DOI 10.1007/s11141-022-10169-0
Radiophysics and Quantum Electronics, Vol. 64, No. 10, March, 2022
(Russian Original Vol. 64, No. 10, October, 2021)

LUMP INTERACTIONS WITH PLANE SOLITONS

Yu. A. Stepanyants,[1,2 *] **D. V. Zakharov,**[3] **and V. E. Zakharov**[4,5]

UDC 530.18 + 532.59 + 538.574

We analyze in detail the interactions of two-dimensional solitary waves called lumps and one-dimensional line solitons within the framework of the Kadomtsev–Petviashvili equation describing wave processes in media with positive dispersion. We show that line solitons can emit or absorb lumps or periodic chains of lumps, as well as interact with each other by means of lumps. Within a certain time interval, lumps or lump chains can emerge between two line solitons and then disappear due to absorption by one of the solitons. This phenomenon resembles the appearance of rogue waves in the oceans. The results obtained are graphically illustrated and can be applicable to the description of the physical processes occurring in plasmas, fluids, solids, nonlinear optical media, etc.

1. INTRODUCTION

Slightly more than half a century ago, in 1970, B. B. Kadomtsev and V. I. Petviashvili published a seminal paper where an equation generalizing the well-known Korteweg–de Vries equation for the two-dimensional case was derived [1]. This equation, later named the Kadomtsev–Petviashvili equation [2], was designated for the description of nonlinear dispersive waves traveling primarily in one direction with a smooth variation in the perpendicular direction. In dimensional form, the Kadomtsev–Petviashvili equation can be written as

$$\frac{\partial}{\partial\xi}\left(\frac{\partial v}{\partial\tilde{\tau}}+c\,\frac{\partial v}{\partial\xi}+\alpha\,v\,\frac{\partial v}{\partial\xi}+\beta\,\frac{\partial^3 v}{\partial\xi^3}\right)=-\frac{c}{2}\frac{\partial^2 v}{\partial\eta^2}, \tag{1}$$

where $v(\xi,\eta,\tilde{\tau})$ is the variable describing the perturbation of a particular field (e. g., the water surface elevation or plasma density, etc.), c is the speed of linear long waves, α and β are the coefficients of nonlinearity and dispersion, respectively, which depend on the particular physical problem. Equation (1) can be transformed to the dimensionless form which is convenient for the further analysis:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\,\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)=-3\gamma\,\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where $u=\alpha v/(6\beta)$, $t=\beta\tilde{\tau}$, $x=\xi-c\tilde{\tau}$, $y=\eta\sqrt{6|\beta|/c}$, and $\gamma=\pm1$ is a dispersion parameter that plays an important role in determining the physical and mathematical properties of the solutions of this equation.

* Yury.Stepanyants@usq.edu.au

[1] School of Mathematics, Physics, and Computing, University of Southern Queensland, Toowoomba, Australia; [2] R. E. Alekseev State Technical University of Nizhny Novgorod, Nizhny Novgorod, Russia; [3] Central Michigan University, Mount Pleasant, USA; [4] Lebedev Institute of the Russian Academy of Sciences; [5] Skolkovo Institute of Science and Technology, Moscow, Russia. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 64, No. 10, pp. 739–755, October 2021. Russian DOI: 10.52452/00213462_2021_64_10_739 Original article submitted July 28, 2021; accepted October 28, 2021.

One of the simplest solutions of the Kadomtsev–Petviashvili equation is a plane soliton which can propagate at a small angle to the x axis:

$$u(x, y, t) = A \operatorname{sech}^2 (kx + ly - \omega t), \tag{3}$$

where k and l are arbitrary parameters, $A = 2k^2$ is the soliton amplitude, and $\omega = 4k^3 + 3\gamma l^2/k$. The velocity of this soliton is written as

$$\mathbf{V} = \left(\frac{\omega k}{k^2 + l^2}, \frac{\omega l}{k^2 + l^2} \right), \quad V \equiv |\mathbf{V}| = \frac{4k^4 + 3\gamma l^2}{k\sqrt{k^2 + l^2}}, \tag{4}$$

where V is the soliton speed.

Note that in the one-dimensional case where $l = 0$ and the Kadomtsev–Petviashvili equation reduces to the Korteweg–de Vries equation, the soliton speed $V = V_{\mathrm{KdV}} = \omega/k = 4k^2 = 2A$. In the two-dimensional case, assuming that $l \ll k$, we obtain from Eq. (4)

$$V = \frac{4k^4 + 3\gamma l^2}{k\sqrt{k^2 + l^2}} \approx 4k^2 \left[1 - \frac{1}{2} \left(1 - \frac{3}{2} \frac{\gamma}{k^2} \right) \frac{l^2}{k^2} \right] = V_{\mathrm{KdV}} \left[1 - \frac{1}{2} \tan^2 \varphi \left(1 - \frac{6\gamma}{V_{\mathrm{KdV}}} \right) \right], \tag{5}$$

where $\varphi = \arctan^{-1}(l/k)$ is the angle between the soliton velocity $\mathbf{V}$ and the x axis. In accordance with the Kadomtsev–Petviashvili approximation, the angle φ must be small, such that $\varphi \ll 1$. Therefore, $V \approx V_{\mathrm{KdV}}$ up to a small correction of order $\sim \varphi^2$.

As was shown in [1], plane solitons are stable with respect to small perturbations along their fronts only if the dispersion parameter $\gamma > 0$. Otherwise, they are unstable and undergo a self-focusing instability. In the case $\gamma > 0$, Eq. (2) is called the KP2 equation. It is strongly integrable in the terminology of [3]. In the case $\gamma < 0$, Eq. (2) is called the KP1 equation, and it is weakly integrable [3]. The development of the self-focusing instability of plane solitons leads to the creation of lumps, i. e., completely localized two-dimensional solitary waves.

The lump solution of the KP1 equation was constructed numerically for the first time by V. I. Petviashvili in his seminal paper [4] (see also [5]). Then such solutions were found analytically in [6]. The solution of Eq. (2) with $\gamma = -1$ describes a symmetric lump moving along the x axis (see Fig. 1*a*):

$$u = 12V \frac{9 + V^2 y^2 - 3V(x - Vt)^2}{[9 + V^2 y^2 + 3V(x - Vt)^2]^2}, \tag{6}$$

where $A = 4V/3$ is the lump amplitude and $V > 0$ is the lump speed. There are more general solutions describing lumps traveling at arbitrary angles to the x axis [6–8]. Lumps interact elastically with each other [6, 7, 9] not even undergoing phase shifts. Due to their nonmonotonic asymptotics with the local minima, they can create stationary multi-lump formations [8–11]. One of the simplest multi-lump formation, the bi-lump, is shown in Fig. 1*b* (the details of solutions can be found in the cited papers).

A single lump is stable with respect to small perturbations [12], whereas multi-lump structures and periodic chains of lumps are unstable [11]. In particular, under a small periodic perturbation along the chain front, a periodic lump chain decays into two new chains, which, in turn, are unstable with respect to small perturbations, and so on. This process repeats again and again with smaller and smaller decay rates [13]. The lifetime of a relatively rarefied chain of lumps with a great distance between them can be fairly long. Therefore, such chains can be observable both in experiments and in nature. Similarly, if the plane soliton has a periodic perturbation along its front, then as a result of the focusing instability, a new plane soliton of a smaller amplitude arises and is accompanied by the periodic chain of lumps moving side by side to each other [13]. Such an instability occurs only with respect to a long perturbation with the wavelength $\Lambda > \Lambda_c$ along the y axis, where Λ_c is inversely proportional to the soliton amplitude and much greater than the soliton width [14].

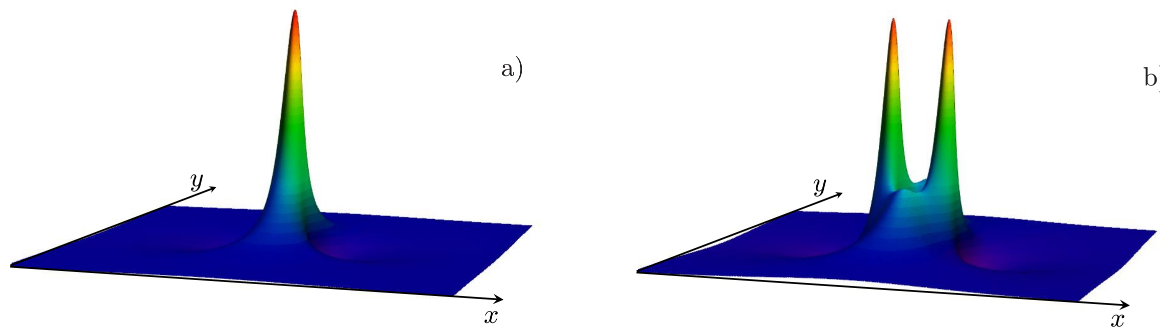

Fig. 1. Three-dimensional plot of the lump solution (6) with $V = 3$ (*a*) and the bi-lump solution with $V = 3$ (*b*).

