the resulting gap in the chain. Figure 10 gives an example of such a solution with rank $M = 1$, order $N = 2$, and depth $S = 1$, with the ψ-function given by

$$\psi(x, y, t) = e^{\frac{1}{4}(-2t+2x+iy)}(-3t + x + iy + 1) + e^{\frac{1}{16}(-t+4x+iy)}.$$

5 | SUMMARY AND CONCLUSION

We have constructed a new family of lump chain solutions of the KP-I equations using the Grammian form of the τ-function. A simple lump chain consists of an infinite line of equally spaced lumps. The lumps propagate with equal velocity, which is in general distinct from the group velocity of the line. The general solution consists of an evolving polyhedral arrangement of lump chains. At a point where three or more lump chains meet, the individual lumps from the incoming chains are redistributed along the outgoing chains, with the number of lumps being locally conserved. The linear structure of the solutions is very similar to that of the line-soliton solutions of KP-II. However, various degenerate configurations may occur for KP-I lump chains that cannot occur for KP-II line-solitons: parallel chains, chains of equal velocity, quasiperiodic superimposed chains, stable points of high multiplicity, and forbidden polyhedral configurations. We have also constructed more general solutions of KP-I using the Grammian method. Such solutions consist of an arrangement of lump chains as described above, together with line-solitons and individual lumps that are emitted and/or absorbed by the lump chains. A detailed classification of the solutions of KP-I that may be obtained by the Grammian method is an interesting and difficult problem, and is beyond the scope of this paper. We plan to return to this problem in future work.

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