# Rational Solutions of Davey-Stewartson

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**Abstract**

We present a derivation of the Davey-Stewartson (DS) equation from its Lax Pair and solve for rational soliton solutions by using the dressing method. We express the Kernel of the Marchenko function in terms of a separable spectral function. We obtain solutions for the general DS in terms of 4 independent arbitrary functions chosen to be polynomial and apply reductions to obtain regular and singular solutions (parameterized by 2 independent functions) with physical application. We explore only when polynomials are linear functions. These solutions can be adapted to other $2+1$ integrable systems for which only the dispersion relation needs to be adapted.

*Keywords: DS,* $2+1$ *integrable systems, Dressing Method*

## 1 Introduction

Zakharov and Schulman a derive the Zakharov-Schulman system; a system of partial differential equations from which one can obtain the *Davey-Stewartson* equations (DS) [9]. Fokas et al [2] presents the general Davey-Stewartson equation as

$$\begin{aligned} iq_t + q_{zz} + q_{\overline{zz}} + 4(f+\overline{q}) = 0 \\ 2f_{\overline{z}} - (|q|^2)_z = 0 \end{aligned}$$

and solves it on the half-plane. The quantity $q$ is a complex amplitude and $f$ describes an underlying flow [4]. Applying the proper reduction one obtains the Nonlinear-Schrodinger (NLS) equation, given by

$$iq_t + q_{zz} + q_{\overline{zz}} = 0\,, \qquad z = x+iy\,, \qquad x, y, \in \mathbb{R}, \quad t>0$$

Anker and Freeman derive Davey-Stewartson solutions by applying the Dressing Method in [1]. Their paper assumes the spectral function is exponential in the physical variables

$$F_{ij} = A_{ij}e^{lx+my+nz}$$

where the phase of a soliton is parameterized by $l, m$, and $n$. In this approach, we use the same technique, but the spectral function, $F$, is assumed to be polynomial. As a result, we obtain rational solutions of the general, unreduced DS equations parameterized by 4 independent arbitrary functions.

Inspired by the Inverse Scattering Transform (IST), the Dressing Method (originally described in [10]) evolves a spectral function in the time variable (a function representing the scattering data) and uses the Gelfand-Levitan-Marchenko (GLM) equation

$$K(x,z) + F(x,z) + \int_x^\infty K(x,s)F(x,s)ds = 0 \tag{1}$$

to transform the spectral function into a time evolved potential which solves a nonlinear PDE. We can obtain the potential by taking a finite number of derivatives of the kernel

$$p(x) = (\partial_x - \partial_z)^n K(x,x)|_{z=x}$$

We present two matrix-valued differential operators $D_1$ and $D_2$ that the spectral function satisfies, $D_1F = 0$ and $D_2F = 0$ and derive the associated differential equation that the kernel $K(x,z;y,t)$ satisfies to obtain solutions

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$p = [I, K(x,x)]$ of DS. This is done by applying the operators $D_1$ and $D_2$ to the GLM equation (1). $D_1$ is a linear differential operator in variables $y, x, z$ and $D_2$ is a linear differential operator in $t, x, z$. The $z$ variable is a dummy variable in the Kernel $K$.

Then we obtain rational function solutions by assuming rank 1 separability of the spectral function $F$, so that $F$ is expressed in terms of four independent functions to be appropriately chosen. This assumption in combination with equation (1) forces $K$ to be separable too. Then one can solve the GLM (1) for $K$ in terms of $F$. Applying reductions correctly, we reduce the number of independent functions and obtain regular and singular solutions. We also obtain known forms of Dromion solutions.

Zakharov and Fokas applied the dressing method to a non-local Riemann-Hilbert problem [3] to derive dromion and line dromion solutions to KP 1 and DS 1 equations. Dynamics of rogue waves in Davey Stewartson 2 equation [5] uses Hirota Bilinear method to derive fundamental rogue waves and their interactions. In [11], Zhang et al. studies lump and breather N-soliton solutions of the nonlocal Davey-Stewarston 2 equation derived using Darboux Transformations. Exact solutions of Davey-Stewartson equations are obtained by Hirota's Bilinear method in [6]. Gilson and Nimmo study asymptotics of dromion solutions in [4]. Lou, in [7], provides a universal formula for creating dromions, dromion lattices, breather, instantons, among other localized solutions. Lou et al. use the Hirota bilinear method and then apply separation of variable to obtain dromion solutions in [8].

The structure of this paper is as follows. In Section 2, we derive DS from the Lax Pair formulation. Section 3 details the obtaining the general solution of the DS equations without reduction. Section 3 contains subsections detailing possible solutions that can be obtained after applying a reduction. Sections 4, 5 give explicit derivation of equations satisfied by the kernel $K$ if $D_1 F = 0$ and $D_2 F = 0$, respectively. Section 6 shows how we can extract the DS Lax Pair from the GLM. i.e. the solutions we obtain are solutions of DS.

## 2 Derivation of Davey-Stewartson

Consider the Lax Pair of the form

$$\Psi_y = L\Psi \tag{2}$$
$$\Psi_t = M\Psi \tag{3}$$

where $L = I\partial_x + A$ and $M = \partial_x^2 + B$. I will use the notation $\partial_x^n := \frac{\partial^n}{\partial x^n}$ to denote the $n^{th}$ derivative with respect to $x$. We define the matrices in our operators by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Notice that $I = \sigma_3$ the Pauli basis matrix, $A$ is the unknown potential of the Dirac operator $L$, and $B$ is a matrix component of the time evolution operator to be determined. To find a solution to the under-determined system above, we impose the condition that partial derivatives commute. i.e. $\Psi_{yt} = \Psi_{ty}$. The corresponding Lax equation is

$$L_t - M_y = [L, M] \tag{4}$$

For a given potential $A$, we will derive the associated time evolution operator $B$ using (4). Note,

$$L_t = A_t \qquad M_y = B_y \tag{5}$$

We compute the commutator $[T, \partial_x^n]$ of operators acting on the scattering potential $\Psi$ in the following way

$$\begin{aligned} [T, \partial_x^n]\Psi &= (T\partial_x^n - \partial_x^n T)\Psi \\ &= T\partial_x^n \Psi - \partial_x^n T\Psi \\ &= T(\partial_x^n \Psi) - \partial_x^n (T\Psi) \\ &= T(\partial_x^n \Psi) - \sum_{k=0}^{n} \binom{n}{k} \partial_x^{n-k} T \partial_x^k \Psi \\ &= \Big( - \sum_{k=0}^{n-1} \binom{n}{k} \partial_x^{n-k} T \partial_x^k \Big) \Psi \end{aligned}$$

For example,

$$\begin{aligned}
[I\partial_x, B]\Psi &= (I\partial_x B - BI\partial_x)\Psi \\
&= I\partial_x(B\Psi) - BI\partial_x\Psi \\
&= I(B_x\Psi + B\partial_x\Psi) - BI\partial_x\Psi \\
&= IB_x\Psi + BI\partial_x\Psi - BI\partial_x\Psi \\
&= IB_x\Psi + [I,B]\partial_x\Psi \\
&= \Big(IB_x + [I,B]\partial_x\Big)\Psi
\end{aligned}$$

By convention, we drop the scattering potential $\Psi$, but it is implied when doing the calculus of operators. Similarly,

$$[A, \partial_x^2] = -A_{xx} - 2A_x\partial_x$$

and using that pure differential operators commute

$$\begin{aligned}
[L, M] &= \Big[I\partial_x + A, \partial_x^2 + B\Big] \\
&= \Big[I\partial_x, \partial_x^2\Big] + \Big[I\partial_x, B\Big] + \Big[A, \partial_x^2\Big] + \Big[A, B\Big] \\
&= IB_x + [I,B]\partial_x - A_{xx} - 2A_x\partial_x + [A,B]
\end{aligned}$$

So that we have

$$L_t - M_y = A_t - B_y \tag{6}$$
$$[L, M] = IB_x + [I,B]\partial_x - A_{xx} - 2A_x\partial_x + [A,B] \tag{7}$$

When equating the above two equations (by (4)) and matching orders of derivatives we see

$$\mathcal{O}(\partial_x^1): \qquad [I,B] - 2A_x = 0 \tag{8}$$
$$\mathcal{O}(\partial_x^0): \qquad A_t - B_y = IB_x - A_{xx} + [A,B] \tag{9}$$