As was recently shown [15], lump chains moving at an angle to each other can interact undergoing spatial phase shifts similarly to interacting plane solitons (see, e. g., [16]). Under certain conditions, two interacting lump chains can form a single lump chain or, vice versa, one lump chain can split into two other lump chains moving at an angle to each other in space. The entire pattern moves stationarily as a plane soliton triad under resonance interaction within the framework of the KP2 equation (see [17] and references therein).

Another interesting observation was made in recent publications [18–20], where it was shown that within the framework of the KP1 equation, there are solutions that represent lump emission from a plane soliton or lump absorption by a plane soliton. A lump can be emitted by one plane soliton and absorbed by another plane soliton. A lump can be obscured between two plane solitons traveling parallel to each other and having equal amplitudes at infinity. There are many other solutions representing a number of plane solitons exchanging lumps.

Here, we review such solutions and analyze in detail the interaction of lumps with plane solitons, using the Grammian form of the τ-function to express the solution of the KP1 equation [15]. We show that such an approach allows one to present different types of soliton and lump interactions in a relatively simple and natural way.

2. INTERACTION OF A LINE SOLITON WITH LUMPS

There are two standard formulas for the solution of the KP1 equation in terms of the τ-function, both of them involving a set of solutions to an auxiliary linear system. Of the two, more commonly forms used for representing the solution is based on the Wronskian:

$$u(x,y,t) = 2\,\frac{\partial^2 \log\tau}{\partial x^2}, \quad \tau(x,y,t) = \mathrm{Wr}(\psi_1,\ldots,\psi_M). \tag{7}$$

Here, $\mathrm{Wr}(\psi_1,\ldots,\psi_M)$ is the Wronskian of a linearly independent set of solutions of the system

$$\psi_y = i\psi_{xx}, \quad \psi_t = -4\psi_{xxx}. \tag{8}$$

Note that this is a set of linear equations which can be solved by separation of variables. Such an approach has been used in [21] for the construction of the simplest solutions within the Zakharov–Shabat general scheme [2].

In our paper, we instead consider solutions defined by the so-called Grammian formula. As before, let $(\psi_1,\ldots,\psi_M)$ be a set of solutions (not necessarily linearly independent) of Eq. (8), and let c_{jk} be a constant $M\times M$ matrix. Then the function given by

$$u(x,y,t) = 2\,\frac{\partial^2 \log\tau}{\partial x^2}, \quad \tau(x,y,t) = \det\left[c_{jk} + \int\limits_{-\infty}^{x} \psi_j(x',y,t)\,\overline{\psi}_k(x',y,t)\,dx'\right] \tag{9}$$

is a solution of the KP1 equation (2). The solution is nonsingular if the Wronskian is everywhere positive, and we will verify this condition each time when applying this formula.

To construct solutions of Eq. (8), let us denote

$$\phi(x,y,t,\lambda) = \lambda x + i\lambda^2 y - 4\lambda^3 t. \tag{10}$$

Then, for any value of λ, the function $\psi(x,y,t) = \exp[\phi(x,y,t,\lambda)]$ satisfies Eq. (8). More generally, let $p_s(x,y,t,\lambda)$ denote a homogeneous polynomial of degree s in x, y, and t that is defined by the formula

$$p_s(x,y,t,\lambda) = \exp[-\phi(x,y,t,\lambda)]\frac{\partial^s}{\partial\lambda^s}\exp[\phi(x,y,t,\lambda)], \tag{11}$$

where $p_0 = 1, \quad p_1 = x+2i\lambda y-12\lambda^2 t, \quad p_2 = p_1^2+2iy-24\lambda t, \ \ldots$ It is easy to see that any function of the form $p_s(x,y,t,\lambda)\exp[\phi(x,y,t,\lambda)]$ satisfies Eq. (8). Using Eq. (9), we construct a wide family of solutions of the KP1 equation by choosing the functions ψ_j as linear combinations of the functions $p_s(x,y,t,\lambda)\exp[\phi(x,y,t,\lambda)]$ for various values of s and λ.

To simplify the exposition, we assume that all $\lambda_j = a_j > 0$ are real-valued. Let $s_1,\ldots,s_M$ be nonnegative integers, and let $b_1,\ldots,b_M$ be real constants. We set

$$\psi_j(x,y,t) = b_j\, p_{s_j}(x,y,t,a_j)\exp[\phi(x,y,t,a_j)], \quad j=1,\ldots,M. \tag{12}$$

In this case, one can readily calculate

$$\int\limits_{-\infty}^{x} \psi_j(x',y,t)\,\overline{\psi}_k(x',y,t)\,\mathrm{d}x' = \frac{b_j b_k}{a_j+a_k} p_{s_j s_k}(x,y,t,a_j,a_k)\exp[\phi(x,y,t,a_j)+\overline{\phi}(x,y,t,a_k)], \tag{13}$$

where $p_{jk}(x,y,t,a,a')$ is a nonhomogeneous polynomial of degree $j+k$, given by

$$p_{jk}(x,y,t,a,a') = \sum_{l=0}^{j+k}\left(\frac{-1}{a+a'}\right)^l \frac{\partial^l}{\partial x^l}[p_j(x,y,t,a)\,\overline{p}_k(x,y,t,a')]. \tag{14}$$

For future use, we present the first few polynomials p_{jk}:

$$p_{00}=1, \quad p_{01} = x - 12a^2 t - \frac{1}{a+a'} - 2iay, \quad p_{10} = x - 12a'^2 t - \frac{1}{a+a'} + 2iay, \tag{15}$$

$$p_{11}|_{a'=a} = \left(x - 12a^2 t - \frac{1}{2a}\right)^2 + 4a^2y^2 + \frac{1}{4a^2}. \tag{16}$$

Inserting Eq. (14) into Eq. (9) and factoring out a common exponential (which disappears after applying the second logarithmic derivative), we obtain the solution in the form

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log\det\left[c_{jk}\exp[-\phi(x,y,t,a_j)-\overline{\phi}(x,y,t,a_k)] + \frac{b_j b_k}{2(a_i+a_j)}p_{s_j s_k}(x,y,t,a_j,a_k)\right]. \tag{17}$$

We now consider several families of solutions of the KP1 equation (2), which are given by the above formula. These solutions describe the elementary processes of interaction of traveling waves, known as line solitons, with localized disturbances, known as lumps.

2.1. Line soliton

The standard line-soliton solution of the KP1 equation is obtained by setting $M = 1$, $a_1 = a$, $c_{11} = 1$, and $s_1 = 0$. In this case, the solution is written as

$$u(x,y,t) = 2\,\frac{\partial^2}{\partial x^2}\log\{\exp[-2F(x,t,a)] + C_0\} = 2a^2\operatorname{sech}^2\left(ax - 4a^3t + \frac{1}{2}\log C_0\right), \quad C_0 = \frac{b_1^2}{2a}, \tag{18}$$

where

$$F(x,t,a) = ax - 4a^3t. \tag{19}$$

The terms $\exp(-2F)$ and C_0 in the logarithm are equal on the vertical line $x = 4a^2t - \log C_0/(2a)$. The solution is supported on a narrow strip centered on this line (see the left-hand panel in Fig. 2 below). Away from the line, one of the two terms in the logarithm is dominant and $u(x,y,t)$ is exponentially small. Hence, the solution represents a solitary wave traveling to the right with the speed $V_\mathrm{s} = 4a^2$.

2.2. One-lump solution

If we put $c_{jk} = 0$, then the τ-function in Eq. (17) is a polynomial. The solution $u(x,y,t)$ is a rational function and consists of a set of localized lumps, which may be bound or undergo anomalous scattering. To obtain the simplest one-lump solution, we set $M = 1$, $a_1 = a$, and $s_1 = 1$, so that

$$u(x,y,t) = 2\,\frac{\partial^2}{\partial x^2}\ln\tau_\mathrm{l}(x,y,t,a), \tag{20}$$

where the τ-function τ_l of the lump is given by Eq. (16) as

$$\tau_\mathrm{l}(x,y,t,a) = p_{11}(x,y,t,a,a) = \left(x - 12a^2t - \frac{1}{2a}\right)^2 + 4a^2y^2 + \frac{1}{4a^2}. \tag{21}$$

The solution is essentially the same as in Eq. (6). It is centered at the point $(x,y) = (12a^2t + 1/2a, 0)$ and represents a lump traveling to the right with the speed $V = 12a^2$. The solution has a local maximum at the center and decays algebraically as $(x^2 + y^2)^{-1}$ at infinity (see Fig. 3).

2.3. Line soliton absorbing or emitting a lump

Let us consider the following set of parameters: $M = 2$, $a_1 = a_2 = a$, $s_1 = 0$, and $s_2 = 1$. We also assume that c_{jk} is the rank one matrix

$$c_{jk} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{22}$$

In this case, after factoring out a constant, the solution reads

$$u(x,y,t) = 2\,\frac{\partial^2}{\partial x^2}\log\Big\{\tau_\mathrm{l}(x,y,t,a) + C_1\exp[2F(x,t,a)]\Big\}, \quad C_1 = \frac{b_1^2}{8a^3}. \tag{23}$$

In contrast to the two previous cases, this solution is nonstationary. For a fixed moment of time t, consider the curve $2F = \log\tau_\mathrm{l} - \log C_1$ in the (x,y) plane along which the two terms in the logarithm are equal. On the right of this curve, the dominant term is $\exp(2F)$, and the solution u is exponentially small. On the left, the dominant term is τ_l, which is given by Eq. (21) and produces a lump solution moving along the x axis with the speed $V_\mathrm{l} = 12a^2$. Along the curve, there is a line soliton (deformed near $y = 0$ by the term τ_l) moving to the right with the speed $V_\mathrm{s} = 4a^2$. Since the lump moves with the speed $V_\mathrm{l} = 3V_\mathrm{s}$, it eventually merges with the line soliton and is absorbed by it.

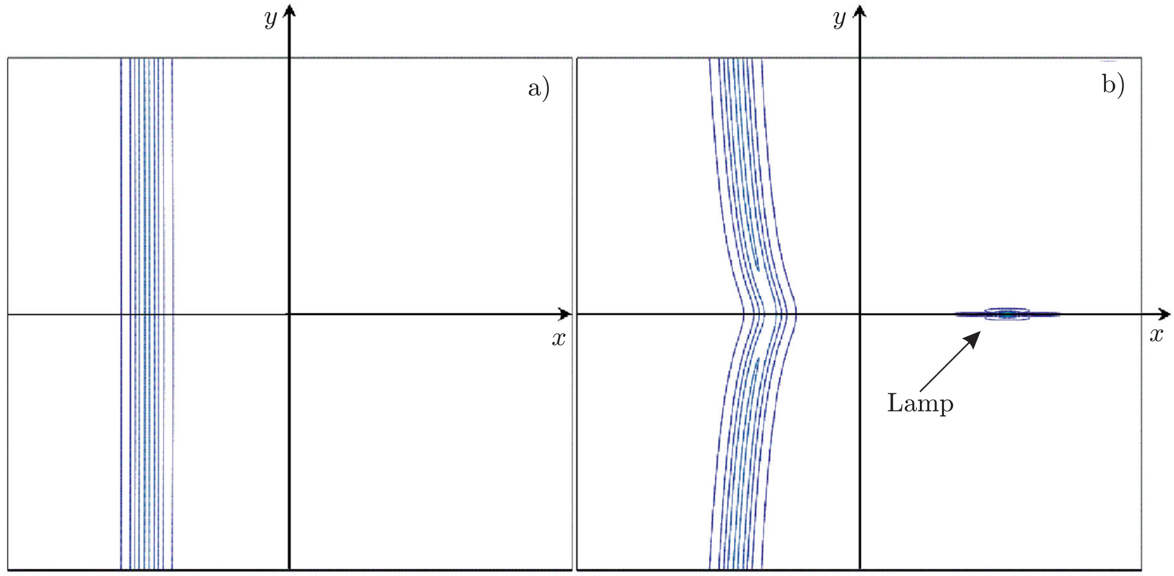

Fig. 2. Slightly perturbed line soliton at $t = -100$ (a) radiating a lump at $t = 0$ (b). The vertical scale on both panels is 5 times greater than the horizontal scale. Panel a: $-430 \le x \le -390$. Panel b: $-30 \le x \le 10$. On both panels, $-100 \le y \le 100$.

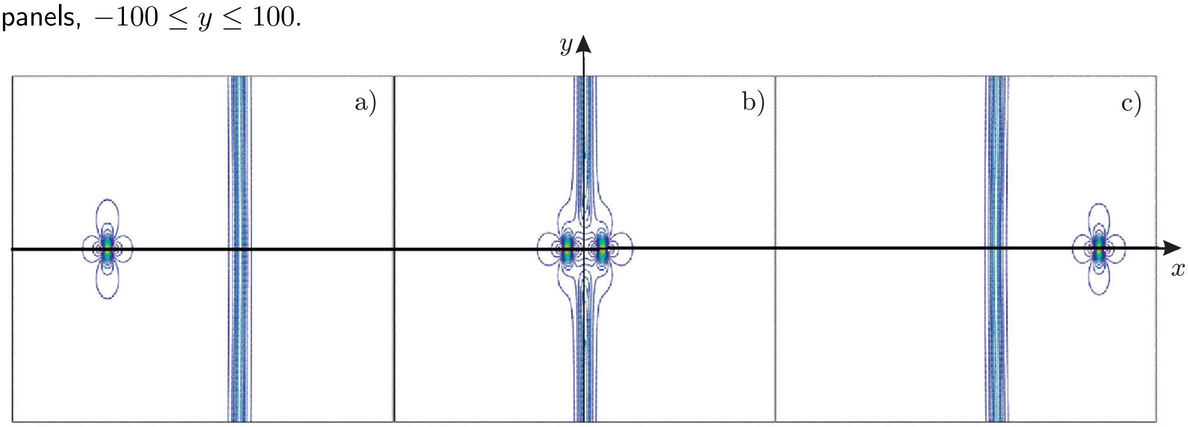

Fig. 3. Line-soliton overtaking by a lump of Eq. (25): $t = -200$ and $-500 \le x \le -300$ (a), $t = -85$ and $-175 \le x \le 25$ (b), and $t = 0$ and $-150 \le x \le 50$ (c). On all panels, $-100 \le y \le 100$.

One can also construct a line soliton emitting a lump. Such a situation occurs if we set $M = 2$, $a_1 = a_2 = a$, $s_1 = s_2 = 1$, and $c_{jk} = \delta_{jk}$, where δ_{jk} is the Kronecker symbol. Indeed, in this case, we factor out the exponential function $\exp(-2F)$, which disappears after differentiation, and obtain

$$u(x, y, t) = 2\,\frac{\partial^2}{\partial x^2}\log\Big\{\exp[-2F(x, t, a)] + \frac{b_1^2 + b_2^2}{2a}\tau_l(x, y, t, a)\Big\}. \tag{24}$$

The solution again consists of a line soliton moving with the speed $V_s = 4a^2$ along the curve where the two terms are equal, and a lump moving with the speed $V_l = 12a^2$. However, unlike the previous case, the term τ_l, which produces a lump, is now dominant on the right of the line soliton. Hence, the line soliton radiates a lump in the process of evolution. Solution (24) is illustrated by the contour lines in Fig. 2 at two time instances with the following parameters: $a = 1$, $b_1 = 0$, and $b_2 = 10^6$. The line soliton with the bent front shown in Fig. 2b asymptotically becomes straight as in Fig. 2a.

2.4. Lump passing through a line soliton

As it has been demonstrated above, a Kadomtsev–Petviashvili lump moving with the speed V_l can be absorbed or emitted by a line soliton moving with the speed V_s if $V_l/V_s = 3$. If $V_l/V_s \neq 3$, then the lump can pass through the line soliton or may create together with it a stationary pattern when $V_l/V_s = 1$.