The first term in (8) evaluates to

$$[I,B] = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix} - \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}b_{11} & b_{12}\\ -b_{21} & -b_{22}\end{bmatrix} - \begin{bmatrix}b_{11} & -b_{12}\\ b_{21} & -b_{22}\end{bmatrix} = \begin{bmatrix}0 & 2b_{12}\\ -2b_{21} & 0\end{bmatrix}$$

so that by (8), we get

$$2\begin{bmatrix}0 & b_{12}\\ -b_{21} & 0\end{bmatrix} - 2\begin{bmatrix}0 & p_x\\ q_x & 0\end{bmatrix} = 0 \implies B = \begin{bmatrix}b_{11} & p_x\\ -q_x & b_{22}\end{bmatrix}$$

Next, we evaluate the terms in (9)

$$A_t - B_y = \begin{bmatrix}0 & p_t\\ q_t & 0\end{bmatrix} - \begin{bmatrix}b_{11y} & p_{xy}\\ -q_{xy} & b_{22y}\end{bmatrix} = \begin{bmatrix}-b_{11y} & p_t - p_{xy}\\ q_t + q_{xy} & -b_{22y}\end{bmatrix}$$

$$IB_x = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}b_{11x} & p_{xx}\\ -q_{xx} & b_{22x}\end{bmatrix} = \begin{bmatrix}b_{11x} & p_{xx}\\ q_{xx} & -b_{22x}\end{bmatrix}$$

$$A_{xx} = \begin{bmatrix}0 & p_{xx}\\ q_{xx} & 0\end{bmatrix}$$

$$\begin{aligned}
[A,B] = AB - BA &= \begin{bmatrix}0 & p\\ q & 0\end{bmatrix}\begin{bmatrix}b_{11} & p_x\\ -q_x & b_{22}\end{bmatrix} - \begin{bmatrix}b_{11} & p_x\\ -q_x & b_{22}\end{bmatrix}\begin{bmatrix}0 & p\\ q & 0\end{bmatrix} \\
&= \begin{bmatrix}-pq_x & b_{22}p\\ b_{11}q & p_xq\end{bmatrix} - \begin{bmatrix}p_xq & b_{11}p\\ b_{22}q & -pq_x\end{bmatrix} \\
&= \begin{bmatrix}-(pq)_x & (b_{22} - b_{11})p\\ (b_{11} - b_{22})q & (pq)_x\end{bmatrix}
\end{aligned}$$

Therefore, by (9), we get

$$\begin{bmatrix} -b_{11y} & p_t - p_{xy} \\ q_t + q_{xy} & -b_{22y} \end{bmatrix} = \begin{bmatrix} b_{11x} - (pq)_x & (b_{22} - b_{11})p \\ (b_{11} - b_{22})q & b_{22x} + (pq)_x \end{bmatrix}$$

from which we derive the following set of equations from the diagonal entries

$$b_{11y} + b_{11x} = (pq)_x \tag{10}$$
$$b_{22y} + b_{22x} = -(pq)_x \tag{11}$$

and from the off-diagonal entries, the terms in $IB_x$ cancel with $A_{xx}$ to obtain the coupled system of PDEs for which we'd like to solve for the quantity $b_{22} - b_{11}$

$$p_t - p_{xy} = (b_{22} - b_{11})p \tag{12}$$
$$q_t + q_{xy} = -(b_{22} - b_{11})q \tag{13}$$

by subtraction of equations (10) and (11), we get

$$(b_{11} - b_{22})_y + (b_{11} - b_{22})_x = 2(pq)_x$$

From the above equations, define $u := b_{22} - b_{11}$ and get a system of coupled PDEs

$$p_t - p_{xy} = up \tag{14}$$
$$q_t + q_{xy} = -uq \tag{15}$$
$$u_y + u_x = 2(pq)_x \tag{16}$$

We can get rid of the presence of $u$ in the above equations by solving for $u$ in (14) and computing its partial derivatives

$$u_x = \frac{(p_{tx} - p_{xxy})p - p_x(p_t - p_{xy})}{p^2}$$
$$u_y = \frac{(p_{ty} - p_{xyy})p - p_y(p_t - p_{xy})}{p^2}$$

Similarly, using equation (15)

$$u_x = -\frac{(q_{tx} + q_{xxy})q - q_x(q_t + q_{xy})}{q^2}$$
$$u_y = -\frac{(q_{ty} - q_{xyy})q - q_y(q_t + q_{xy})}{q^2}$$

Summing the partial derivatives, using the property (16), we obtain

$$\Big(\partial_x + \partial_y\Big)(p_t - p_{xy})p - (p_t - p_{xy})\Big(\partial_x + \partial_y\Big)p = 2p^2(pq)_x$$
$$\Big(\partial_x + \partial_y\Big)(q_t + q_{xy})q - (q_t + q_{xy})\Big(\partial_x + \partial_y\Big)q = -2q^2(pq)_x$$

We can write equations for $p$ and $q$ in the form

$$\Big[\Big(\partial_x + \partial_y\Big), p_t - p_{xy}\Big]p = 2p^2(pq)_x$$
$$\Big[\Big(\partial_x + \partial_y\Big), q_t + q_{xy}\Big]q = -2q^2(pq)_x$$

Applying the reduction $p = \pm\bar{q}$, we obtain the Davey-Stewartson equation

$$\Big[\Big(\partial_x + \partial_y\Big), p_t \pm p_{xy}\Big]p \mp 2p^2(|p|^2)_x = 0 \tag{17}$$

Notice applying the reduction to our system of coupled PDEs (14,15,16) results in the form

$$p_t + p_{xy} = up \tag{18}$$
$$u_y + u_x = 2(|p|^2)_x \tag{19}$$

Using a variable substitution $z = x + iy$, and replacing $t \to it$ and $y \to iy$, the above equations are equivalent to that used by Fokas et al.

## 3 Solutions of Davey-Stewartson equation

Suppose we require the function $F$ to satisfy the linear equation $D_1F = 0$ where

$$D_1F := \partial_y F + I\partial_x F + \partial_z F I = 0 \tag{20}$$

where $I = \sigma_3$ is the Pauli basis matrix. By applying $D_1$ to the Marchenko equation,

$$K(x,z) + F(x,z) + \int_x^\infty K(x,s)F(x,s)ds = 0 \tag{21}$$

we can derive the equation that $K$ satisfies

$$\partial_y K + I\partial_x K + \partial_z K I + [I, K(x,x)]K = 0 \tag{22}$$

or symbolically, $D_1K + [I, K(x,x)] = 0$. We denote $p(x) := K(x,x)$ to be a solution of (17). Now suppose that $F$ is off-diagonal and $K$ is unknown and expressed in the form

$$F = \begin{bmatrix} 0 & F_{12} \\ F_{21} & 0 \end{bmatrix} \qquad K = \tilde{K} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

After substitution of these matrices into equation (21), we obtain the following entry-wise equations from the top row

$$K_{11}(x,z) + \int_x^\infty K_{12}(x,s)F_{21}(s,z)ds = 0 \tag{23}$$

$$K_{12}(x,z) + F_{12}(x,z) + \int_x^\infty K_{11}(x,s)F_{12}(s,z)ds = 0 \tag{24}$$

We also note

$$[I, K(x,x)] = 2\begin{bmatrix} 0 & K_{12}(x,x) \\ -K_{21}(x,x) & 0 \end{bmatrix}$$

We begin by assuming $F$ is a rank 1 separable function. Let $F(x,z) = f(x)g(z)$ where $f$ and $g$ are diagonal and off-diagonal matrices, respectively. Notice that their product will be an off-diagonal matrix of the form of $F$.

$$f(x) = \begin{bmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{bmatrix} \qquad g(z) = \begin{bmatrix} 0 & g_{12}(z) \\ g_{21}(z) & 0 \end{bmatrix}$$

so that,

$$\implies F(x,z) = \begin{bmatrix} 0 & f_1(x)g_{12}(z) \\ f_2(x)g_{21}(z) & 0 \end{bmatrix} \tag{25}$$