To construct such a solution, let us set $M = 2$, $a_2 \neq a_1$, $s_0 = 1$, and $s_1 = 1$ and assume that c_{jk} is a rank one matrix (22). In this case, one can obtain the following solution:

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \log\Big\{\tau_1(x, y, t, a_2) + C_2\tau_1(x - x_0, y, t, a_2)\exp[2F(x, t, a_1)]\Big\}, \tag{25}$$

where

$$C_2 = \frac{(a_1 - a_2)^2 b_1^2}{a_1(a_1 + a_2)^2}, \qquad x_0 = \frac{2a_1}{a_1^2 - a_2^2}. \tag{26}$$

In this solution, there is a line soliton along the vertical line $2F = -\log C_2$, which moves with the speed $V_s = 4a_1^2$. On both sides of the soliton, there is a one-lump solution moving with the same speed $V_l = 12a_2^2$ and having a relative phase shift x_0 between them.

If $a_2 < a_1 < a_2\sqrt{3}$, then the solution consists of a lump overtaking a line soliton from the left and reappearing on the right of the line soliton with the phase shift x_0. If $a_1 > a_2\sqrt{3}$, then the line soliton overtakes the lump, which reappears behind it. In either case, there exists a relatively short time interval during which both lumps are visible. This situation is demonstrated in Fig. 3.

In the intermediate case $a_1 = a_2\sqrt{3}$, the two lumps are traveling with the same speed as the line soliton, and the solution is stationary. The stationary solution is shown in Fig. 4 for the following parameters:

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{\sqrt{12}},$$

$$b_1 = \frac{\left(\sqrt{3}+1\right)^2}{2} \exp\left[-\frac{\sqrt{3}}{2}\left(\sqrt{3}+1\right)\right] \approx 0{,}35,$$

$$b_2 = \sqrt[4]{3}.$$

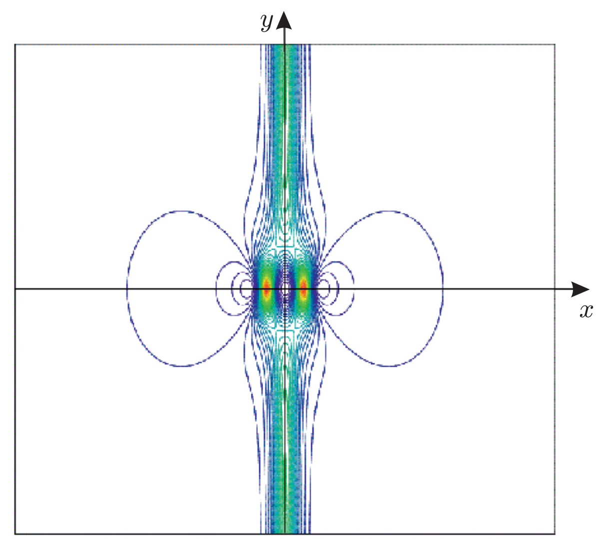

Fig. 4. Two lumps having equal amplitudes and bounded with a line soliton: $-50 \leq x \leq 50$ and $-50 \leq y \leq 50$.

In this case, the τ-function has the form

$$\tau(\xi, y) = 3(\xi + 3)^2 + y^2 + 9 + \exp(\xi)\left[3(\xi - 3)^2 + y^2 + 9\right], \tag{27}$$

where $\xi = x - Vt$ and $V = 1$. This and some other stationary solutions were obtained in [22]. In particular, solutions representing a chain of lumps intersecting a plane soliton at an angle were found in that paper. These solutions will be described in Sec. 3.

In the case $a_1 \to a_2$, the phase shift x_0 indefinitely increases, the line soliton and the right-hand lump disappear at plus infinity, and only the left-hand lump remains. For $a_1 < a_2$, the solution again consists of a lump overtaking a line soliton.

2.5. Two weakly bound line solitons exchanging a lump

We now consider solution (17) with $M = 2$, $a_1 = a_2 = a$, $s_1 = 0$, $s_2 = 1$, and $c_{jk} = \delta_{jk}$. In this case, a straightforward calculation yields

$$u(x,y,t) = 2\,\frac{\partial^2}{\partial x^2}\log\Big\{\exp[-2F(x,t,a)] + Q(x,y,t) + C_3\exp[2F(x,t,a)]\Big\}, \tag{28}$$

where Q and C_3 are expressed in terms of the one-lump function τ_l (see Eq. (21)) as follows:

$$Q(x,y,t) = \frac{b_2^2}{2a}\tau_l(x,y,t,a) + \frac{b_1^2}{2a}, \quad C_3 = \frac{b_1^2 b_2^2}{16a^4}. \tag{29}$$

The structure of solution (28) is determined by the relative values of the three terms under the logarithm. At a fixed moment of time t, the (x,y) plane is divided into three regions, in each of which one of the three terms in the logarithm of Eq. (28) is dominant:

$$\begin{aligned}\Delta_1(t) &= \big\{(x,y) : \exp(-2F) \geq \max[Q, C_3\exp(2F)]\big\},\\ \Delta_2(t) &= \big\{(x,y) : Q \geq \max[\exp(-2F), C_3\exp(2F)]\big\},\\ \Delta_3(t) &= \big\{(x,y) : C_3\exp(2F) \geq \max[\exp(-2F), Q]\big\}.\end{aligned} \tag{30}$$

To determine the shape of the regions $\Delta_i(t)$, consider the three functions $\exp(-2F)$, Q, and $C_3\exp(2F)$ on the x axis for fixed values of y and z. It is clear that $\exp(-2F)$ is dominant when $x \to -\infty$, whereas $C_3\exp(2F)$ is dominant when $x \to +\infty$. At the same time, there is always an intermediate region where Q is the dominant term. Indeed, the minimum value of the function $\max(\exp(-2F), C_3\exp(-F))$ is $\sqrt{C_3}$, whereas

$$Q(x,y,t) \geq \frac{1}{2}\left(\frac{b_2^2}{4a^3} + \frac{b_1^2}{a}\right) \geq \sqrt{\frac{b_2^2}{4a^3}\cdot\frac{b_1^2}{a}} = 2\sqrt{C_3}. \tag{31}$$

The length of this intermediate interval on the x axis is smallest when $y = 0$ and increases logarithmically when $|y| \to \infty$. It follows that the region Δ_2 is a vertical strip separating the region Δ_1 on the left and the region Δ_3 on the right.

Thus, solution (28) is as follows. In the interior of the regions Δ_1 and Δ_3, where one of the exponential functions, either $\exp(-2F)$ or $C_3\exp(2F)$, is dominant in Eq. (17), the solution is exponentially small. In the region Δ_2, the solution is approximately equal to $u \approx 2\,\partial^2\log Q/\partial x^2$. In the limiting case $b_1/b_2 \to 0$, it is a lump traveling to the right with the velocity $V_l = 12a^2$, while for $b_1 > b_2$, the lump is flattened and becomes invisible when $b_1/b_2 \to +\infty$. Along the curves $\Delta_1 \cap \Delta_2$ and $\Delta_2 \cap \Delta_3$, there are two line solitons traveling to the right with the same velocity $V_s = 4a^2$. Due to the lump influence, the solitons are deformed and are closest at $y = 0$, and the distance between them increases logarithmically as $|y| \to +\infty$. Since $V_l = 3V_s$ as before, the lump is emitted by the left-hand line soliton and absorbed by the right-hand line soliton. We note that solution (23) can be obtained in the limit $b_2 \to +\infty$, while solution (24) corresponds to the limit $b_1 \to 0$. In both limits, one of the two line solitons disappears at infinity.

The solution is shown in Fig. 5 for $a = 1$, $b_1 = 10^2$, and $b_2 = 10^6$ and in Fig. 6 for $a = 1$ and $b_1 = b_2 = 1$. In the former case, a lump is clearly seen between two line solitons (see Fig. 5*b*), whereas in the latter case, the lump is flattened and cannot be distinguished from the interacting line solitons.

We now interpret the obtained solution. Two parallel KP1 line solitons having equal amplitudes and separated by a large distance can be described by the Korteweg–de Vries equation in the ideal case where their fronts are not perturbed along the y direction. In such a case, they will undergo an exchange-type interaction: a portion of the energy from the rear soliton will be transmitted to the front soliton. Since the front soliton has a larger amplitude, the distance between the solitons will increase linearly in time, proportionally to the amplitude difference.

The solution that we construct consists of two parallel line solitons of equal amplitude that are

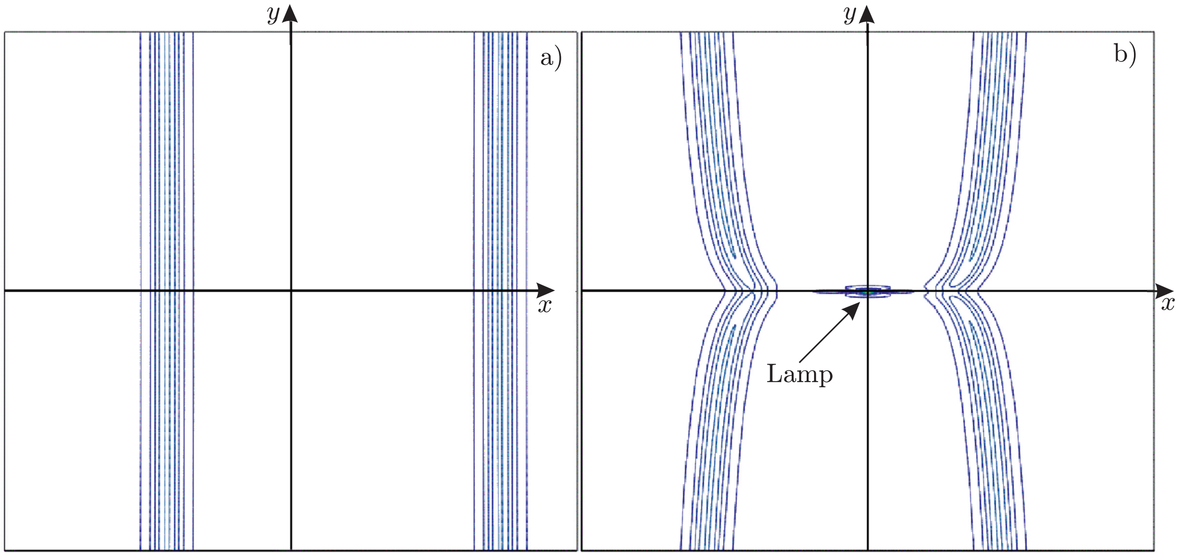

Fig. 5. Two quasi-parallel solitons of equal amplitudes at $t = -100$ (a). In the process of evolution, the left-hand soliton radiates a lump at $t = -1$, which is eventually absorbed by the right-hand soliton. On both panels, the vertical size is 5 times longer than the horizontal size. Panel a: $-431.5 \le x \le -391.5$. Panel b: $-31.5 \le x \le 8.5$. On both panels, $-100 \le y \le 100$.

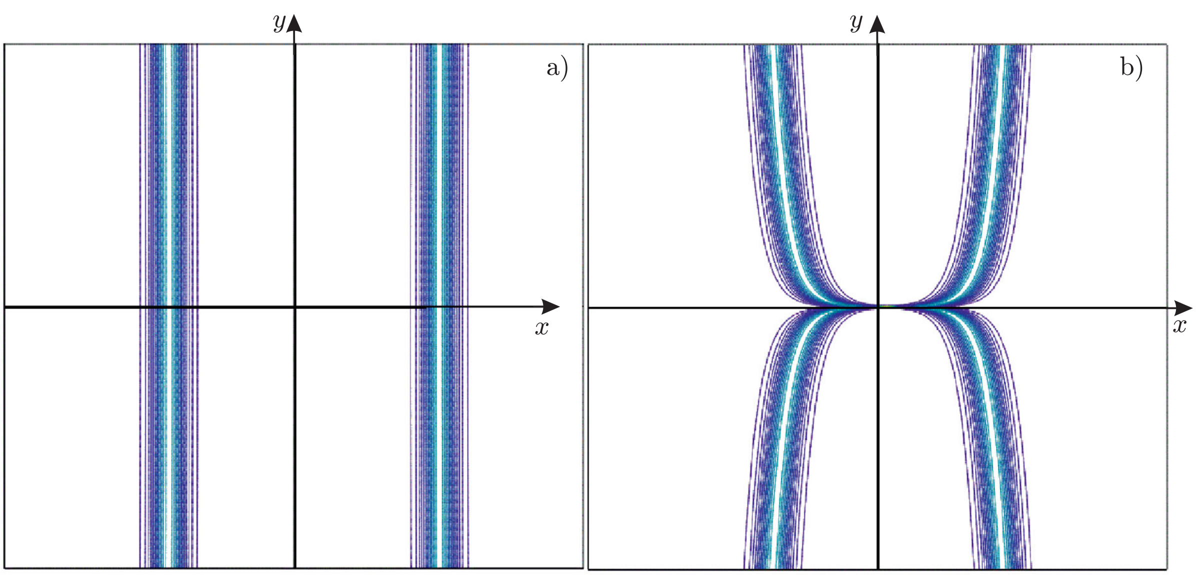

Fig. 6. Two quasi-parallel solitons of equal amplitudes at $t = -1000$ (a) and $t = 0$ (b). A flat-head lump is obscured between the solitons in the field of their tails. The vertical size on both panels is 50 times longer than the horizontal size. Panel a: $-4020 \le x \le -3980$. Panel b: $-20 \le x \le 20$. On both panels, $-1000 \le y \le 1000$.

infinitesimally perturbed at $t \to -\infty$. In this case, the one-dimensional Korteweg–de Vries equation is insufficient and the two-dimensional KP1 equation must be used. The perturbation increases with time and becomes a lump. The distance between the solitons decreases logarithmically with time until the solitons exchange a lump. Then the solitons diverge logarithmically with time. At $t \to +\infty$, we have again two parallel line solitons of equal amplitude.

3. INTERACTION OF A LINE SOLITON AND LUMP CHAINS

In Sec. 2, we presented solutions of the KP1 equation in the form of line solitons interacting with individual lumps. The KP1 equation also has solutions describing infinite sequences of lumps, known as lump chains [15,23,24]. Lump chains can interact with line solitons in a manner similar to individual lumps. Below we present several examples of such interactions in terms of the Grammians and the corresponding τ-functions.

3.1. Lump chain

To construct an isolated lump chain, we follow the method of [15] and consider, as the generating function, the Grammian τ-function (9) with $M = 1$ and $c_{11} = 0$. Let us choose two real-valued spectral parameters $a_1 < a_2$, real-valued phases ρ_1 and ρ_2, and the ψ-function in the form

$$\psi(x,y,t) = \psi_1(x,y,t) = \sqrt{2a_1}\exp[\phi(x,y,t,a_1)+\rho_1] + \sqrt{2a_2}\exp[\phi(x,y,t,a_2)+\rho_2]. \tag{32}$$

Then, we can calculate the corresponding τ-function as

$$\tau_{\mathrm{c}}(x,y,t) = \int\limits_{-\infty}^{x} |\psi(x',y,t)|^2\,\mathrm{d}x = \exp(2F_1)+\exp(2F_2)+C_4\exp(F_1+F_2)\cos\left[\left(a_1^2-a_2^2\right)y\right], \tag{33}$$

where

$$F_1(x,t) = a_1x - 4a_1^3t + \rho_1, \quad F_2(x,t) = a_2x - 4a_2^3t + \rho_2, \quad C_4 = 4\frac{\sqrt{a_1a_2}}{a_1+a_2}. \tag{34}$$

The two exponentials $\exp(2F_1)$ and $\exp(2F_2)$ are equal along the vertical line in the (x,y) plane:

$$x = V_{\mathrm{c}}t - \rho_{12}, \quad V_{\mathrm{c}} = 4(a_1^2 + a_1a_2 + a_2^2), \quad \rho_{12} = \frac{\rho_1-\rho_2}{a_1-a_2}. \tag{35}$$

Away from this line, one of the two exponentials is dominant and the corresponding solution $u(x,y,t)$ is exponentially small. Meanwhile, it can be shown that the term containing the cosine function is never dominant. Hence, the solution is concentrated in a narrow strip along this line and consists of an evenly spaced sequence of lumps. The lumps propagate to the right with the speed V_{c}. By an appropriate choice of the spectral parameters, this solution can be represented in two forms [22]. One of them describes a line of lumps periodic in the y direction and moving along the x direction:

$$\tau = \cosh\left(k\xi\sqrt{V_{\mathrm{c}}}\right) - \sqrt{\frac{1-4k^2}{1-k^2}}\cos\left(y\frac{kV_{\mathrm{c}}}{\sqrt{3}}\sqrt{1-k^2}\right), \tag{36}$$

where $\xi = x - V_{\mathrm{c}}t$ and $|k| < 1/2$. Another solution can be obtained from the previous one by replacing $k = i\kappa$. It describes a line of lumps periodic in the direction of motion (x direction):

$$\tau = \cos\left(\kappa\xi\sqrt{V_{\mathrm{c}}}\right) + \sqrt{\frac{1+4\kappa^2}{1+\kappa^2}}\cosh\left(y\frac{\kappa V_{\mathrm{c}}}{\sqrt{3}}\sqrt{1+\kappa^2}\right). \tag{37}$$