Then equations satisfied by the factors of $F$ when acted upon by $D_1$ (defined by (20)) are $D_1F = D_1(f \cdot g) = 0$

$$\partial_y f_1 + \partial_x f_1 = 0, \qquad \partial_y f_2 - \partial_x f_2 = 0 \tag{26}$$

$$\partial_y g_{12} - \partial_z g_{12} = 0, \qquad \partial_y g_{21} + \partial_z g_{21} = 0 \tag{27}$$

creating four unknown functions, $f_1 = f_1(y-x), f_2 = f_2(y+x), g_{12} = g_{12}(y+z), g_{21} = g_{21}(y-z)$. We assume $K$ is also separable, $K = \tilde{K}_1(x)g(z)$, where $\tilde{K}_1$ has the form

$$\tilde{K}_1 = \begin{bmatrix} \tilde{K}_{11}(x) & \tilde{K}_{12}(x) \\ \tilde{K}_{21}(x) & \tilde{K}_{22}(x) \end{bmatrix}$$

so that

$$\implies K(x,z) = \tilde{K}_1(x)g(z) = \begin{bmatrix} \tilde{K}_{11}(x) & \tilde{K}_{12}(x) \\ \tilde{K}_{21}(x) & \tilde{K}_{22}(x) \end{bmatrix} \begin{bmatrix} 0 & g_{12}(z) \\ g_{21}(z) & 0 \end{bmatrix} = \begin{bmatrix} \tilde{K}_{12}(x)g_{21}(z) & \tilde{K}_{11}(x)g_{12}(z) \\ \tilde{K}_{22}(x)g_{21}(z) & \tilde{K}_{21}(x)g_{12}(z) \end{bmatrix} \tag{28}$$

Therefore,

$$K(x,s)F(s,z) = \begin{bmatrix} \tilde{K}_{11}(x)g_{12}(s)f_2(s)g_{21}(z) & \tilde{K}_{12}(x)g_{21}(s)f_1(s)g_{12}(z) \\ \tilde{K}_{21}(x)g_{12}(s)f_2(s)g_{21}(z) & \tilde{K}_{22}(x)g_{21}(s)f_1(s)g_{12}(z) \end{bmatrix} \tag{29}$$

By substituting equations (28), (25), and (29) into (21), we extract the following equations from the top row

$$\tilde{K}_{12}(x) + \tilde{K}_{11}(x)\int_x^\infty g_{12}(s)f_2(s)ds = 0$$
$$\tilde{K}_{11}(x) + f_1(x) + \tilde{K}_{12}(x)\int_x^\infty g_{21}(s)f_1(s)ds = 0$$

and from the bottom row

$$\tilde{K}_{22}(x) + f_2(x) + \tilde{K}_{21}(x)\int_x^\infty g_{12}(s)f_2(s)ds = 0$$
$$\tilde{K}_{21}(x) + \tilde{K}_{22}(x)\int_x^\infty g_{21}(s)f_1(s)ds = 0$$

Denote the integrals $A = \int_x^\infty g_{12}(s)f_2(s)ds$ and $B = \int_x^\infty g_{21}(s)f_1(s)ds$ and we rewrite the above system as

$$\tilde{K}_{11}A + \tilde{K}_{12} = 0$$
$$\tilde{K}_{11} + \tilde{K}_{12}B = -f_1$$

or consider it in matrix multiplication form

$$\begin{bmatrix} A & 1 \\ 1 & B \end{bmatrix}\begin{bmatrix} \tilde{K}_{11} \\ \tilde{K}_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix}$$

Denote the determinant of the above matrix as $\Delta := AB - 1$ and apply Kramer's rule

$$\tilde{K}_{11} = \frac{1}{\Delta}\begin{vmatrix} 0 & 1 \\ -f_1 & B \end{vmatrix} = \frac{f_1}{\Delta}, \qquad \tilde{K}_{12} = \frac{1}{\Delta}\begin{vmatrix} A & 0 \\ 1 & -f_1 \end{vmatrix} = \frac{-Af_1}{\Delta}$$

Similarly, we obtain the following quantities from the bottom row equations

$$\tilde{K}_{21} = \frac{1}{\Delta}\begin{vmatrix} -f_2 & 1 \\ 0 & B \end{vmatrix} = \frac{-Bf_2}{\Delta}, \qquad \tilde{K}_{22} = \frac{1}{\Delta}\begin{vmatrix} A & -f_2 \\ 1 & 0 \end{vmatrix} = \frac{f_2}{\Delta}$$

Therefore, we can obtain soliton solutions of the DS equation in terms of 4 unknown functions, $f_1, f_2, g_{12}, g_{21}$ that each have implicit dependence on $y$ determined by equations (26) and (27)

$$p(x; y) = 2K_{12}(x, x; y) = 2\tilde{K}_{11}(x; y)g_{12}(x; y) \tag{30}$$
$$= \frac{2f_1(y - x)g_{12}(y + x)}{\left(\int_x^\infty g_{12}(y + s)f_2(y + s)ds\right)\left(\int_x^\infty g_{21}(y - s)f_1(y - s)ds\right) - 1} \tag{31}$$

Similarly,

$$q(x; y) = -2K_{21}(x, x; y) = -2\tilde{K}_{22}(x; y)g_{21}(x; y) \tag{32}$$
$$= \frac{-2f_2(y + x)g_{21}(y - x)}{\left(\int_x^\infty g_{12}(y + s)f_2(y + s)ds\right)\left(\int_x^\infty g_{21}(y - s)f_1(y - s)ds\right) - 1} \tag{33}$$

### 3.1 Example

Consider the example when our 4 independent functions are linear functions (we omit the case when they are constant). Let $f_1(x) = mx$, $f_2(x) = nx$, $g_{12}(x) = ax$ and $g_{21}(x) = bx$. So that $F$ looks like

$$F(x, z) = \begin{bmatrix} mx & 0 \\ 0 & nx \end{bmatrix}\begin{bmatrix} 0 & az \\ bz & 0 \end{bmatrix} = \begin{bmatrix} 0 & amxz \\ bnxz & 0 \end{bmatrix}$$

We can express the integrals $A$ and $B$ as follows

$$A = \int_x^\infty n(y + s)a(s + y)ds = an\int_x^\infty (y^2 + 2sy + s^2)ds = an\left(y^2x + x^2y + \frac{1}{3}x^3\right)$$
$$B = \int_x^\infty b(s - y)m(s - y) = bm\int_x^\infty (y^2 - 2sy + s^2)ds = bm\left(y^2x - x^2y + \frac{1}{3}x^3\right)$$

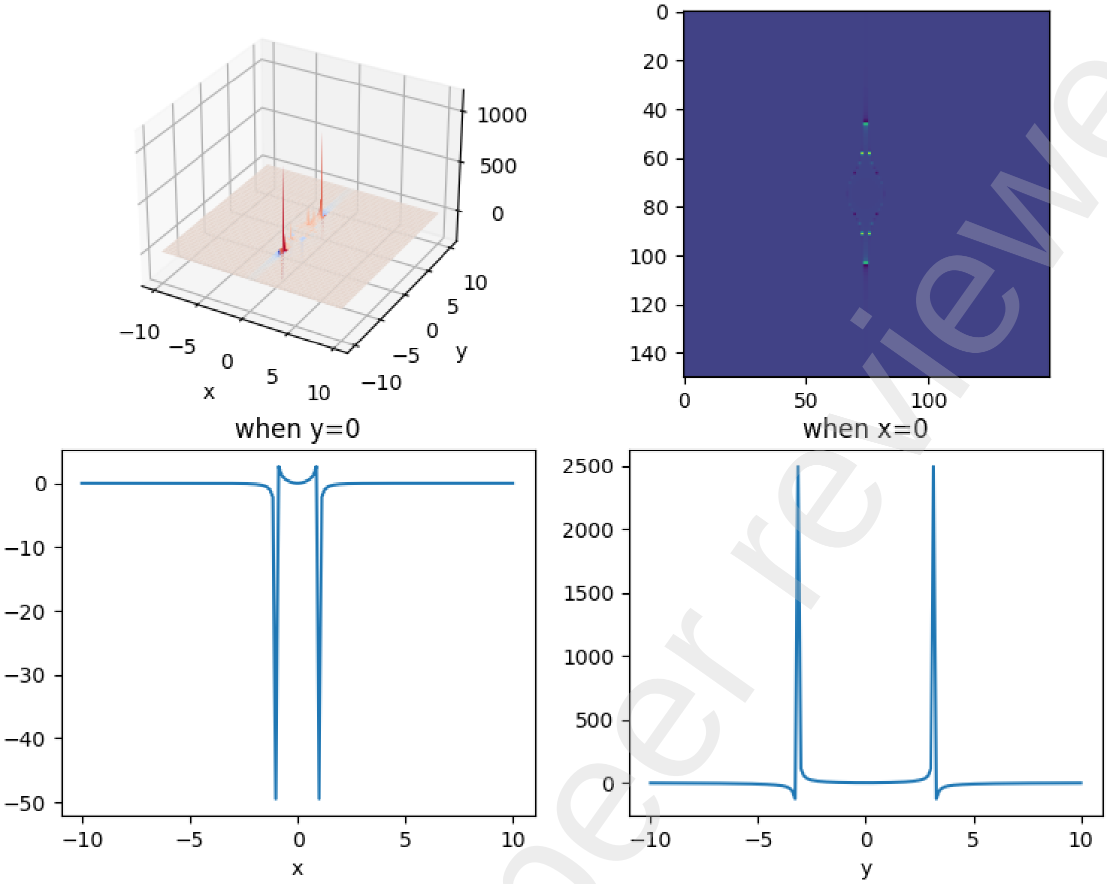


Figure 1: The top is a 3D surface of the singular solution (35). The bottom left is a cross-section when $y = 0$. The bottom right is a cross-section when $x = 0$.

Then we can express a solution

$$p(x, y) = \frac{2m(y - x)a(y + x)}{abmn\left(y^2x + x^2y + \frac{1}{3}x^3\right)\left(y^2x - x^2y + \frac{1}{3}x^3\right) - 1} \tag{34}$$

$$= \frac{2am(y^2 - x^2)}{abmn\left(y^4x^2 - \frac{1}{3}y^2x^4 + \frac{1}{9}x^6\right) - 1} \tag{35}$$

### 3.2 Complexification and reduction

Suppose that we make the variable $y$ purely imaginary ($y \to iy$). Then our four independent functions can be expressed as

$$f_1 = f_1(iy - x)\,, \qquad f_2 = f_2(x + iy)$$
$$g_{12} = g_{12}(z + iy)\,, \qquad g_{21} = g_{21}(iy - z)$$

Let's define the complex variable $\xi = x + iy$ and apply the reductions $p = \pm\bar{q}$ to equation (31) and (33) to obtain

$$\frac{1}{\Delta} f_2(\bar{\xi}) g_{21}(-\xi) = \frac{1}{\Delta} f_1(-\bar{\xi}) g_{12}(\xi) \tag{36}$$

If the following reductions are imposed

$$f_1(-\xi) = \pm\overline{f_2}(\xi) \tag{37}$$
$$g_{12}(-\xi) = \pm\overline{g_{21}}(\xi) \tag{38}$$

Note that there are 4 reductions due to the choice of signs in equations (37) and (38). To obtain solutions, we apply one such substitution of the reduction into the solution, and apply complexification of the variable $y \to iy$. If the signs of the reduction match $-,-$ or $+,+$, we obtain singular solutions

$$p_{\text{sing}}(x;y) = \frac{2\overline{f}_2(x-iy)\overline{g}_{21}(-x-iy)}{\Big(\int_x^\infty \overline{g}_{21}(-iy-s)f_2(iy+s)ds\Big)\Big(\int_x^\infty g_{21}(iy-s)\overline{f}_2(s-iy)ds\Big)-1} \tag{39}$$
$$= \frac{2f_2(x+iy)g_{21}(-x+iy)}{\Big(\int_x^\infty g_{21}(iy-s)f_2(iy+s)ds\Big)\Big(\int_x^\infty g_{21}(iy-s)f_2(s+iy)ds\Big)-1} \tag{40}$$
$$= \frac{2f_2(x+iy)g_{21}(-x+iy)}{\Big(\int_x^\infty g_{21}(iy-s)f_2(iy+s)ds\Big)^2-1} \tag{41}$$

If we consider reductions with opposite signs $-,+$ or $+,-$, we obtain regular solutions

$$p_{\text{reg}}(x;y) = \frac{-2\overline{f}_2(x-iy)\overline{g}_{21}(-x-iy)}{\Big(\int_x^\infty \overline{g}_{21}(-iy-s)f_2(iy+s)ds\Big)\Big(\int_x^\infty g_{21}(iy-s)\overline{f}_2(s-iy)ds\Big)-1}$$
$$= \frac{-2f_2(x+iy)g_{21}(-x+iy)}{-\Big(\int_x^\infty g_{21}(iy-s)f_2(iy+s)ds\Big)\Big(\int_x^\infty g_{21}(iy-s)f_2(s+iy)ds\Big)-1}$$
$$= \frac{2f_2(x+iy)g_{21}(-x+iy)}{\Big(\int_x^\infty g_{21}(iy-s)f_2(iy+s)ds\Big)^2+1}$$

One can see how similar these two solutions are. Now we've expressed our solution in terms of 2 independent functions (instead of 4). If we suppose $f_2(\xi)=\xi, g_{21}(\xi)=\xi$ are identity functions, we get

$$p(x;y) = \frac{2(\overline{x-iy})(\overline{-x-iy})}{\Big(\int_x^\infty (\overline{-iy-s})(iy+s)ds\Big)\Big(\int_x^\infty (iy-s)(\overline{s-iy})ds\Big)-1} \tag{42}$$
$$= \frac{2(x+iy)(-x+iy)}{\Big(\int_x^\infty (iy-s)(iy+s)ds\Big)\Big(\int_x^\infty (iy-s)(s+iy)ds\Big)-1} \tag{43}$$
$$= \frac{-2(x^2+y^2)}{\Big(-\int_x^\infty (y^2+s^2)ds\Big)\Big(-\int_x^\infty (y^2+s^2)ds\Big)-1} \tag{44}$$
$$= \frac{-2(x^2+y^2)}{\Big(y^2x+\frac{1}{3}x^3\Big)^2-1} \tag{45}$$