Both of these solutions reduce to the solution for a lump when $k \to 0$ or $\kappa \to 0$. Then the degenerate solution becomes

$$\tau = 3V\xi^2 + V^2y^2 + 9. \tag{38}$$

3.2. Line soliton radiating a lump chain

Let us choose again two real-valued spectral parameters $a_1 < a_2$ and real-valued phases ρ_1 and ρ_2, but now we modify the τ-function in Eq. (33) by setting $c_{11} = 1$:

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2}\log\Big\{1 + \exp(2F_1) + \exp(2F_2) + C_4\exp(F_1+F_2)\cos\big[\big(a_1^2 - a_2^2\big)\,y\big]\Big\}. \tag{39}$$

The new τ-function $\tau(x,y,t) = 1 + \tau_c(x,y,t)$ has three terms that may be dominant, namely, 1, $\exp(2F_1)$, and $\exp(2F_2)$. The relative values of these terms depend on the relationship between x and t:

$$\begin{array}{rll} \exp[2F_2(x,t)] > \exp[2F_1(x,t)] & \text{if} & x > V_c t - \rho_{12}, \\ \exp[2F_2(x,t)] < \exp[2F_1(x,t)] & \text{if} & x < V_c t - \rho_{12}; \\ \exp[2F_1(x,t)] > 1 & \text{if} & x > 4a_1^2 t - \rho_1/a_1, \\ \exp[2F_1(x,t)] < 1 & \text{if} & x > 4a_1^2 t - \rho_1/a_1; \\ \exp[2F_2(x,t)] > 1 & \text{if} & x > 4a_2^2 t - \rho_2/a_2, \\ \exp[2F_2(x,t)] < 1 & \text{if} & x > 4a_2^2 t - \rho_2/a_2. \end{array}$$

Since $4a_1^2 < 4a_2^2 < V_c \equiv 4(a_1^2 + a_1 a_2 + a_2^2)$, for sufficiently large positive t, the dominant terms in the τ-function are as follows in the order of increasing x:

$$\begin{array}{lll} 1 & \text{if} & x < 4a_1^2 t - \rho_1/a_1; \\ \exp(2F_1) & \text{if} & 4a_1^2 t - \rho_1/a_1 < x < V_c t - \rho_{12}; \\ \exp(2F_2) & \text{if} & x > V_c t - \rho_{12}. \end{array} \tag{40}$$

Along the vertical line $x = 4a_1^2 t - \rho_1/a_1$, we have $1 \sim \exp(2F_1) \gg \exp(2F_2)$. This corresponds to a line soliton moving with the speed $V_{s1} = 4a_1^2$. Similarly, along the vertical line $x = 4a_1^2 t - \rho_1/a_1$, we have $\exp(2F_1) \sim \exp(2F_2) \gg 1$, so that $\tau(x,y,t) \approx \tau_c(x,y,t)$, which corresponds to a lump chain moving with the speed V_c. Away from these two lines, the solution is exponentially small.

For sufficiently large negative t, the dominant terms in the τ-function are

$$1 \text{ if } x < 4a_2^2 t - \rho_2/a_2; \qquad \exp(2F_2) \text{ if } x > 4a_2^2 t - \rho_2/a_2. \tag{41}$$

This corresponds to a line soliton moving with the speed $V_{s2} = 4a_2^2$ along the vertical line $x > 4a_2^2 t - \rho_2/a_2$ where $1 = \exp(2F_2) \gg \exp(2F_1)$. The solution is exponentially small away from this line.

Hence, the solution has the following structure. For $t < 0$, there is a line soliton traveling with the speed $V_{s2} = 4a_2^2$. At a certain moment of time, the soliton radiates a vertical chain of lumps periodic in the y direction, which propagates away from the original soliton with the speed $V_c > V_{s2}$ and the other line soliton moving with the lower speed $V_{s1} = 4a_1^2$. Such a solution was obtained for the first time in [13]. We note that if the spectral parameters λ_1 and λ_2 are complex-valued, then the solution consists of a bent line soliton emitting a lump chain at an angle (see [15]). As λ_1 and λ_2 become real-valued, the triple point at which the lump chain is emitted goes to infinity.

3.3. Line soliton absorbing a lump chain

There is an inverse process when a line soliton absorbs a periodic lump chain. To derive such a solution, we choose the following parameters: $a_1 < a_2$, the real phases ρ_1 and ρ_2, and $M = 2$. We also assume that c_{jk} is a rank one matrix with $c_{22} = 1$ and the other elements $c_{jk} = 0$. Then we choose the following auxiliary functions:

$$\begin{aligned} \psi_1(x,y,t) &= \sqrt{2a_1}\exp[\phi(x,y,t,a_1)] + \sqrt{2a_2}\exp[\phi(x,y,t,a_2)], \\ \psi_2 &= \sqrt{2a_1}\exp[\phi(x,y,t,a_1)]. \end{aligned} \tag{42}$$

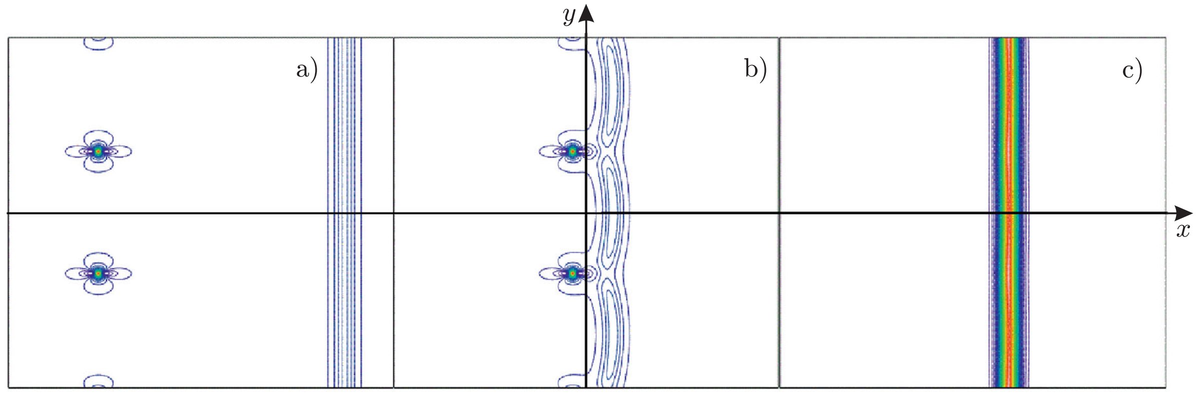

Fig. 7. Absorption of a periodic chain of lumps by a line soliton. The amplitudes of lumps are $A_l = 3.2$, the amplitude of the initial line soliton is $A_{s1} = 2a_1^2 = 0.32$, the amplitude of the resulting line soliton is $A_{s2} = 2a_2^2 = 0.5$, $t = -30$, and $-100 \le x \le 0$ (a), $t = 0$ and $-50 \le x \le 50$ (b), and $t = 30$ and $-25 \le x \le 75$ (c). On all panels, $-100 \le y \le 100$.

A straightforward calculation shows that the τ-function in this case is given by

$$\tau(x,y,t) = \exp(2F_1) + \exp(2F_2) + C_4 \exp(F_1 + F_2) \cos\left[\left(a_1^2 - a_2^2\right) y\right] + C_5 \exp(2F_1 + 2F_2), \; C_5 = \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2}. \tag{43}$$

The three terms to be compared in the τ-function are $\exp(2F_1)$, $\exp(2F_2)$, and $C_5 \exp(2F_1 + 2F_2)$. An analysis similar to that presented above shows that for the large positive t there exists a single line soliton moving with the speed $V_{s2} = 4a_2^2$, while for the large negative t there exists a line soliton moving with the speed $V_{s1} = 4a_1^2$ and a lump chain moving with the speed V_c. At a certain moment of time, the lump chain is absorbed by the line soliton, causing its speed to increase from V_{s1} to V_{s2}. This process is illustrated by Fig. 7. In Fig. 7*a* for $t = -30$, one can see a fragment containing two lumps on the left and a line soliton at a long distance from the lump chain on the right. In Fig. 7*b* for $t = 0$, one can see the lump chain approaching the line soliton which becomes noticeably modulated. In Fig. 7*c* for $t = 30$, there is only one line soliton which has absorbed the lump chain and now moves with a higher speed and a larger amplitude.