if both $f_2 = c$ and $g_{21} = d$ are constants, we get

$$p(x;y) = \frac{2\overline{cd}}{|cd|^2x^2-1} \tag{46}$$

if $f_2$ is linear and $g_{21} = c \in \mathbb{C}$ is constant, we'd get

$$p(x;y) = \frac{2c(x+iy)}{|c|^2(iy+\frac{1}{2}x)^2x^2-1} \tag{47}$$

if one assumes that $f_2$ is constant and $g_{21}$ is linear, one obtains

$$p(x;y) = \frac{2\overline{c}(-x+iy)}{|c|^2(iy-\frac{1}{2}x)^2x^2-1} \tag{48}$$

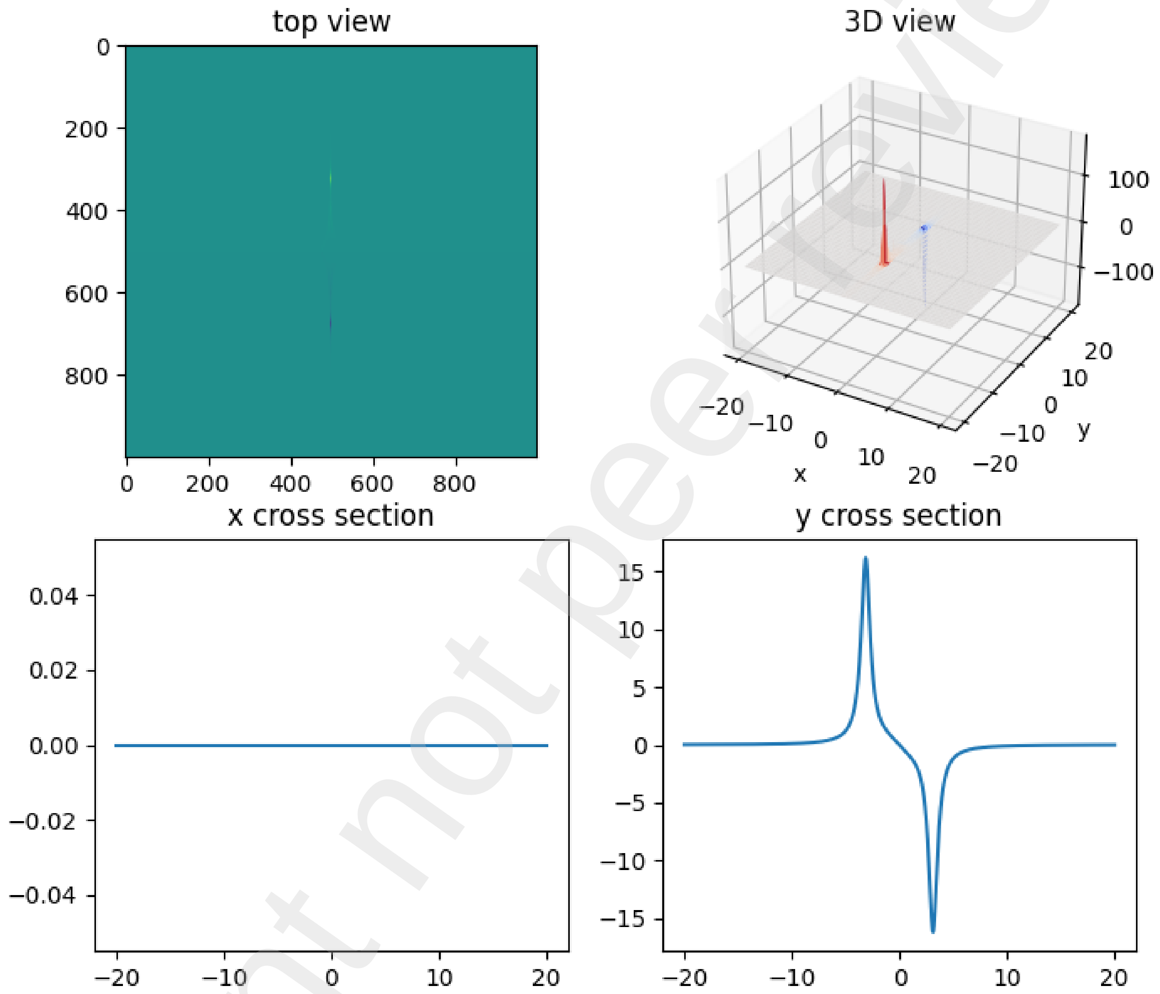


Figure 2: Using linear functions in (42) with parameters $m_1 = i$, $m_2 = 1$, $c_1 = 1$, $c_2 = 0$.

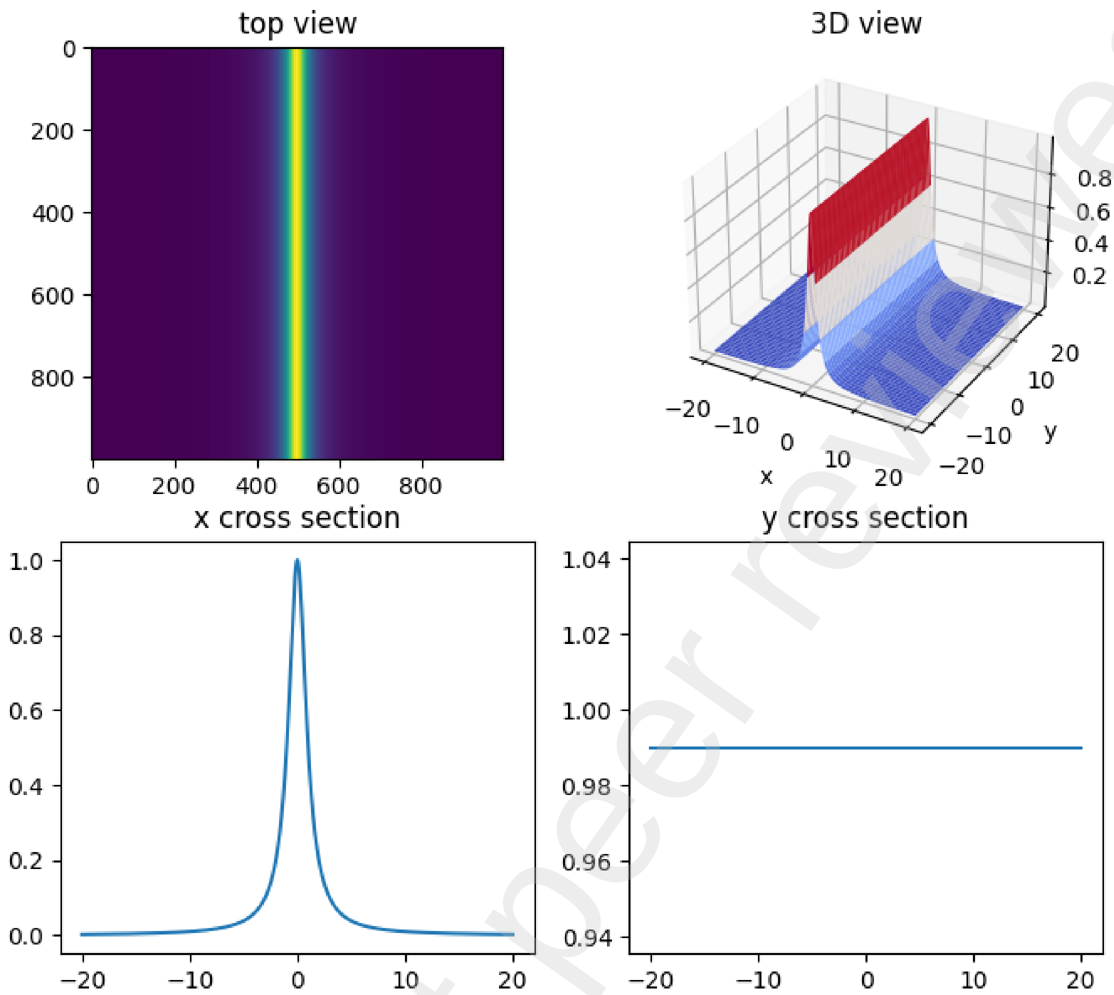


Figure 3: Choose $m_1 = m_2 = 0$ and $c_1 = c_2 = 1$

### 3.3 Dromion solution

We can obtain known dromion solutions by imposing the conditions that $\overline{f}_2 = g_{12}$ and $\overline{f}_1 = \pm g_{21}$ which turns equation (31) into

$$p(x,y) = \frac{2f_1(iy-x)f_2(iy+x)}{\left(\int_x^\infty |f_2(iy+s)|^2 ds\right)\left(\int_x^\infty |f_1(iy-s)|^2 ds\right) \pm 1}$$

Suppose we chose $f_1(\xi) = m_1\xi + c_1$ and $f_2(\xi) = m_2\xi + c_2$, we can recreate many solutions. See figures (3,4,5)

## 4 Matrix valued differential Operator, $D_1$

Consider the differential operator

$$D_1 F = \partial_y F + I\partial_x F + \partial_z F J$$

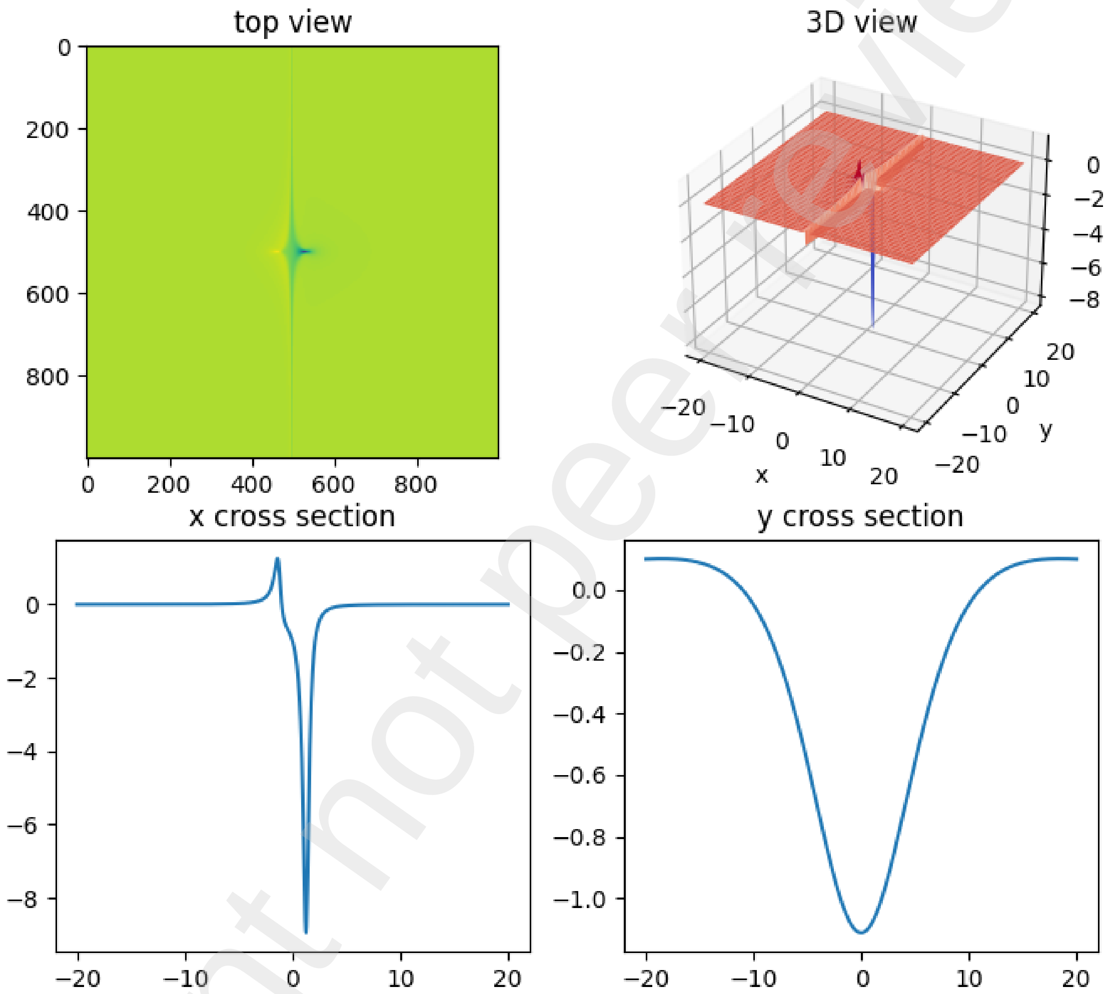


Figure 4: Choose $m_1 = 0$, $m_2 = -1$ and $c_1 = c_2 = 1$

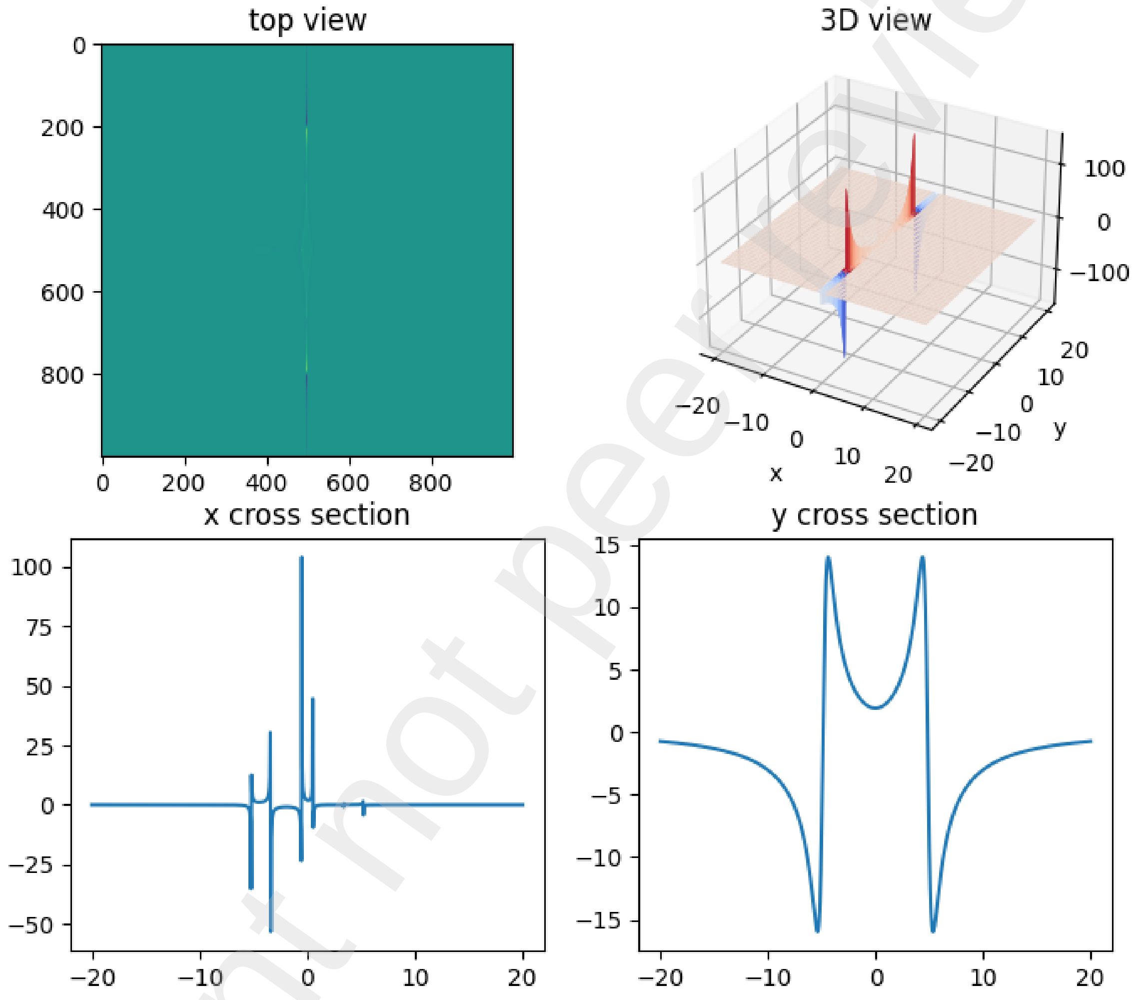


Figure 5: Choose $m_1 = 1$, $m_2 = \frac{1}{3}$ and $c_1 = -2$, $c_2 = 1$

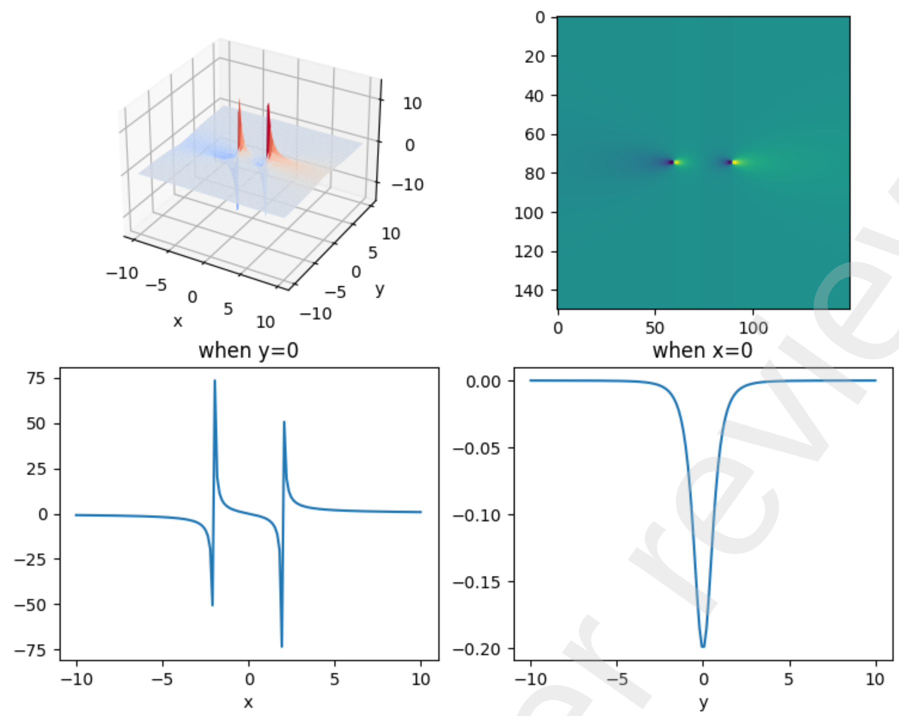


Figure 6: Two lump soliton solution from equation (47) with $c = 1$

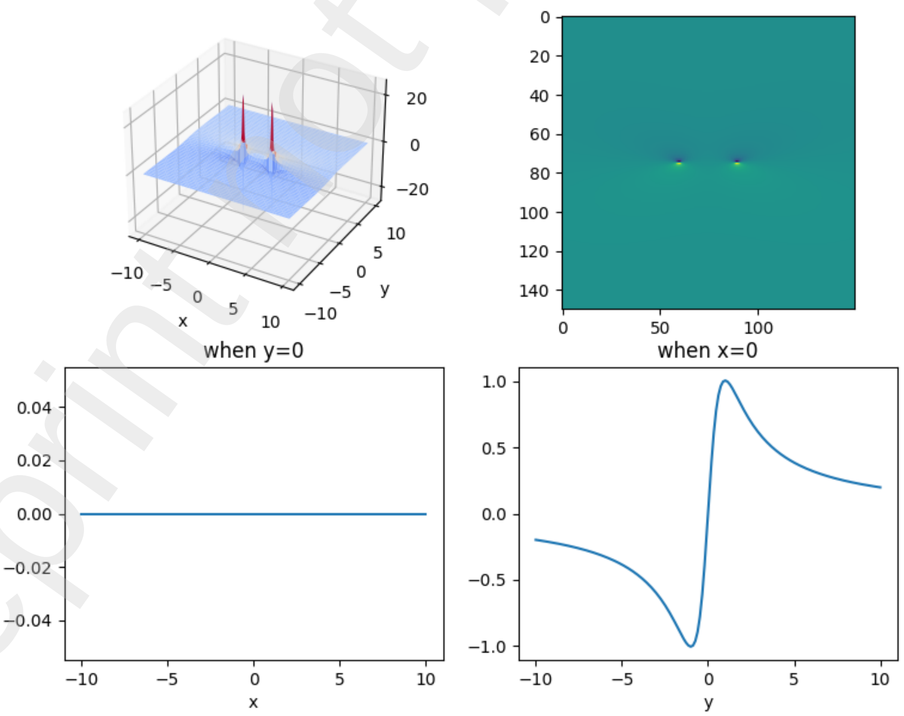


Figure 7: Two lump soliton solution from equation (47) with $c = i$