3.4. Line solitons exchanging a lump chain

Finally, let us construct a solution consisting of two line solitons exchanging a chain of lumps in the process of their interaction. As above, we set $M = 2$, choose the spectral parameters $a_1 < a_2$ and the real phases ρ_1, ρ_2, and ρ_3, and use the auxiliary functions

$$\psi_1(x,y,t) = \sqrt{2a_1} \exp[\phi(x,y,t,a_1) + \rho_1] + \sqrt{2a_2} \exp[\phi(x,y,t,a_2) + \rho_2],$$
$$\psi_2(x,y,t) = \sqrt{2a_1} \exp[\phi(x,y,t,a_1) + \rho_3]. \tag{44}$$

We now set $c_{jk} = \delta_{jk}$. A calculation shows that the τ-function in this case is

$$\tau(x,y,t) = 1 + (1 + C_6) \exp(2F_1) + \exp(2F_2) + C_4 \exp(F_1 + F_2) \cos\left[\left(a_1^2 - a_2^2\right) y\right] + C_5 C_6 \exp(2F_1 + 2F_2), \tag{45}$$

where

$$C_4 = 4\frac{\sqrt{a_1 a_2}}{a_1 + a_2}, \quad C_5 = \frac{(a_1 - a_2)^2}{(a_1 + a_2)^2}, \quad C_6 = \exp[2(\rho_3 - \rho_1)].$$

The four terms in the τ-function that may be dominant are 1, $(1 + C_6)\exp(2F_1)$, $\exp(2F_2)$, and

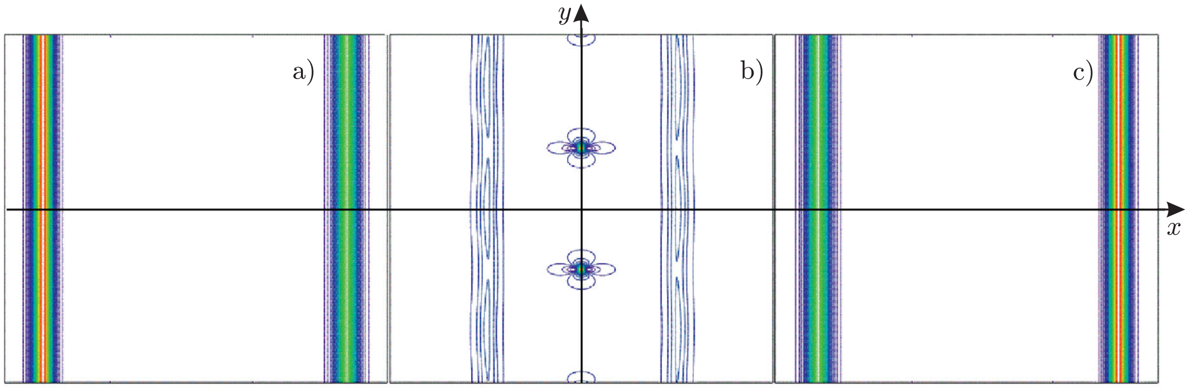

Fig. 8. Exchange-type interaction of two line solitons with the parameters $a_1 = 0.4$ and $a_2 = 0.5$ (the corresponding amplitudes $A_{s1} = 0.32$ and $A_{s2} = 0.5$) if the lump amplitudes amount to $A_l = 3.2$: $t = -80$ and $-90 \le x \le 10$ (a), $t = 16$ and $-14.4 \le x \le 85.6$ (b), and $t = 120$ and $50 \le x \le 150$ (c). On all panels, $-100 \le y \le 100$.

$C_5C_6\exp(2F_1 + 2F_2)$, and the solution is nonvanishing near the lines along which two of these terms are equal and greater than the other two.

To simplify the exposition, we set $\rho_1 = -\log(1 + C_6)/2$ and $\rho_2 = 0$, which can be achieved by translating in x and t. To find the dominant term, we analyze the possible equalities (we consider only the case where C_6 is small, so the dominant pair cannot be 1 and $C_5C_6\exp(2F_1 + 2F_2)$):

$$\begin{aligned}
1 = (1 + C_6)\exp(2F_1) \quad & \text{when} \quad x = x_1(t) = 4a_1^2 t;\\
1 = \exp(2F_2) \quad & \text{when} \quad x = x_2(t) = 4a_2^2 t;\\
(1 + C_6)\exp(2F_1) = C_5C_6\exp(2F_1 + 2F_2) \quad & \text{when} \quad x = x_3(t) = 4a_2^2 t + [\log(1 + C_6^{-1}) - \log C_5]/(2a_2);\\
\exp(2F_2) = C_5C_6\exp(2F_1 + 2F_2) \quad & \text{when} \quad x = x_4(t) = 4a_1^2 t + [\log(1 + C_6^{-1}) - \log C_5]/(2a_1);\\
(1 + C_6)\exp(2F_1) = \exp(2F_2) \quad & \text{when} \quad x = x_5(t) = V_c t.
\end{aligned}$$

Carefully analyzing the relative values of the terms for different x, we see that there are three possibilities, depending on the time t.

1. If $t < 0$, then 1 is dominant for $x < x_2(t)$, $\exp(2F_2)$ is dominant for $x_2(t) < x < x_4(t)$, and $C_5C_6\exp(2F_1 + 2F_2)$ is dominant for $x_4(t) < x$. Then it follows that there exist a line soliton moving with the speed $V_{s2} = 4a_2^2$ on the vertical line $x = x_2(t)$ and a parallel soliton moving with the speed $V_{s1} = 4a_1^2$ along $x = x_4(t)$, as is seen in Fig. *a*.

2. Let $t_0 = \left(\log(1 + C_6^{-1}) - \log C_5\right)/8a_1a_2(a_1 + a_2)$. If $0 < t < t_0$, then 1 is dominant for $x < x_1(t)$, $(1 + C_6)\exp(2F_1)$ is dominant for $x_1(t) < x < x_5(t)$, $\exp(2F_2)$ is dominant for $x_5(t) < x < x_4(t)$, and $C_5C_6\exp(2F_1 + 2F_2)$ is dominant for $x_4(t) < x$. Then there are line solitons moving along $x = x_1(t)$ and $x = x_4(t)$ with the same speed $V_l = 4a_1^2$, while along the line $x = x_5(t)$ there is a lump chain moving with the speed V_c, as is seen in Fig. 8*b*.

3. If $t > t_0$, then 1 is dominant for $x < x_1(t)$, $(1 + C_6)\exp(2F_1)$ is dominant for $x_1(t) < x_3(t)$, and $C_5C_6\exp(2F_1 + 2F_2)$ is dominant for $x_3(t) < t$. Hence, there exist a line soliton moving with the speed $V_{s1} = 4a_1^2$ along $x = x_1(t)$ and a line soliton moving with the speed $V_{s2} = 4a_2^2$ along $x = x_3(t)$, as is seen in Fig. 8*c*.

Summarizing the described process of the soliton interaction, we see that the solution has the following structure. A line soliton with the speed $V_{s2} = 4a_2^2$ at minus infinity approaches a slower moving soliton with the speed $V_{s1} = 4a_1^2$. At $t = 0$, the left-hand soliton emits a lump chain and slows down to $V_{s1} = 4a_1^2$. The lump chain propagates with the speed V_c and at $t = t_0$ is absorbed by the right-hand soliton, accelerating it up to the speed $V_{s2} = 4a_2^2$. The lifespan t_0 of the lump chain increases logarithmically, as $C_6 \to 0$. In this limiting case, the solution based on the τ-function (45) reduces to the solution (39) describing radiation of a lump chain by a line soliton.