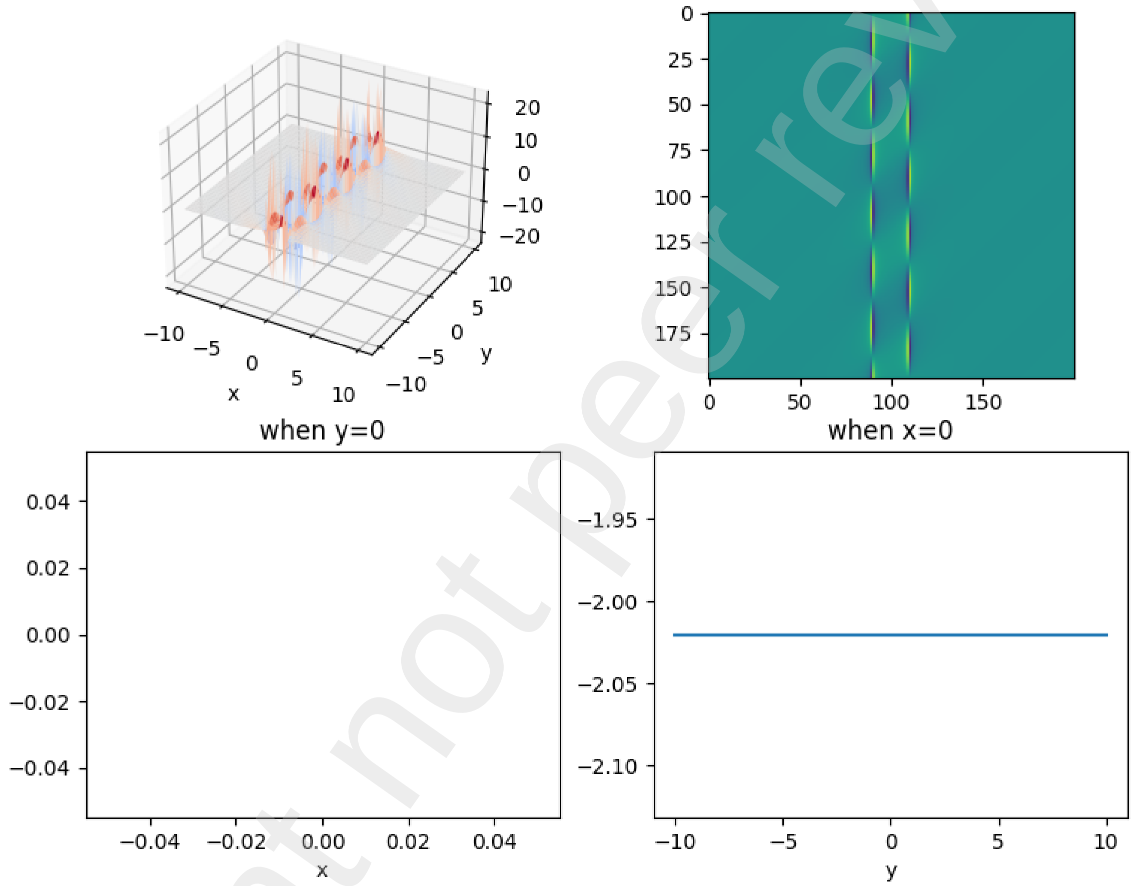


Figure 8: Example of solution (46)

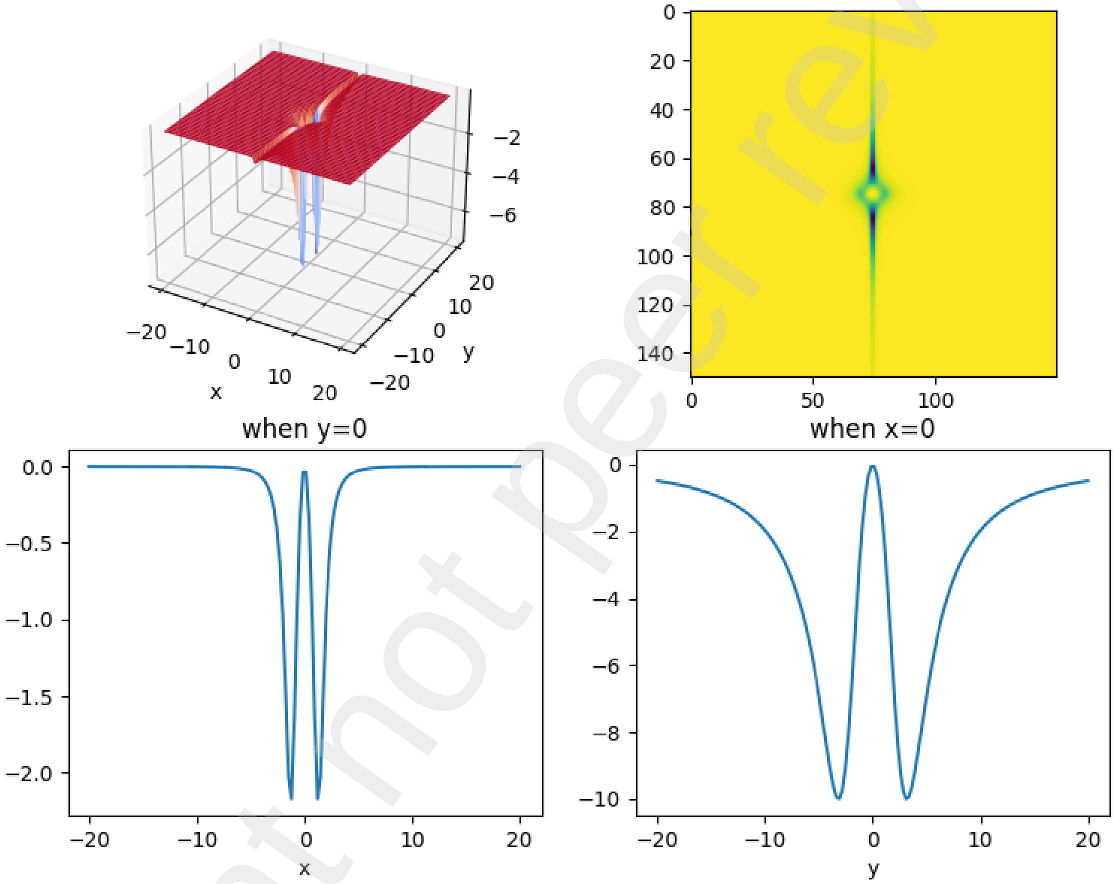


Figure 9: Example dark soliton from equation (45)

where $I = \sigma_3$ and $J$ is unknown. We impose that $D_1 F = 0$ (i.e. F satisfies this linear relation). We will derive the equation that $K$ satisfies. When $D_1$ is applied to the matrix valued version of equation (21), we must compute

$$D_1\Big(\int_x^\infty K(x,s)F(s,z)ds\Big) = \int_x^\infty \Big(\partial_y K + I\partial_x K\Big)F(s,z)ds + IK(x,x)F(x,z) + \int_x^\infty K(x,s)\Big(\partial_y F + \partial_z F J\Big)ds$$

We note that if $D_1 F(x,z) = 0$, so does $D_1 F(s,z) = 0$. so we use the identity $\partial_y F + \partial_z F J = -I\partial_s F$

$$D_1\Big(\int_x^\infty K(x,s)F(s,z)ds\Big) = \int_x^\infty \Big(\partial_y K + I\partial_x K\Big)F(s,z)ds + IK(x,x)F(x,z) + \int_x^\infty K(x,s)\Big(-I\partial_s F\Big)ds$$

Then apply integration by parts

$$\int_x^\infty K(x,s)\Big(-I\partial_s F\Big)ds = -K(x,x)IF(x,z) + \int_x^\infty \partial_s K(x,s)IF(s,z)ds$$

So that the full expression is

$$\begin{aligned} D_1\Big(\int_x^\infty K(x,s)F(s,z)ds\Big) &= \int_x^\infty \Big(\partial_y K + I\partial_x K\Big)F(s,z)ds + IK(x,x)F(x,z) - K(x,x)IF(x,z) \\ &\quad + \int_x^\infty \partial_s K(x,s)IF(s,z)ds \\ &= \int_x^\infty \Big(\partial_y K + I\partial_x K + \partial_s K I\Big)F ds + [I, K(x,x)]F \\ &= \int_x^\infty D_1 K F ds + [I, K(x,x)]F \end{aligned}$$

Note that for the operator acting on $K$ to match the operator acting on $F$, we require $J = I$. Therefore, applying $D_1$ to (21), using $F = -K - K * F$, and denoting $u(x) = K(x,x)$

$$D_1 K + D_1 K * F - [I, u(x)](K + K * F) = 0 \tag{49}$$

If we define the augmented operator $\tilde{D}_1 := D_1 - [I, u(x)]$, then $K$ satisfies

$$\tilde{D}_1 K + \tilde{D}_1 K * F = 0 \tag{50}$$

$$\implies \tilde{D}_1 K = 0 \tag{51}$$

Note that equation (50) represents an integral equation which holds true for all $F$. By the Fredholm alternative, we obtain equation (51)

## 5 Matrix valued differential operator, $D_2$

Now suppose we consider the differential operator $D_2$ defined by

$$D_2 F = \partial_t F + \partial_x^2 F - \partial_z^2 F = 0$$

We will derive the corresponding equation that $K$ satisfies. As usual, we apply $D_2$ onto equation (21). Below we list the partial derivatives of (21) making up $D_2$

$$\begin{aligned} K_t + F_t + K_t * F + K * F_t &= 0 \\ K_{xx} + F_{xx} + 2K_x F + K F_x + K_{xx} * F &= 0 \\ K_{zz} + F_{zz} + K * F_{zz} &= 0 \end{aligned}$$

Note that inside the integral when $x = s$, $F$ satisfies $F_t + F_{ss} - F_{zz} = 0$. We express $F_{zz} = F_t + F_{ss}$, substitute this into the last equation above to get

$$\partial_z^2 K(x,z) + \partial_z^2 F(x,z) + \int_x^\infty K(x,s)\Big(\partial_t F(s,z) + \partial_s^2 F(s,z)\Big)ds = 0$$
$$\implies \partial_z^2 K(x,z) + \partial_z^2 F(x,z) + \int_x^\infty \partial_s^2 K(x,s)F(s,z)ds$$
$$+ \int_x^\infty K(x,s)\partial_t F(s,z)ds + \int_x^\infty K(x,s)\partial_s^2 F(s,z)ds = 0$$
$$\implies \partial_z^2 K(x,z) + \partial_z^2 F(x,z) + \int_x^\infty \partial_s^2 K(x,s)F(s,z)ds$$
$$+ \int_x^\infty K(x,s)\partial_t F(s,z)ds + K(x,x)\partial_x F(x,z) - \partial_x K(x,x)F(x,z) = 0$$

Express the last equation above symbolically

$$K_{zz} + F_{zz} + K * F_t + KF_x - K_x F + K_{ss} * F = 0$$

Thus, apply $D_2$ to equation (21),

$$D_2K + D_2F + D_2K * F + K * F_t + 2K_xF + KF_x - K * F_t - KF_x - K_xF = 0$$
$$D_2K + D_2F + D_2K * F + K_xF = 0$$

Now we can express $F = -(K + K * F)$ and substitute into the above equation

$$D_2K + D_2F + D_2K * F - K_x(K + K * F) = 0$$
$$D_2K + D_2K * F - K_xK - K_xK * F = 0$$
$$D_2K - K_xK + (D_2K - K_xK) * F = 0$$

For which we can define the augmented operator $\tilde{D}_2 := D_2 - K_x(x,x)$ such that $K$ satisfies

$$\tilde{D}_2K + \tilde{D}_2K * F = 0 \tag{52}$$
$$\implies \tilde{D}_2K = 0 \tag{53}$$

Therefore, $K$ satisfies the PDE

$$\partial_t K + \partial_{xx}K - \partial_{zz}K - \partial_x K(x,x)K = 0 \tag{54}$$

The corresponding potential is $p(x) = \partial_x K(x,x)$.

## 6 Extracting the Lax Pair

If we were to suppose $K(x,z;y) = \Psi(x,y)\Psi_0(z,y)$ and substitute this into equation (51), we obtain

$$\Psi_y\Psi_0 + \Psi\Psi_{0y} + I\Psi_x\Psi_0 + \Psi\Psi_{0z}I - [I,u]\Psi\Psi_0 = 0$$
$$(\Psi_y + I\Psi_x - [I,u]\Psi)\Psi_0 + \Psi(\Psi_{0y} + \Psi_{0z}I) = 0$$

resulting in the following system of equations

$$\Psi_y + I\Psi_x - [I,u]\Psi = 0 \tag{55}$$
$$\Psi_{0y} + \Psi_{0z}I = 0 \tag{56}$$

Substituting a similar ansatz, $K(x,z;t) = \Psi(x,t)\Psi_0(z,t)$, into equation (54) we obtain

$$\Psi_t\Psi_0 + \Psi\Psi_{0t} + \Psi_{xx}\Psi_0 - \Psi\Psi_{0zz} - \Big(\Psi_x\Psi_0\Big)\Big|_{z=x}\Psi\Psi_0 = 0$$
$$\Big(\Psi_t + \Psi_{xx} - \big(\Psi_x\Psi_0\big)\Big|_{z=x}\Psi\Big)\Psi_0 + \Psi\Big(\Psi_{0t} + \Psi_{0zz}\Big) = 0$$

from which we can derive the system of equations

$$\Psi_t + \Psi_{xx} - \left(\Psi_x \Psi_0\right)\Big|_{z=x} \Psi = 0 \tag{57}$$

$$\Psi_{0t} + \Psi_{0zz} = 0 \tag{58}$$

From equations (55) and (57) we can extract the Lax pair

$$L_1 := I\partial_x - [I, u]$$

$$L_2 := \partial_{xx} - (\Psi_x \Psi_0)\Big|_{z=x}$$

so that our Lax equations can be expressed as

$$\Big(\partial_y - L_1\Big)\Psi = 0$$

$$\Big(\partial_t - L_2\Big)\Psi = 0$$

## 7 Conclusion

We've derived the Davey-Stewartson equation from within the Lax Pair framework. Then we've applied the dressing method to express the Kernel of the Marchenko equation in terms of a rank 1 separable spectral function. Potential well solutions of the Davey-Stewartson are extracted from some number of derivatives of the diagonal of the Kernel and are parameterized by 4 arbitrary functions. We impose reductions to construct normal solutions as well dromion solutions in terms of 2 arbitrary independent functions. We choose these functions to be linear and constant and plot examples of rational solutions only in the case that the independent functions are linear functions. However, this class of functions is much bigger and one can choose any polynomial.

The solutions presented here are static in time and are general solutions of the Dirac operator. We do not study time dependence in this paper. However, time dependence would be extracted in the same way spatial dependence was attained using equations (26) and (27). Instead, we'd apply $D_2(f \cdot g)$ and note that with the second order dispersion relation, our independent functions evolve according to the heat kernel (for real $t$) and according to the Schrodinger equation (for imaginary $t$). We'd expect this surface to oscillate in time in the latter case.

If we were to change the dispersion relation in the Lax Pair, say equip the Dirac operator with the 3rd order dispersion (giving the $2+1$ MKDV equation), the solutions presented here still hold. The only difference is the time dependence of the independent functions.

## 8 Acknowledgments

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