The solution describing absorption of a lump chain by a line soliton and based on the τ-function (43) can be obtained in another limiting case. To this end, we present the τ-function (45) in the form

$$\tau(x,y,t) = C_6\left\{\frac{1}{C_6}\left[1+\exp(2F_1)+\exp(2F_2)\right]+\exp(2F_1) + \frac{C_4}{C_6}\exp(F_1+F_2)\cos\left[\left(a_1^2-a_2^2\right)y\right]+C_5\exp(2F_1+2F_2)\right\},$$

assuming that the coordinate x in the functions $F_1(x,t)$ and $F_2(x,t)$ is shifted to the right by an arbitrary value x_0. The constant factor C_6 can be omitted as it does not contribute to the solution in terms of $u(x,y,t)$. Consider then the limit when $\rho_3 \to \infty$, $\rho_2 \to -\infty$, $\rho_1 \to \infty$, and $x_0 \to -\infty$ such that $\rho_1 = \rho_3 - p_1 + p_2$, $\rho_2 = a_2(-\rho_3 + 2p_1 - p_2)/a_1 - p_3$, and $x_0 = (-\rho_3 + 2p_1 - p_2)/a_1$, where p_1, p_2, and p_3 are some constants. Then, $C_6 \to \infty$ and the τ-function reduces to Eq. (43).

Note that the described process of exchange-type interaction of two line solitons is very similar to the interaction of two line solitons within the framework of the Korteweg–de Vries equation when the ratio of their amplitudes at infinity is such that $A_1/A_2 < 2.62$. Within the framework of the KP1 equation, both of these processes can occur. The interaction of the Korteweg–de Vries type occurs when two line solitons are unperturbed at infinity, whereas the interaction of the Kadomtsev–Petviashvili type by means of a lump chain is apparently a special case where one of the line solitons has a specific infinitesimal modulation along its front. However, there is one important feature that demonstrates a large difference in the interaction of line solitons of the Korteweg–de Vries type and the Kadomtsev–Petviashvili type. It is well known (see, e. g., [16]) that Korteweg–de Vries solitons undergo a phase shift after interaction, and the phase shift of each soliton is determined entirely by the spectral parameters a_1 and a_2:

$$(\Delta x_{\mathrm{KdV}})_{1,2} = \frac{1}{a_{1,2}}\log\left|\frac{a_1-a_2}{a_1+a_2}\right|. \tag{46}$$

However, when two KP1 line solitons exchange a lump chain, the phase shift is determined by not only the spectral parameters, but also the parameters ρ_1 and ρ_3:

$$(\Delta x_{\mathrm{KP}})_i = (\Delta x_{\mathrm{KdV}})_i - \frac{1}{2a_i}\log\left\{1+\exp[2(\rho_1-\rho_3)]\right\}, \quad i=1,2. \tag{47}$$

This phase shift may be arbitrarily large. By analyzing the phase shift at plus and minus infinity, we are able to recognize whether the KP1 solitons interacted according to the Korteweg–de Vries approximation or they exchanged a lump chain.

4. CONCLUSIONS

Thus, in this paper, we have described the elementary acts of interactions of line solitons with lumps and with each other by means of lumps within the framework of the KP1 equation. Such interactions are impossible within the framework of the KP2 equation applicable to media with the negative dispersion. Our description is based on the presentation of solutions in terms of the τ-function and the Grammian. We have studied lump emission and absorption by a line soliton, interaction of a lump and a line soliton, resonant interaction of line solitons through a lump and a lump chain, and emission and absorption of a periodic chain of lumps by a line soliton. In a similar way, one can study more complex dynamics of line solitons and lumps. Some results in this direction have been obtained in [18–20].

Note that the "sudden" appearance of a lump between two line solitons and its subsequent disappearance after absorption by the second line soliton can be treated formally as the rogue-wave formation. Indeed, in our variables, the lump amplitude is eight times greater than the amplitude of a plane soliton, whereas according to the widely accepted criterion [25], the rogue wave is such a wave whose amplitude is two or more times greater than the average amplitude of background waves.

In conclusion, it is worth noting that the KP1 equation has infinitely many integrals of motion, although the set of such integrals is apparently incomplete [3]. The first integrals of this set (mass, momentum, and energy) play an important role in physical applications. In this set, the simplest one is the mass-conservation integral over the whole (x, y) plane:

$$I_m = \iint\limits_D u(x,y,t)\,dx\,dy.$$

This improper double integral does not satisfy the condition of the Fubini theorem for the lump solution (6) because the function $u(x,y,t)$ does not vanish sufficiently rapidly when $x^2+y^2\to\infty$. Therefore, the result of integration depends on the order of integration over x and y. The higher-order integrals, including the integral of energy and the Hamiltonian, which contain $u(x,y,t)$ in degrees higher than 2, are well defined and can be evaluated with the help of the Fubini theorem.

Yu. A. Stepanyants acknowledges the funding provided by the Council of the grants of the President of the Russian Federation for the state support of Leading Scientific Schools of the Russian Federation (project No. NSH–70.2022.1.5). V. E. Zakharov acknowledges the funding provided by the Russian Science Foundation (project No. 19–72–30028).

REFERENCES

1. B. B. Kadomtsev and V. I. Petviashvili, *Sov. Phys. Doklady*, **15**, 539–542 (1970).
2. V. E. Zakharov and A. B. Shabat, *Func. Anal. Appl.*, **8**, No. 3, 226–236 (1974).
3. V. E. Zakharov, *Stud. Appl. Math.*, **122**, 219–234 (2009). https://doi.org/10.1111/j.1467-9590.2009.00430.x
4. V. I. Petviashvili, *Fiz. Plazmy*, **2**, No. 3, 469–472 (1976).
5. V. I. Petviashvili and O. V. Pokhotelov, *Solitary Waves in Plasmas and in the Atmosphere*, Gordon and Breach, Philadelphia, PA (1992).
6. S. V. Manakov, V. E. Zakharov, L. A. Bordag, et al., *Phys. Lett. A*, **63**, No. 3, 205–206 (1977). https://doi.org/10.1016/0375-9601(77)90875-1
7. Z. Lu, E. M. Tian, and R. Grimshaw, *Wave Motion*, **40**, No. 2, 123–135 (2004). https://doi.org/10.1016/j.wavemoti.2003.12.017
8. N. Singh and Y. Stepanyants, *Wave Motion*, **64**, 92–102 (2016). https://doi.org/10.1016/j.wavemoti.2016.03.005
9. W. Hu, W. Huang, Z. Lu, and Y. Stepanyants, *Wave Motion*, **77**, 243–256 (2018). https://doi.org/10.1016/j.wavemoti.2017.12.004
10. D. E. Pelinovsky and Yu. A. Stepanyants, *JETP Lett.*, **57**, No. 1, 24–28 (1993).
11. K. A. Gorshkov, D. E. Pelinovskii, and Yu. A. Stepanyants, *J. Exp. Theor. Phys.*, **77**, 237–253 (1993).
12. E. A. Kuznetsov and S. K. Turitsyn, *Sov. Phys.–JETP*, **55**, No. 5, 1457–1463 (1982).
13. D. E. Pelinovsky and Yu. A. Stepanyants, *J. Exp. Theor. Phys.*, **77**, No. 4, 602–615 (1993).
14. V. Zakharov, *JETP Lett.*, **22**(7), 172–175 (1975).
15. C. Lester, A. Gelash, D. Zakharov, and V. Zakharov, *Stud. Appl. Math.*, **147**, No. 1, 1425–1442 (2021). https://doi.org/10.1111/sapm.12420

16. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, PA (1981).

17. Y. A. Stepanyants, *Symmetry*, **12**, No. 10, 1586 (2020). https://doi.org/10.3390/sym12101586

18. J. Rao, K. Chow, D. Mihalache, and J. He, *Stud. Appl. Math.*, **147**, No. 3, 1007–1035 (2021). https://doi.org/10.1111/sapm.12417

19. L. Guo, A. Chabchoubb, and J. He, *Physica D*, **426**, 132990 (2021). https://doi.org/10.1016/j.physd.2021.132990

20. J. Rao, J. He, and B. A. Malomed, *J. Math. Phys.*, **63**, 013510 (2022). https://doi.org/10.1063/5.0064304

21. R. S. Johnson and S. Thompson, *Phys. Lett. A*, **66**, No. 4, 279–281 (1978).

22. L. A. Abramyan and Yu. A. Stepanyants, *Radiophys. Quantum Electron.*, **28**, No. 1, 20–26 (1985). https://doi.org/10.1007/BF0103745320–28

23. A. A. Zaitsev, *Sov. Phys. Doklady*, **28**, 720–724 (1983).

24. S. K. Gdanov and B. A. Trubnikov, *JETP Lett.*, **39**, No. 3, 129 (1984).

25. C. Kharif, E. Pelinovsky, and A. Slunyaev, *Rogue Waves in the Ocean*, Springer, New York (2009). https://doi.org/10.1007/978-3-540-88419-4